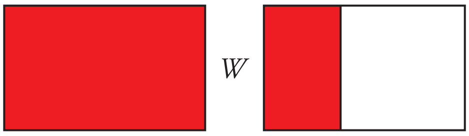
MathSnacks the Golden Ratio

by Marty Ross, Burkard Polster, and QED (the cat)

Repeating Rectangle

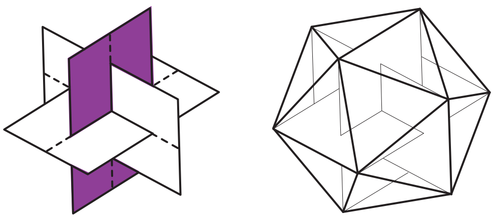


A rectangle is *golden* if removing a square leaves a smaller rectangle of the same proportions. So, if L and W are the dimensions of such a rectangle, then

$$\frac{L-W}{W}=\frac{W}{L}\,.$$

Golden rectangles are famous for their role in aesthetics, though this is as much numerology as mathematics.*

Incredible Icosahedron



The *mathematics* of golden rectangles contains genuine beauty. Defining the *golden ratio* ϕ (Phi) to be L/W, we see

$$\phi-1=\frac{1}{\phi}\,.$$

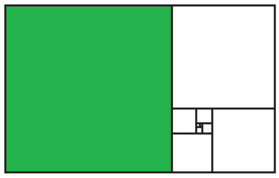
This equation, together with the Pythagorean Theorem, shows how to physically make an *icosahedron*, by slotting three golden rectangles together, as pictured.*

Fibonacci Formula

$$\frac{\phi^n+(1-\phi)^n}{\sqrt{5}}$$

The sequence 1, 1, 2, 3, 5, 8, ... is the famous *Fibonacci sequence*. The next Fibonacci number is 5+8=13, and so on. What if we want the 1000th Fibonacci number, or in general the nth one? We can churn them out, one by one, or we can use the magical formula above.*

Ideal Irrational



ϕ is the number most easily proved to be irrational. For, supposing ϕ *is* rational, we could make a golden rectangle with sides L and W *integers*. And so, the first square pictured also has integer sides, of length $L-W$. Then, the smaller square still has integer sides, and so does the next smaller one, and the next one, and so on. But this infinite diagram, with *all* sides *positive* integers, is clearly impossible: so, the original golden rectangle could *not* have had integer sides, and thus ϕ is irrational!

Ingenious Infinitum

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}}}}}}$$

But what *is* ϕ? We can multiply the equation $\phi=1+1/\phi$ by ϕ, and then solve the resulting quadratic equation to give

$$\phi=\frac{1+\sqrt{5}}{2}$$

Or, since $\phi=1+1/\phi$, we can substitute $1+1/\phi$ for ϕ, giving

$$\phi=1+\cfrac{1}{1+\frac{1}{\phi}}$$

Substituting again and again and again..... ,

Brilliant Books*

J.H. Conway & R.K. Guy, The Book of Numbers, Copernicus, 2002.
M. Livio, The Golden Ratio: The Story of Phi..., Broadway, 2003.