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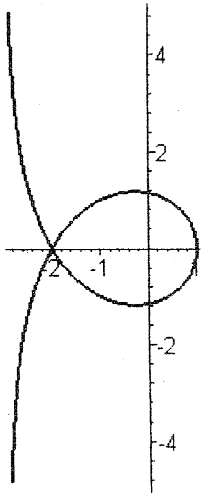

School of Mathematical Sciences – Monash University

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Function is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. It was founded in 1977 by Prof G B Preston, and is addressed principally to students in the upper years of secondary schools, but also more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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THE FRONT COVER

Our cover picture for this issue shows one of the standard forms of a curve known as the *trisectrix of Maclaurin*. It relates to several important themes in the development of Mathematics.

First, the name "trisectrix" relates to the classical problem of finding a ruler and compass method by which to trisect an arbitrary angle. There have been several articles on this topic in *Function*, most recently the history column for August 1999. That article included an account of the proof that the task is impossible. Even before this proof was discovered, however, there was a widespread suspicion that ruler and compass methods alone were not able to perform the operation. Thus other approaches were mooted, and this curve embodies one of them.

The curve was first studied by Colin Maclaurin (1698-1746), a Scottish mathematician most remembered as an early champion of the Calculus against the attacks mounted upon it by Bishop Berkeley. His *Treatise of Fluxions*, the first systematic exposition of the subject, was much praised at the time, although later opinion is less flattering. One recent author, Clifford Truesdell, has described it as "so dense, torve [grim of aspect] and ugsome as scarcely to have been read by anyone but its author and would plunge any beginner into the slough of despond". Possibly this is a little harsh! Maclaurin is also commemorated in the name given to a form of Taylor's Theorem, a standard result in Calculus.

The other body of theory that the curve represents is the topic of cubic equations. Again, these have featured in *Function* on several occasions, most recently in a brief news item in the issue for August 2003. We here discuss the connection between cubic equations and the problem of angle trisection. The connection appeared in the earlier 1999 article, but there only a specific cubic and angle were considered.

There are several standard equations for the curve we exhibit on this number's cover. Different authors use different orientations and different choices of origin. Most include a parameter a, which may however, without any loss of generality, be set equal to one, which is done here. With these understandings, the equation is

$$y = \pm(x+2)\sqrt{\frac{1-x}{x+3}} \tag{1}$$

The effect of the ± sign is seen very clearly in the symmetry that the curve displays about the x-axis.

The curve has several other properties that are straightforward to prove. There is a vertical asymptote at $x = -3$, and the curve is defined for all x in the domain $-3 < x \leq 1$. Apart from the axes, any straight line through the origin cuts the curve in three points. The point $(-2, 0)$ is a point of self-intersection, where there are two distinct values of the slope and thus two distinct tangents. Beyond these obvious features, there are others of interest to mathematicians, but they will not concern us here.

Equation (1) is by no means the only mathematical approach to this curve. Even with the same conventions on orientation, etc, there are others that are also useful. For our present purposes, the one to watch is the use of *polar co-ordinates*. This takes any point P on the curve and uses the distance OP of that point from the origin as one co-ordinate. This is given the label r. The other co-ordinate is the value of the angle between OP and the positive x-axis. This angle is traditionally called θ. In these co-ordinates the equation of the curve is much simpler. It becomes

$$r\cos\left(\frac{\theta}{3}\right) = 1. \tag{2}$$

To connect the two equations takes some work. The x and y of the first equation are related to the r and θ of the second by means of the equations:

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$$

If we now substitute these values into Equation (1) and simplify, we reach

$$r^3\cos\theta + 3r^2 - 4 = 0.$$

This is a cubic equation, and the idea is now to solve it for r to reach a simpler connection between r and θ.

This leads us to the standard theory for the cubic. Begin by dividing throughout by $4r^3$ and putting $u = 1/r$. This gives after some extremely simple rearrangement:

$$u^3 - \frac{3}{4}u = \frac{\cos\theta}{4}$$

This is one of the standard forms of the cubic, which can always be written in a form in which there is no term in the square of the unknown. The more general version of this standard form is

$$u^3 + pu = q.$$

The method of solution is to put $Q = \frac{p}{3}$, $R = \frac{q}{2}$ and $D = Q^3 + R^2$. D is the so-called *discriminant*; it plays the same role as the discriminant $\Delta\left(= b^2 - 4ac\right)$ in the more familiar quadratic equations in that it determines the character of the roots. If $D > 0$, there is one real root and two complex ones; if $D = 0$, there are three real roots, but at least two of them are equal; if $D < 0$, there three unequal real roots.

In our case, $Q = -\frac{1}{4}$, $R = \frac{\cos\theta}{8}$ and $D = -\frac{\sin^2\theta}{64}$ which is clearly negative. We thus expect three real roots, and the standard way to proceed is by means of a trigonometric substitution. Here however, this is to put the cart before the horse, as the substitution was developed specifically to reduce the general form to the special case

$$u^3 - \frac{3}{4}u = \frac{\cos\theta}{4}$$

here under discussion.

In general, we substitute

$$\cos\theta = \frac{R}{\sqrt{-Q^3}}$$

but if we do this here, we merely find that $\cos\theta = \cos\theta$. This is good in a way, as it can act as a check on the work, but it tells us that some new insight is needed to complete the solution. However this is easily rectified, as the point of the general substitution was to reach exactly the equation

$$u^3 - \frac{3}{4}u = \frac{\cos\theta}{4}$$

The reason that this is a good equation to have is that it is known that, for any angle α,

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha.$$

So now compare this with the previous equation, and set $\alpha = \frac{\theta}{3}$. From the comparison we learn that $u = \cos\left(\frac{\theta}{3}\right)$.

This completes the proof of Equation (2).

And now we may see how the trisectrix works. The distance from O to the point P is the reciprocal of $\cos\left(\frac{\theta}{3}\right)$. We can mark on the diagram the point P, measure its distance from the origin, and compare the result with the distance x, the x-co-ordinate of the point P. Then

$$\left.\begin{aligned} r\cos\theta &= x \\ r\cos(\theta/3) &= 1 \end{aligned}\right\}$$

and therefore $x\cos\left(\frac{\theta}{3}\right) = \cos\theta$. $\cos\left(\frac{\theta}{3}\right)$ thus appears as the ratio of two known quantities and so its value may be determined. This in turn enables us to construct the angle $\frac{\theta}{3}$. So, although we cannot use ruler and compass alone to trisect an angle, we can do quite well if our toolkit also includes a trisectrix.

The cover picture was produced using MAPLE. Equation (2) gave a better picture than did Equation (1). The commands that produced the diagram read:

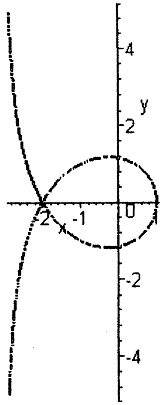
```
>with(plots);
>polarplot(sec(theta/3),theta=-4.18..4.18,scaling=constrained,
     color=black,thickness=2);
```

[Here sec is an abbreviation for 1/cos, and the value 4.18 was chosen for consistency with the picture below, deriving from Equation (1).]

Equation (1) was graphed equally painlessly:

```
>with(plots);
>implicitplot((y^2)*(3+x)=((x+2)^2)*(1x),x=-3..1,y=5..5,scaling=
    constrained,numpoints=10000,color=black,thickness=2);
```

Here is the result. We thought it less attractive than the other.



A final question for readers to ponder over. A cubic has three roots and in this case they are all real. These must correspond to the three intersections between the trisectrix and the straight line through O making an angle θ with the positive x-axis. How are the other two points to be interpreted?

Readers seeking further information could do worse than to consult the websites

http://www-groups.dcs.st-and.ac.uk/history/Curves/Trisectrix.html

with a linked page on the life of Maclaurin, and

http://mathworld.wolfram.com/MaclaurinTrisectrix.html

which derives in large measure from the other, but is more detailed and has links to useful explanatory pages.

MATHEMATICAL CLASSIC

Hermann Bondi on the Night Sky

[From time to time, *Function* reprints an excerpt from a famous mathematical classic. In recent years, we have had George Boole on the laws of thought and Florian Cajori on how the ancient Babylonians solved quadratic equations. Here in the same spirit is Hermann Bondi on "Olbers' Paradox" and the prediction of the expanding universe. It comes from his book *The Universe at Large*, a reprint of an early (1959) account first appearing in *The Illustrated London News*.]

"The question ... arises of whether ... very distant stars, [that] would individually be too faint to be seen, might not be so exceedingly numerous as to provide an even background illumination of the night sky? This is the question that the German astronomer Olbers asked [in] 1826. [He argued as follows.]

"On the basis of ... four assumptions,

> "[1. Distant stars are separated by the same average distance as nearby stars; 2. Distant stars have the same average brightness as nearby stars; 3. Light is propagated in distant regions of the universe in obedience to the same laws as govern it nearby; 4. (Implicitly) There are no large-scale motions of the universe, which is essentially static.]

it is easy to work out the background light of the sky. Imagine a vast spherical shell surrounding us. .... The thickness of the shell is supposed to be small compared with its radius; but the whole shell is supposed to be so enormous that there are vast numbers of stars within the shell. How many stars are there in this shell? If we call the radius of the shell R and its thickness H, then we can readily see that the surface of the sphere on which the shell is built is $4\pi R^2$ and thus the volume of the shell is, to a sufficient approximation, $4\pi R^2 H$. If, now, N is the number of stars per unit volume, then the number of stars in the volume $4\pi R^2 H$ will be $4\pi R^2 HN$. How much light will all the stars in the shell send out? If the average rate at which an individual star sends out light is L, then all the stars in the shell put together will send out $4\pi R^2 HNL$. However, what interests us is not how much light all these stars send out, but how much light we receive from them. Consider the light of an individual star in the shell. By the time the light reaches us, it will have spread out over a sphere of surface $4\pi R^2$. That is to say, the light of each individual star has to be divided by $4\pi R^2$ to tell us the intensity of light from it which is received here. This is true of all the stars in the shell, and, therefore, the total light we receive from all the stars in the shell is the total light they send out divided by $4\pi R^2$. This division leads to the cancellation of the factor $4\pi R^2$ and we are left with *HNL*.

"It will be seen that this does not involve the radius of the shell at all. The amount of light we receive from any shell of equal thickness is the same irrespective of the radius of the shell. If, therefore, we add shell after shell, then, since we get the same amount of light from each shell, the amount received will go up and up without limit. [Even if we take account of the fact that some stars block the light emitted by others, we find that we get from] all these shells of stars a flood of light equal to 50,000 times sunlight when the sun is at its zenith. On this basis, then, it should be incredibly bright both day and night. Everything would be burned up; it would correspond to a temperature of over 10,000 degrees Fahrenheit [over 5500 degrees Celsius]. Naturally, this remarkable result astonished Olbers, and he tried to find a way out. .... [Various] ways have been tried, but none of them works. We are, therefore, inevitably led to the result that, on the basis of Olbers' assumptions, we should be receiving a flood of light which is not, in fact, observed."

Bondi then goes on to discuss the assumptions detailed at the start of this excerpt, and to decide that it is the fourth (implicit) assumption that is in error. This leads him to the deduction that the universe must be expanding, as modern theory now completely accepts. As he writes: "... the forecasts of the theory do not agree with observation, and thus the assumptions on which the theory is based must be wrong. .... By this method of empirical disproof, we have discovered something about the universe and have so made cosmology a science."

BOOK REVIEW

Vasile Berinde: *Exploring, Investigating and Discovering in Mathematics* (Birkhäuser, 2004) 246 + xx pp. Reviewed By Michael A B Deakin.

The author of this new book is a highly experienced problem setter and solver based in Eastern Europe. The original edition was published in 2001 in Romanian, and this version claims to be a literal translation. It would seem to be so, as the English is at times stilted rather than idiomatic. This, however, is my only criticism of a work that deserves notice for its content, its valuable asides and its overall structure. It comprises a collection of interesting and challenging problems, but it also fulfills a useful reference role for anyone concerned with the posing and the solution of mathematical problems.

The level of Mathematics will strike Australian readers as high, and this may well reflect the difference in standards between schools in this country and those of Eastern Europe. (Regular readers of *Function* will recall how well teams from Eastern Europe perform in the International Mathematical Olympiads.)

The book is arranged into 24 chapters, and concludes with a brief but interesting addendum. Much of the material is drawn from the Romanian journal *Gazeta Matematica*, an approximate counterpart of *Function*, but which was founded in 1895, and is still going strong. The most frequently cited author is Berinde himself, and his comments are always to the point.

Each chapter begins with a "Source Problem", and follows this up with variations on the theme established by it. Solutions are given to the source problem and its variants in such a way as to instruct not only in the relevant Mathematics, but also in the strategy of seeking solutions.

I will not list all 24 of the themes discussed, but will rather concentrate on a single example that should interest *Function's* readers. Chapter 12 concerns numbers in arithmetic progression. Suppose the sequence $\{a_p\}$ makes up an arithmetic progression for all $p \geq 1$. The source problem asks for a proof that $a_p - 3a_{p+1} + 3a_{p+2} - a_{p+3} = 0$ for all positive integral *p*. The proof is straightforward, and readers are invited to construct it for themselves. But now, Berinde generalises the problem and asks for a proof that

$$a_p - \binom{n}{1}a_{p+1} + \binom{n}{2}a_{p+2} - \ldots + (-1)^n\binom{n}{n}a_{p+n} = 0.$$

(The notation here is that explained in this issue's History Column.)

With this problem solved, Berinde goes on to speculate on the converse proposition: *If the equation above is true, then is the sequence* $\{a_p\}$ *an arithmetic progression?* It turns out that it is, but this in its turn suggests a further related problem, based on Equation (2) of the History Column in this issue. This then leads to a further converse result and another generalisation. And still we are not finished; further generalisations and converses appear. All in all, the source problem generates nine further related problems. The final section details the provenance of all the various problems and gives references for further reading. (The references, however, may not be so easy to come by. They are mostly in Romanian and in books or journals published in Romania!)

This chapter is typical. The format and organisation of each of the others follows similar lines. My one general comment would be that, just as the level of Mathematics assumed is higher than that of Australian schools, so too are the demands made in other directions. The reader is assumed familiar with a range of notational conventions and technical terms that lie outside what our own schools teach. This is indeed the language of today's Mathematics, and so perhaps Australia has some catching up to do.

The final section, the Addendum, deserves especial notice. It is the shortest of all and runs to a mere four pages. It offers three general principles that a problem solver should respect (along, of course, with accuracy and rigour). These are: (1) "The solution should be clearly constructed, highlighting the various stages, so that the basic lines constitute a well-traced sketch for solving the problem", (2) "[Once] we have solved a problem and the solution has a coherent method, we should investigate to what extent that method can be applied to problems related to the initial one", (3) "[Once] we have solved the problem, we should not consider the solution complete until" we have analysed all its components, examined the status of all the arguments employed and checked the role played by all the data supplied.

Very sound advice, and usefully amplified. It makes a nice finish to a very nice book.

LETTERS TO THE EDITOR

More on Currency Conversion

I read with interest the article on "The Doubly Golden Euro" in last February's issue. But readers should know that the golden ratio, τ, is not the only number whose reciprocal is easily calculated. At a time when the Australian dollar was worth about 71 US cents, I realised that the US dollar was about \$1.41 in our money.

This is because $1.41 \approx \sqrt{2}$ and $\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$.

Similarly, because $\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$, we can calculate that when our dollar was down to 58 US cents, their dollar was worth 1.73 of ours.

There is a whole family of such reciprocals. The next one, based on the number 4, is very simple, and leads into a second family where the number to be inverted is a simple whole number ratio. For example 0.75 is $\dfrac{3}{4}$ and its reciprocal $\dfrac{4}{3} \approx 1.33$. Similarly, $0.625 = \dfrac{5}{8}$ and $\dfrac{8}{5} = 1.60$. (Not all that far from the golden ratio, this one!)

A third family can be constructed from the binomial approximation $\dfrac{1}{1 \pm \varepsilon} \approx 1 \mp \varepsilon$ for small values of ε. This has worked at times for the Swiss Franc and the Singapore dollar in relation to our dollar.

Bernard Anderson
Portland College

[*It seems likely that the entire range $0 < x < 1$ can be covered by the use of devices such as Mr Anderson describes. We leave the exploration of this question to readers. Eds.*]

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More News from Wales

My friend and co-worker Kim Dean has asked me to pass on to *Function's* readers the discoveries I have made in going through the manuscripts left behind at the death of our mutual friend Dai Fwls ap Rhyll. Kim wrote a year ago about the death of this controversial genius; his loss is mourned by those of us who knew him, and by a wider public as well. Once his death was confirmed, the relevant authorities acceded to my request and appointed me as his literary executor. It is in this capacity that I write to you.

The task is far from easy. In 2002, Kim did a wonderful job making sense of a painfully scrawled document that nevertheless discovered serious flaws in established mathematical formulae. I have had to follow in the same tradition and it has been heavy going!

Of those documents that I have had some success in deciphering, perhaps the most interesting is his discussion of Euclid's parallel postulate. Surely everyone will know the basic background. Euclid's "fifth postulate" became the basis on which he formulated his account of parallels. It was later replaced by the so-called "Playfair's Axiom", which states:

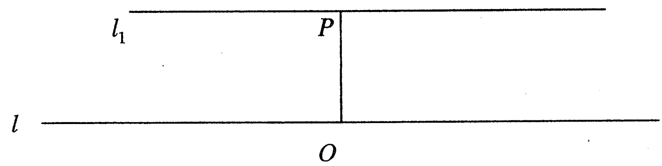
> Given any line l and any point P not on it, there is exactly one line through P parallel to l.

The development of non-euclidean geometries depended on the supposed impossibility of demonstrating the truth of this axiom from the other axioms and postulates in Euclid's *Elements*.

All this theory, I can now confidently say, is wrong! Dr Fwls has produced a simple proof of Playfair's Axiom based only on material in the *Elements*. Nor is the new material abstruse or difficult. Like all of Dr Fwls' work, it uses only simple mathematical concepts, rarely if ever straying beyond the confines of what is routinely taught in High School Mathematics courses.

Here is how Dr Fwls proves Playfair's Axiom.

From the point P, drop a perpendicular to the line l, meeting it in a point O. Now through P draw a line l_1 perpendicular to OP. The situation is as shown in the diagram below:



Consider now the question of whether l and l_1 intersect. Suppose that they do, and suppose, for definiteness, that they do so in a point A to the left (in the diagram) of OP. Then, by symmetry, they will also meet in a point B to the right of OP (and indeed equidistant from that line).

Then the points A, O, B are collinear, as all three lie on the line l. Similarly the points A, P, B are also collinear as all three lie on the line l_1. Thus the two different lines l, l_1 meet in two distinct points A, B, and this is clearly impossible in view of Euclid's other assumptions.

It follows that l_1 must be parallel to l. Furthermore, the construction admits only of a single line l_1. It follows that the line l_1 is a unique parallel to line l passing through the point P.

Thus Playfair's Axiom is proved.

Where generations of eminent mathematicians have strayed into fanciful realms, Dr Fwls has here introduced a refreshing note of common sense.

Sue de Nimmes,

Llanfairpwllgwyngyll-
gogerychwyndrobwyll-
antisiliogogogoch,

Wales

HISTORY OF MATHEMATICS

Meditations on a Tattoo

Michael A B Deakin

I was out shopping when I saw a fellow with a tattoo on his shoulder. That in itself is hardly remarkable, but *this* tattoo took the form of a mathematical equation. It read:

$$K_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \tag{1}$$

First let me explain the notation. $\binom{n}{\nu}$ is the "combinatorial symbol". It stands for the number of ways in which a subset of ν elements may be chosen from a full set of n such elements. [This is what most mathematicians would regard as the standard notation, although the older ${}^nC_\nu$ retains some currency in high schools, and various mathematical software packages use yet others.]

The numbers $\binom{n}{\nu}$ will be most familiar to *Function's* readers as the entries in Pascal's Triangle:

0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
	etc

Here the column on the extreme left gives the number of the row (starting, please note, from 0). Within each individual row, number the entries 0, 1, 2, etc. Then $\binom{n}{\nu}$ is the νth entry in the nth row. $\binom{4}{2} = 6$, etc.

The notation $\sum_{\nu=0}^{n}$ is a shorthand for addition; it tells us to insert in order the values 0, 1, 2, …, n for ν into what follows, here $\binom{n}{\nu}$, and to add up all the results. So, what is meant here is:

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}$$

This amounts to adding up the entries in the nth row of Pascal's Triangle. When we do this, we find:

Row 0	1
Row 1	2
Row 2	4
Row 3	8
Row 4	16,

which strongly suggests the generalisation

$$\text{Row } n \qquad 2^n.$$

In the notation I have just been explaining, this generalisation would read

$$\sum_{\nu=0}^{n}\binom{n}{\nu}=2^n. \tag{2}$$

A proof of this result may readily be supplied. The *binomial theorem* concerns the expansion of $(a+b)^n$, and it may be approached by considering the product $(a+b)(a+b)\ldots(a+b)$, where there are n factors involved. When the multiplications are carried out, the result will be a sum of terms each of which is of the form $a^\nu b^{n-\nu}$, and, as the ν a's may be chosen from the n factors in $\binom{n}{\nu}$ different ways, there are $\binom{n}{\nu}$ such terms. Thus the binomial theorem, in the notation I have just described, reads

$$\left(a+b\right)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} a^{\nu} b^{n-\nu} .$$

Now simply put $a = b = 1$ into this formula to reach Equation (2).

These were the thoughts that occupied me when I saw the tattoo. I made bold to speak to its owner along these lines and suggest the left-hand side of the equation might more accurately read, not K_n but 2^n. My suggestion was loftily dismissed, but I did learn from him that the formula had been copied from Wittgenstein's *Tractatus*.

This is the short name of an influential book by the philosopher Ludwig Wittgenstein. Its full title is *Tractatus Logico-philosophicus*, and it was the first of his many books, but the only one published in his lifetime. It is a somewhat quirky production, consisting of a sequence of oracular utterances ("propositions"), numbered in a manner reminiscent of the Dewey decimal system for cataloguing books.

The underlying rationale is explained in a footnote, which reads:

> "Note. The decimal numbers assigned to individual propositions indicate the logical importance of the propositions, the stress laid on them in my exposition. The propositions *n*.1, *n*.2, *n*.3, etc. are comments on proposition no. *n*; the propositions *n.m*1, *n.m*2, etc. are comments on proposition no. *n.m*; and so on."

The *Tractatus* was written in German, but there are two English translations, of which the second is regarded as more accurate, although the first is better known. In the older, more familiar, of the two translations, the first few of these propositions are:

1. The world is everything that is the case.
1.1 The world is a totality of facts, not of things.
1.11 The world is determined by the facts, and by their being *all* the facts.
1.12 For the totality of facts determines both what is the case, and what is not the case.
1.13 The facts in logical space are the world.

These propositions describe a view of the world that Wittgenstein was concerned to promote. Most of us would see the world as being made up of *things* – what Wittgenstein later calls “simples”. He, however, saw a new approach to Philosophy, basing his analyses on the language in which our thoughts are expressed. The *Tractatus* takes a very simplified model of language. It considers the basic unit of discourse to be the straightforward assertion, which may be either true or false. Then if there are two such assertions, there are four possibilities: both true; first true, second false; first false, second true; both false. And so on for more than two assertions. It is in this connection he gives Equation (1) as his Proposition No 4.27.

Quite how Wittgenstein meant his equation to be interpreted is a matter for some analysis. The symbols $\binom{n}{\nu}$ and $\sum_{\nu=0}^{n}$ have standard interpretations, and any mathematician will understand what they mean. K_n however does not. Thus Wittgenstein is not asserting a result – as he would have been had he written Equation (2); rather he must be using the equation in a *definitional mode*, defining what he means by K_n.

It is rather surprising that Wittgenstein did not immediately recognise that K_n had the value 2^n. In a career that spanned many professions (primary school teacher, soldier, architect, professor of Philosophy) he had also worked as an aeronautical engineer and so become interested in a roundabout way in Mathematics and its foundations. This had led him to work under the philosopher-mathematician Bertrand Russell, who supervised his graduate study. Eventually the *Tractatus* was accepted as his doctoral dissertation. It is also surprising that Russell seems to have missed the point also.

The viewpoint of the *Tractatus* is aptly named “logical atomism”. When he first wrote it, Wittgenstein believed it to offer a solution to all the problems of Philosophy. Its approach was certainly new and the work has been most influential. The emphasis on language was in itself a departure, but the insistence on *facts* instead of *things* being the stuff of reality was more radical.

We can get a feel for the new viewpoint if we reflect on the analysis of Rubik’s Cube. (See *Function*, October 1981 and February 1982.) The key to unravelling the puzzle is to analyse the effect of *operations* on the cube, rather than the state it is in. The rise of abstract algebra lent some readiness to this type of thought, and it informs some of the metaphysical thinking of Alfred North Whitehead, the mathematician who supervised Bertrand Russell's own studies and who collaborated with Russell in the writing of *Principia Mathematica.*

So perhaps the time was ripe for this novel approach. "The facts in logical space are the world." Of all the statements that may be *imagined* to be true (logical space), those that actually *are* constitute the world. The *Tractatus* thus limits discussion to the uses of language in making assertions. Not all of the things we might like to say fall into this category. These Wittgenstein dealt with in his final Proposition:

> 7 Whereof one cannot speak, thereof one must be silent.

Indeed, Wittgenstein is said to have condensed the entire *Tractatus* into one compound sentence:

> "What can be said at all can be said clearly, and what we cannot talk about we must pass over in silence."

The first part of this sentence summarises Proposition 4.116; the second is Proposition 7; the entire sentence is given at the website

http://www.brewbooks.com/ret/tlp/ref_tlp.html

Later, Wittgenstein came to reject the narrow focus of the *Tractatus*, and paid more attention to the complexities of real language. As he wrote in introducing his later (posthumous) work *Philosophical Investigations*:

> "... since beginning to occupy myself with philosophy again, sixteen years ago, I have been forced to recognise grave mistakes in what I wrote in that first book. I was helped to realise these mistakes – to a degree which I myself am hardly able to estimate – by the criticism which my ideas encountered from [the mathematician] Frank Ramsey, with whom I discussed them in innumerable conversations during the last two years of his life."

Real languages display a complexity that has been appreciated even more in the years since Wittgenstein. The following example (a well-known one) comes not from him but out of the early experiments in machine translation. The sentence

"Time flies like an arrow"

has of course the meaning we usually attribute to it, but we can also analyse it to mean

"Get out there pronto and start timing flies"

or as a (bizarre) analogue of

"Fruit flies like a banana".

Yet we never consider these other possible readings; we know at once what is meant.

However, the tattooed man did not enter into all these subtleties. He seemed to ascribe a mystical significance to the formula, which he saw as expressing the manifold possibilities the world has to offer. (It is most doubtful that Wittgenstein meant what the man thought he meant.) "Wittgenstein wouldn't make a mistake," he declared.

No, this is not exactly a mistake, but we might class it as an oversight. Max Black, in his very detailed and thorough work *A Companion to Wittgenstein's "Tractatus"*, remarks laconically "K_n: has the value 2^n".

A similar oversight is to be found in the later Proposition 4.42, which gives the formula $\sum_{K=0}^{K_n} \binom{K_n}{K} = L_n$. Black likewise provides the formula for L_n, which I invite readers to discover for themselves.

However, Wittgenstein *did* make mistakes in his mathematical formulae. His *Philosophical Investigations* was left incomplete at the time of his death, although the incompleteness was only minor. However it contains a formula that the editors (Anscombe & Rhees) who prepared the text for publication felt compelled to correct. What should be written as $2x + 1$ appeared in the manuscript as $x^2 + 1$. The error and its correction occur in §226, on p 86e. It seems a curious mistake, but perhaps we can take a charitable view and assume that Wittgenstein would have made the correction himself, had he lived to do so!

NEWS ITEMS

Yet more on the Collins Case

In June 2003, we ran an account of a US court case involving expert statistical evidence: the so-called "Collins Case". A couple were arrested and charged with a crime because their description matched exactly that of the perpetrators. The odds of any randomly selected couple matching this description were assessed as 1 in 12,000,000. At their trial, this figure was quoted as the probability of their innocence. This identification of one set of odds with another is however wrong; in fact it is now called the "prosecutor's fallacy".

In our June description, we reported on an analysis by T Rolf Turner of the University of New Brunswick. Turner likened the situation to that of a very large number of beads in a barrel in which almost all the beads are white, but there is a very small probability that any given bead may be red. Turner assumed the probability of any given bead's being red as $p = \frac{1}{12{,}000{,}000}$ and took the number of beads to be N, whose value he took to be 2,000,000. (These numbers arise from evidence given at the trial and the subsequent appeal and, although their accuracy has been questioned, they are accepted for the sake of argument in almost all discussions of the case.)

We thus have a situation where a very large number N is balanced against a very small probability p. This type of problem leads to the *Poisson distribution* with a parameter m equal in value to the product Np. m is the expected number of beads in the barrel; its value here is 1/6. The Poisson distribution however requires modification in this instance, to take account of the undoubted fact that a couple *did* match the unlikely description: i.e. there was known to be at least one red bead in the barrel.

Turner's analysis has more recently been questioned by Halvor Mehlum of the University of Oslo. The distinction between Turner's account and Mehlum's is subtle and serves to illustrate the difficulty involved in applying probability theory to real life situations. Mehlum argues with Turner on the basis that "[the] only information that Turner extracts from the circumstances is that there is at least one red bead". He calls this piece of data F_0. Mehlum would replace this piece of data with a stronger one: a "bead drawn at random is red". He calls this F_1.

He argues that the second interpretation better applies to the case in hand. This is justified by means of a verbal argument, essentially based on the assumed independence of the description of the perpetrators and the commission or not of the crime. This is what justifies his "drawn at random" wording. In *Function*, we wrote: "It is established by careful investigation that the barrel *does indeed* contain a red bead". For Mehlum, this "careful investigation" consists of drawing out a bead at random, finding it to be red and then putting it back into the barrel.

Now write $\Pr\{r = n \mid F_0\}$ for the probability that there are n red beads in the barrel, given that F_0 is true, and $\Pr\{r = n \mid F_1\}$ for the probability that there are n red beads in the barrel, given that F_1 is true.

The theory of the Poisson distribution now gives:

$$\Pr\{r = n \mid F_0\} = \frac{e^{-m}}{1 - e^{-m}} \frac{m^n}{n!} \quad \text{and} \quad \Pr\{r = n \mid F_1\} = \frac{e^{-m} m^{n-1}}{(n-1)!}$$

On the basis of *his* distribution (the first of these), Turner calculated the probability that Mr and Mrs Collins were innocent as 4.3%, a figure that makes it unlikely, but large enough to constitute "reasonable doubt". Mehlum however, on the basis of *his* distribution (the other), reaches the figure of 7.9%, somewhat more favorable to the accused.

Perhaps we might reflect that Turner's analysis makes the less tendentious assumption F_0, which we can most surely accept. Mehlum makes the stronger assumption F_1 and as he himself comments, this enables him to come up with the higher figure. Mehlum also relates the case to the so-called "island problem" (see *Function*, April 1980).

A general point might perhaps be in order. Turner's assumption F_0 is incontrovertible, and even this (weaker) assumption is enough to allow the accused the benefit of the doubt. Mehlum's assumption F_1 is perhaps more precise, but it is less assured. Were one arguing the case in court, perhaps the simpler course to follow would be to use Turner's analysis, even though Mehlum's is likely to be more accurate!

Means, more Means and still more Means

Mathematicians are a little suspicious of the word "average" and often refer to the concept involved as a "measure of central tendency". Three such measures are the *mean*, the *median* and the *mode*. These three were the subject of an early article in *Function* (February 1979). However, the word *mean* is itself ambiguous, and four different meanings of this word were discussed in our issue for August 1991.

There are however many more "means" and a recent article in *Mathematics Magazine* explores some of them. *Mathematics Magazine* is an approximate counterpart of *Function*, published by the Mathematical Association of America (MAA). Whereas *Function* is meant for senior secondary students, *Mathematics Magazine* is addressed to undergraduates. However, there is some overlap in standards and material. In their issue for October 2003, they publish an article that readers of *Function* could follow and enjoy.

The author is Howard Eves, who in the late 1960s produced his book *In Mathematical Circles*, a real treasure of anecdotes and miscellanea of a mathematical nature, that sold so well that it led to the production of several sequels. (The entire set has just been reissued by the MAA.)

In this article, Eves considers seven of these "means". If a and b are two positive numbers, then their different means are:

Arithmetic	$A(a,b)=\dfrac{a+b}{2}$
Geometric	$G(a,b)=\sqrt{ab}$
Harmonic	$H(a,b)=\dfrac{2ab}{a+b}$
Heronian	$N(a,b)=\dfrac{a+\sqrt{ab}+b}{3}$
Contraharmonic	$C(a,b)=\dfrac{a^2+b^2}{a+b}$
Root Mean Square	$R(a,b)=\sqrt{\dfrac{a^2+b^2}{2}}$
Centroidal	$T(a,b)=\dfrac{2\left(a^2+ab+b^2\right)}{3(a+b)}$

Our 1991 article considered four of these: A, G, H and R , and included a proof that $H < G < A < R$. Eves has a further set of inequalities: $H < G < N < A$.

We leave readers to look at Eves' article for themselves, but make some comments of our own on the matters related to those he raises.

You may care to prove that all these means lie between a and b and that, as long as a and b are both positive and unequal, then all the means are different. Can you provide a complete listing of the seven means in order of their size?

Four of the means are special cases of "power means":

$$P_n(a,b) = \left(\frac{a^n + b^n}{2}\right)^{1/n}$$

A is the case $n = 1$, R is the case $n = 2$, and H is the case $n = -1$. Although it is far from obvious, G is the case $n = 0$. For a proof of this, see *Function*, *Volume* **16**, *Part* **1**, p 23. The inequalities given in our 1991 article can then be seen as special cases of a more general result:

$P_n(a,b)$ in an increasing function of n.

(And *this* in its turn is a special case of an even more general result known as *Jensen's Inequality*.)

It will be noted that the various means are not independent of one another. The most obvious connection is that $C = \dfrac{R^2}{A}$ but there are also others. If we extend the definitions to more than two numbers, we would have formulae such as $A(a,b,c) = \dfrac{a+b+c}{3}$ and then we could write, for example,

$$N(a,b) = A(a,b,G(a,b)).$$

The list goes on.

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COMPUTERS AND COMPUTING

The "Kahane Function"

In 1990, W M Kahane addressed the International Congress of Mathematicians in Kyoto, and proposed a number of tests for computer accuracy. One of these was briefly reported in *Function* in our issue for February 1991.

It goes like this. Define

$$f_n(x) = \left(x^{2^{-n}}\right)^{2^n}$$

where x is a positive number and n is a positive integer.

Now, very clearly, we have $f_n(x) = x$, as a straightforward consequence of the index laws. However, Professor Kahane was concerned to study $f_n(x)$ not in this ideal sense, but rather as evaluated by a computer. The extent to which the value differs from x provides a measure of the computer's inaccuracy.

We can easily compute $f_n(x)$ on a hand-held calculator by entering x, pressing the $\sqrt{\ }$ button n times and then pressing the ^{2} button n times. In the earlier *Function* article, it was reported that a Casio *fx*-570 gave $f_{29}(2) = 1.805$ and $f_{30}(2) = 1$. This is actually quite good accuracy. By comparison, a Casio *fx*-82L gives $f_{10}(2) = 1.9999997$.

The reason for the error is not hard to trace. Whatever number x we begin with, the sequence of values of $\sqrt{\sqrt{\ldots\sqrt{x}}}$ will tend towards 1, and after sufficiently many presses, the calculator will record it as either exactly 1 or possibly 1.00.....1, if $x > 1$, or as 0.99....9, if $x < 1$. Once this point is reached, we get a value of either 1 itself or else a power of $1 \pm 0.00....1$, rounded to whatever accuracy the computer can achieve.

The dependence on x is not especially sensitive, unless of course x is very near 1. If we start with $x = 4$, for example, we would expect to find that $f_9(4) \approx f_{10}(2)^2 \approx 1.9999997^2$ and so we do. The Casio *fx*-82L returns a value 3.9999996. Each squaring of the value of x corresponds closely to a reduction of 1 in the value of n.

If we try the test on MAPLE, the result depends on how the calculation is configured. If we simply ask the machine to compute (x^(2^(-n)))^(2^n) for a specific value of n, it will return the answer x even for values of n as high as 1000. It knows about the identity (although if we type in n in place of the specific value, it can't simplify the resulting expression).

However, if we make the package actually compute the intermediate value, as the hand-held calculator has to do, then the performance is not particularly good. On the default setting (ten significant figures, one more than the Casio *fx*-82L), we find:

```
> 2^(2^(-5));
                      (1/32)
                     2

> evalf(%);

                     1.021897149

> %^(2^5);

                     2.000000022
```

whereas with $x = 2$, $n = 5$, the Casio returns the value 2.

EXCEL, by contrast, does exceedingly well. Starting again with $x = 2$, and continuing to take square roots and to square, we find that the square roots are recorded as 1 from $n = 22$ onwards, but that nonetheless the value 2 for $f_n(2)$ is still returned until $n = 33$. Even after this, the value returned is still reasonably close to 2, and so it continues, but with the error getting larger. When $n = 52$, the value returned is the truncated decimal expansion of the number e, the base of the natural logarithms. This may say something about how the EXCEL spreadsheet is programmed.

Readers are invited to explore the Kahane function for themselves. There's a lot to discover, both theoretically and experimentally!

PROBLEMS AND SOLUTIONS

We begin with the solutions to the problems set in August 2003.

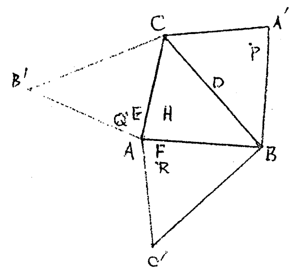
Solution to Problem 27.4.1

This problem, taken from *The Australian Senior Mathematics Journal*, may be approached in various different ways. It read:

> Through the vertices of a triangle ABC, draw its circumcircle. Form three other circles by reflecting this circle in each of AB, BC, CA. Show that these three circles all pass through a common point.

We received solutions from Bernard Anderson, Keith Anker, John Barton and Julius Guest. Anderson and Guest used different approaches based on co-ordinate geometry. Anker used an ingenious argument based on the angle in a semicircle's being a right angle. The elegant solution given here is Barton's. He wrote:

"As well as reflecting the circumcircle ABC in, say, side AB, reflect the other two sides BC, CA as well. And similarly for each of the other sides BC, CA chosen as the reflecting agent. Let the reflections of the vertices A, B, C be denoted , respectively by A', B', C'. Then AA', BB', CC' coincide respectively with the altitudes AD, BE, CF of the triangle ABC, and hence are concurrent at the orthocentre H."



Barton went on to note that there is a standard theorem (to be found, for example, in C V Durell's *Modern Geometry* as Theorem 20) to the effect that, if the circumcircle ABC cuts an altitude AD at P, then $HD = DP$. Similarly $HE = EQ$ and $HF = FR$. Thus all three circles intersect at H.

Solution to Problem 27.4.2

This problem was based on an article by T Eisenberg in the *International Journal for Mathematical Education in Science and Technology*. It read:

> *A*, *b* and *N* are non-negative real numbers with $b \neq 0$. Let $a_0 = a$, $b_0 = b$ and then form the sequences $\{a_n\}$, $\{b_n\}$ according to the rules $a_{n+1} = a_n + Nb_n$, $b_{n+1} = a_n + b_n$. Prove that as $n \to \infty$, $\frac{a_n}{b_n}$ tends to a limit and determine the value of that limit.

We received solutions from Bernard Anderson, Keith Anker, John Barton, Julius Guest and Joseph Kupka. All gave similar analyses, although details differed. What follows is a composite.

First note that $b_n \neq 0$. This may very easily be proved by induction. Now write $x_n = \frac{a_n}{b_n}$. Then $x_{n+1} = \frac{x_n + N}{x_n + 1}$. Now *if* x_n tends to a limit *L*, then *L* must satisfy $L = \frac{L+N}{L+1}$ and this entails that $L = \sqrt{N}$.

It remains to show that the sequence indeed converges. There are four cases: (1) $N = 0$, (2) $0 < N < 1$, (3) $N = 1$, (4) $N > 1$. Of these, the first and third are easy and the details are left to the reader. In Case (2),

$$x_{n+1} - \sqrt{N} = \frac{x_n + N}{x_n + 1} - \sqrt{N} = \left(1 - \sqrt{N}\right)\left(x_n - \sqrt{N}\right)\frac{1}{x_n + 1}.$$

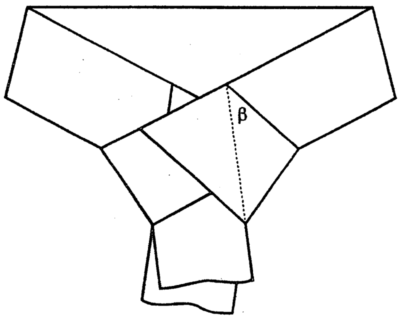
Now $x_n > 0$, and therefore $\left|x_{n+1} - \sqrt{N}\right| < \left(1 - \sqrt{N}\right)\left|x_n - \sqrt{N}\right|$. It follows that for any positive integer *p*, $\left|x_{n+p} - \sqrt{N}\right| < \left(1 - \sqrt{N}\right)^p\left|x_n - \sqrt{N}\right|$. Now letting $p \to \infty$, $\left(1 - \sqrt{N}\right)^p \to 0$, so that as *n* increases, $x_n - \sqrt{N}$ tends to zero.

In Case (4), begin by writing $\nu = \frac{1}{N}$ and interchanging the roles of x_n and $\frac{1}{x_n}$. The argument then proceeds as in Case (2).

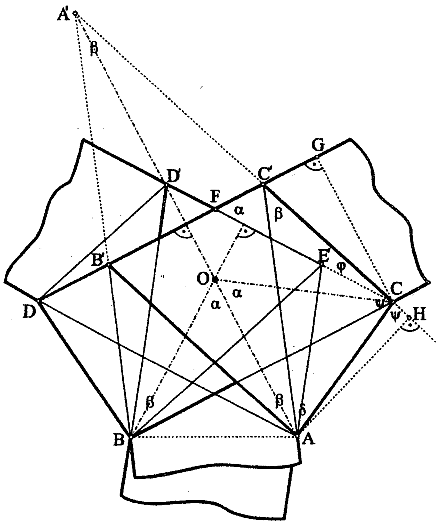
Solution to Problem 27.4.3

This problem was submitted by Avni Pllana of Austria, who also submitted a solution, as did Keith Anker, whose solution was different. The problem read:

> A strip of paper is folded into a "Tie" as shown below. Determine the angle β.



The two solvers seem to have interpreted the problem differently. Pllana relates it to the diagram below, where the knot is indicated by the



more prominent lines. (Compare the two figures.) He then argues as follows.

"From the constant width it follows that

$$\overline{AH} = \overline{CG} \,, \tag{1}$$

and

$$\overline{BB'} = \overline{B'C'} = \overline{CC'} \,. \tag{2}$$

"The folded triangle $f\Delta AB'BCC'$ can be viewed as the isosceles triangle $\Delta A'BC$ folded around the side $\overline{B'C'}$ in such a way that Equation (2) holds. Then

$$f\Delta AB'BCC' \cong f\Delta BD'DAE' \tag{3}$$

"Now we can formulate the problem this way: Find the angle β such that when the isosceles $\Delta A'BC$ folded around $\overline{B'C'}$ satisfying Equation (2), and rotated until $\overline{AC}$ coincides with $\overline{AB}$, that is $f\Delta AB'BCC'$ coincides with $f\Delta BD'DAE'$, the extended side $\overline{D'E'}$ passes through the vertex C.

"Therefore from $\Delta CC'F$ it follows that $\alpha + \varphi + \frac{\pi}{2} + \frac{\beta}{2} = \pi$, i.e.

$$\alpha + \varphi + \frac{\beta}{2} = \frac{\pi}{2} . \tag{4}$$

"From $\Delta ACC'$ we have

$$\psi = \beta + \delta \,. \tag{5}$$

"The Sine rule applied to $\Delta ACC'$ yields

$$\frac{\overline{CC'}}{\sin\delta} = \frac{\overline{AC}}{\sin\beta} \,. \tag{6}$$

"Applying Equations (1) and (5) to ΔACH and $\Delta CC'G$ we have

$$\overline{AC}\sin(\beta+\delta)=\overline{CC'}\sin(\frac{\pi}{2}-\frac{\beta}{2}), \text{ i.e. } \overline{AC}=\frac{\overline{CC'}\cos\frac{\beta}{2}}{\sin(\beta+\delta)}. \tag{7}$$

"Substituting from Equation (7) into Equation (6) and rearranging yields

$$\sin\beta\sin(\beta+\delta)=\sin\delta\cos\frac{\beta}{2}. \tag{8}$$

"From the isosceles ΔAOC it follows that $2(\frac{\beta}{2}+\delta)+\alpha=\pi$, or

$$\alpha=\pi-\beta-2\delta. \tag{9}$$

"From $\Delta ACC'$ it follows that $\beta+\delta+\psi+\varphi=\pi$,
or using Equation (5)

$$\varphi=\pi-2(\beta+\delta). \tag{10}$$

"Now from Equations (4), (9) and (10) we get

$$\delta=\frac{1}{8}(3\pi-5\beta). \tag{11}$$

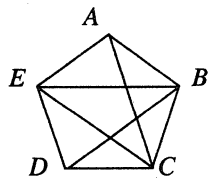
"Substitution from Equation (11) into Equation (8) finally yields

$$\sin\beta\sin\left\{\frac{3}{8}(\pi+\beta)\right\}=\sin\left\{\frac{3}{8}(\pi-\frac{5}{3}\beta)\right\}\cos\frac{\beta}{2}, \tag{12}$$

an equation in the unknown angle β."

Pllana then solved this equation numerically to find $\beta=39.8956^\circ$.

Anker interpreted the problem differently. He commented that he hoped the knot was still loose, as "otherwise the problem is impossible". He had the knot forming part of a regular pentagon, as shown overleaf where the four of its five diagonals are also drawn. The "knot" is identified with the trapezium $BCDE$ and the angle β with the angle BAC. In this case, by a familiar result for the regular pentagon, the angle is 36°.



Solution to Problem 27.4.4

Problem 27.4.4 has quite a long history of which some was detailed last August. It is simply stated:

$$\text{Simplify } \sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}\ .$$

We received solutions from John Barton and Julius Guest. Here is Guest's.

Write the first term as x and the second as y. Then $x^3 = 2+\sqrt{5}$ and $y^3 = 2-\sqrt{5}$. It follows that $x^3+y^3=4$ and $(xy)^3=-1$. The second of these equations may be simplified to $xy=-1$, if we confine ourselves to real arithmetic. And also

$$x^3+y^3=(x+y)(x^2-xy+y^2)=(x+y)\{(x+y)^2-3xy\}.$$

Now put $z = x + y$ to reach the cubic equation in z: $z^3+3z=4$.

This has the sole real root $z = 1$, and this is the required simplification.

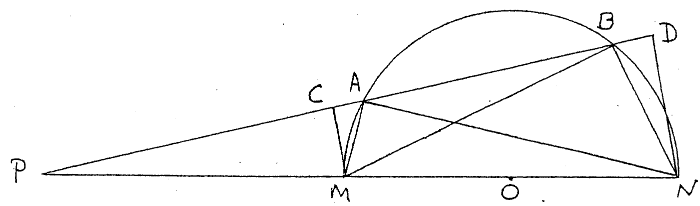
[*The restriction to real arithmetic may be seen as implicit in the notation in which the problem is expressed. However, the expression* $\left(2+5^{1/2}\right)^{1/3}+\left(2-5^{1/2}\right)^{1/3}$ *could be said to admit complex values. Eds*]

As usual, we close with four new problems.

Problem 28.2.1, submitted jointly by Willie Yong (Singapore), Jim Boyd and Richard Palmaccio (both USA)

In the diagram below, A and B are points on a semicircle resting on the diameter MN, whose mid-point is O. AB and MN are extended to meet in P. Prove that

$$\frac{AM \times BM}{AN \times BN} = \frac{PM}{PN}$$



Problem 28.2.2, based on a note in *School Science and Mathematics*, October 2003, pp 309-310.

Let

$$A = \frac{1}{2}\left\{\sqrt[3]{2}\left[\sqrt[3]{1+\sqrt{5}} + \sqrt[3]{1-\sqrt{5}}\right]\right\}$$

$$B = \frac{1}{2}\left\{\sqrt[3]{14+6\sqrt{5}} + \sqrt[3]{14-6\sqrt{5}} - \sqrt[3]{6+2\sqrt{5}} - \sqrt[3]{6-2\sqrt{5}}\right\}$$

$$C = \frac{1}{10}\left\{\sqrt[6]{3+\sqrt{5}} - \sqrt[6]{3-\sqrt{5}}\right\}\left\{\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}}\right\}\sqrt{5}.$$

Prove that $A = B = C$.

Problem 28.2.3, from *Mathematical Bafflers*, ed Angela Dunn

Solve in integers x and y the equation $x^2 = \dfrac{y^2}{y+4}$.

Problem 28.2.4, adapted from the same source

A and B are integers as also are both of the roots of both the quadratic equations

$$x^2 + Ax + B = 0 \text{ and } x^2 + Bx + A = 0.$$

Find A and B.

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A Correction

Our regular correspondent Keith Anker writes to point out an error in the comment at the bottom of page 163 of our issue for October 2003. That comment mentions a "triangle [with] sides 4, 2, 1" and another whose sides are twice these. Anker says he would like to see a picture of these triangles. So would we. It would indeed be most intriguing! The point is that these lengths cannot possibly be the sides of a triangle. For three lengths to form a triangle, the sum of the two smaller lengths must exceed the longest. This does not happen here!

The simplest triangle satisfying the given requirements derives from a difference of two cubes of 19 ($=3^3 - 2^3$), not 7, as was claimed. We then have for the first triangle, $a = 18$, $b = 12$, $c = 8$, and for the second, sides of 27, 18 and 12.

Our apologies for the error.

It remains to say that the mistake was not the fault of our other regular correspondent David Shaw, who pointed out the possible generalisation of Problem 27.2.3. Rather it was an aberration caused when the chief editor sought to amplify Mr Shaw's absolutely correct and pertinent comment.

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