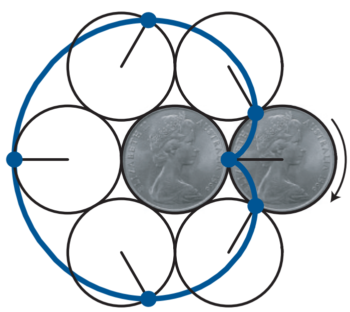
Solutions to last issue's Mathsnacks Coin Puzzles

by Burkard Polster, Marty Ross and QED (the cat)

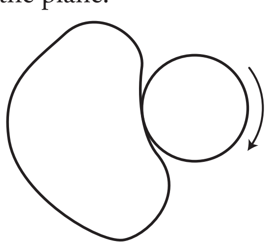
If a coin rolls without slipping around another coin of the same size, how many times will it rotate while making one revolution? Just once, or twice? What if the fixed coin is twice the diameter of the rotating coin? How many revolutions will the coin make if it rolls around 2 coins (all coins being of the same size) that are place side by side? Or n coins?

Solution: We'll give the solutions to our three problems in terms of circles instead of coins. Let's mark a point on the perimeter of the rolling coin/circle, start the circle rolling, and track the marked point.



As the diagram suggests, the rolling coin will rotate twice. If the fixed circle is twice the diameter of the rolling circle, the Coffee Cup Curve (*Mathsnacks*, this issue) suggests that the rolling coin will rotate three times.

In general, consider rolling our circle around a closed curve in the plane.



If, on its journey around the curve, the circle touches every point of the curve, then

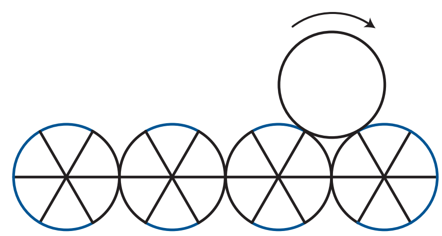
$$\text{rotation number} = \frac{\text{length fixed curve}}{\text{length rolling circle}} + 1.$$

Here is a simple way to prove this. Cut the curve open at some point and uncoil it into a straight line segment. Rolling the circle along this segment it will rotate (length curve)/(length circle) times. Keeping the circle attached to one end of the segment, we then recoil the segment back into the curve, which contributes the final rotation. The same argument shows that if we roll the circle inside the curve, it will rotate

$$\text{rotation number} = \frac{\text{length fixed curve}}{\text{length rolling circle}} - 1$$

In the case of two circles, in which the fixed circle is x times the diameter of the rolling circle, the formulas tell us that if the rolling circle rolls outside or inside, it rotates $x + 1$ or $x - 1$ times, respectively. You can convince yourself that this is true for the cases $x = 1$, 2, 3, 4 using the diagrams on this page and on the *Mathsnacks* backpage.

Next, let's roll a circle around four circles of the same size, placed side by side.



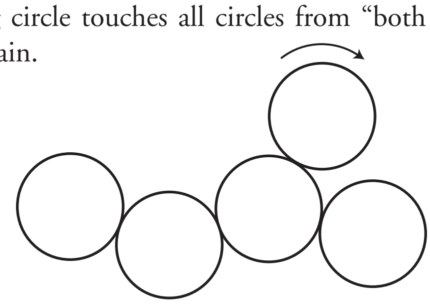
The rolling is along arcs, each one sixth of a circle, and it is easy to see that the total number of arcs travelled is

$$4 + 2 + 2 + 4.$$

This suggests a rotation number of 12/6 = 2. However, consistent with the case of a single circle, we must double this number to get the correct answer: four revolutions. This immediately generalizes to the following formula for the case of n side-by-side circles:

$$\text{rotation number} = \frac{2(4 \cdot 2 + 2(n-2))}{6} = \frac{2n+4}{3}$$

It is fairly easy to see that this formula remains true if the n circles form a chain but are not necessarily fully aligned. However, for the formula to work it is necessary that on its journey around the chain, the rolling circle touches all circles from "both sides" of the chain.



Some of these and other wonderful coin puzzles are discussed at length by Martin Gardner in his classic puzzle collection *Mathematical Carnival*.