3 Shadows of hovering cubes (25 marks)

A *unit* cube is hovering in space. (Well, it *might* happen!)

a) Prove that the *area* of its orthogonal projection onto the xy-plane is always equal to the *length* of its orthogonal projection onto the z-axis,[3] regardless of how the cube is positioned in space. Pretty amazing, isn't it?

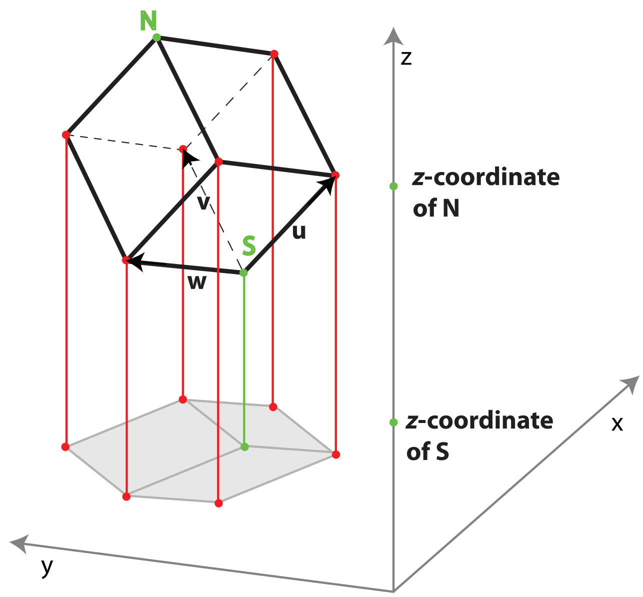

Figure 5: Shadows of a floating cube. The orthogonal projection onto the xy-plane is the gray hexagon. The length of the orthogonal projection/shadow onto the z-axis is the difference between the z-coordinates of the points N and S.

Hints: 1. Express the area and length in question in terms of the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$. 2. Note that because we are dealing with a unit cube $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, $\mathbf{v} = \mathbf{w} \times \mathbf{u}$ and $\mathbf{u} = \mathbf{v} \times \mathbf{w}$.

(10 marks)

[3]The orthogonal projection of a point (a, b, c) onto the xy-plane is the point $(a, b, 0)$. The orthogonal projection of a shape in space onto the xy-plane is simply the collection of the orthogonal projections of all the points of this shape onto the xy-plane.

This means that if we take the (flat) ground we're standing on to be the xy-plane, then the orthogonal projection of a shape onto the ground is just the shadow of the shape cast by the sun when it is directly overhead, with its rays hitting the ground at right angles.

The orthogonal projection of the point (a, b, c) onto the z-axis is the point $(0, 0, c)$. The orthogonal projection of our shape onto the z-axis is the collection of the orthogonal projections of all the points of the shape onto the z-axis.