# Function

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*Function* is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. It was founded in 1977 by Prof G B Preston, and is addressed principally to students in the upper years of secondary schools, but also more generally to anyone who is interested in mathematics.

*Function* deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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* $17 for *bona fide* secondary or tertiary students.

# THE FRONT COVER

Our front cover for this issue is a photograph of the French mathematician Jean-Pierre Serre, who has been awarded the inaugural Abel Prize by the Norwegian Academy of Arts and Letters. We ran a brief news-item on the establishment of this prize in our issue for February 2002, but let us briefly recap.

There has never been a Nobel Prize in Mathematics, and this lack is the subject of stories in our issues for April 1987 and April 1992. One of the attempts to fill the gap was the establishment of the Fields Medal, and this development is also recounted in those earlier articles.

In status, the Fields Medal is probably equivalent to the Nobel Prize, but it differs in two important respects. First, the Nobel Prize endows the recipient with a handsome monetary grant; the Fields Medal does not. Secondly, the Fields Medal (unlike the Nobel Prize) is explicitly intended to foster the development of promising young mathematicians. Recipients must be under the age of 40.

The Abel Prize, by contrast, is generous. It carries a value of 6 million Norwegian kroner (about $A1.25m), and there is no age limitation placed upon it. The prize aims to help raise the standing of Mathematics in the community at large, and more particularly to stimulate the interest of children and young people in pursuing Mathematics.

The name honors the memory of Niels Henrik Abel, a brilliant Norwegian mathematician of the early 19$^{th}$ Century, whose bicentennial occurred last year, coinciding with the establishment of the award. He was born into poverty and lived a hand-to-mouth existence throughout his short life which ended when he was only 26, dying of (probably) tuberculosis. His most enduring result is the proof that the general quintic (5$^{th}$-degree) equation cannot be solved by the normal processes of algebra (is not solvable by radicals, as the jargon has it).

Serre, the recipient of the award named after Abel, is a mathematician of outstanding depth and breadth in his interests. Here we can only describe this work in general terms as its details are extremely technical. He is perhaps best known for his development of algebraic approaches to deep geometric problems in higher dimensions. His work in this area introduced a systematic view of earlier results, which it clarified very considerably.

He has also been active in Number Theory. His work here has been seen as extending Abel's insights on $5^{th}$-degree equations, in particular to polynomial equations in two variables. It is also seen as fundamental to much modern research in Number Theory, including Wiles' solution of the Fermat problem (see *Function*, April 1994).

He was born in France in 1926, showed early promise, and was awarded the advanced degree of D. Sc. in 1951. After brief periods at other institutions, in 1956 he took up a professorship at the Collège de France, a post he still holds.

His work has been widely honored by both the French government and by the mathematical community. In 1954, he won the Fields Medal, and he remains the youngest person ever to do so. In connection with the Abel bicentennial, the University of Oslo joined the ranks of the many universities that have awarded honorary degrees to him.

Biographies of both Serre and Abel may be found on the net. A good place to start is

http://www.groups.dcs.st-and.ac.uk/history

and follow the prompts from there. The above summary and the cover photograph are both derived from the *Notices of the American Mathematical Society* for June-July, 2003. This is available online. Go to

http://www.ams.org/notices/

and again follow the prompts.

# HORNER IN THE CORNER

## John A Shanks, University of Otago, NZ

No, not Little Jack Horner, but William George Horner, born in Bristol, England in 1786. While he never went to University and cannot be regarded as a mathematician, he was undoubtedly a clever fellow. In fact at the age of 14 he became an assistant master at Kingswood school and at 18 the headmaster! He became famous for his invention of the zoetrope (see Figure 1), which he called a "daedaleum" Latin for "ingenious device". This consists of a small drum with an open top, supported on a central axis. A sequence of pictures on a strip of paper is placed around the inside of the drum, and slots are cut at equal distances around the outer surface of the drum, just above the pictures. When the drum is spun, a viewer looking through the slots sees a rapid progression of images and the impression of animated motion. His model is regarded as an important step in the development of motion pictures.

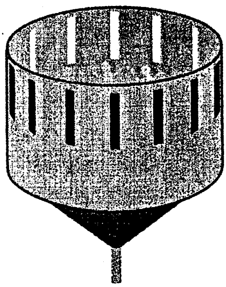

Figure 1

Horner's only significant contribution to Mathematics is now known as **Horner's Method**. This term is used somewhat ambiguously as there are two methods which bear this name. The first is his method for finding the roots (solutions) of polynomial equations (presented to the Royal Society of London in 1819), which involves finding a root digit by digit, producing a set of related polynomial equations using a table of calculated coefficients. (This was a rediscovery of a technique known to the Chinese mathematician Zhu Shijie some 500 years earlier.) As an offshoot of this method Horner used an efficient process for evaluating a polynomial using what we also call *synthetic division.* It is this latter Horner's method of "nested evaluation" that is the subject of this article. The "corner" of the title arises in a method proposed by the French mathematician M E Lill.

The polynomial $p(x) = 3x^4 - 4x^3 - x^2 + 5x + 6$ for example can be written in nested form like this:

$$\begin{aligned} p(x) &= \left(3x^3 - 4x^2 - x + 5\right)x + 6 \\ &= \left(\left(3x^2 - 4x - 1\right)x + 5\right)x + 6 \\ &= \left(\left(\left(3x - 4\right)x - 1\right)x + 5\right)x + 6 \end{aligned}$$

The evaluation of the polynomial in this form requires only four multiplications and four additions, and an algorithm based on this idea is very widely used in computing. This way of writing the polynomial implies that we calculate the bracketed expressions from the innermost outward. For example, to evaluate $p(2)$ one would calculate as follows. (Note how the result at the end of each line is used to begin the next.)

$$3 \times 2 - 4 = 2$$

$$2 \times 2 - 1 = 3$$

$$3 \times 2 + 5 = 11$$

$$11 \times 2 + 6 = 28$$

28 is the required value.

These calculations are often displayed in a table (as explained below).

$$\begin{array}{c|ccccc} & 3 & -4 & -1 & 5 & 6 \\ 2 & & 6 & 4 & 6 & 22 \\ \hline & 3 & 2 & 3 & 11 & 28 \end{array}$$

The first row contains the coefficients of the polynomial, and the 2 on the left is the $x$-value at which we are evaluating $p(x)$. The bottom row consists of the sums of each column; the first entry is always the same as the leading co-efficient, while the other entries can be seen to coincide with the intermediate values 2, 3, 11 and 28 that we calculated above. Each entry in the second row (after the first which is always blank) is the product of the previous sum with the $x$-value.

In 1867, the French mathematician M E Lill suggested a graphical method for solving polynomial equations. This obscure but ingenious technique can never produce roots with great accuracy but it does give a novel insight into how the value of a polynomial $p(x)$ changes as the value of $x$ changes, and a unique visual representation of this dependence. As we will see, its basis is none other than a clever application of Horner's nested evaluation method.

## Lill's Method

A sequence of perpendicular straight lines is drawn to represent a given polynomial. The lengths of the lines are taken in proportion to the magnitudes of the coefficients of the polynomial. The first line can be drawn horizontally left to right. Each additional line is attached to the previous one, turning right (i.e. clockwise) if there is no change of sign in the coefficients, and turning left where there *is* a change of sign. Figure 2 shows some examples of the resulting patterns of lines for the polynomials

$$\text{(a) } x^3+2x^2-x-1, \qquad \text{(b) } x^3+2x+2, \qquad \text{(c) } 2x^4-x^3-x^2-3x-4.$$

(It should be stressed that, although our examples all have integer co-efficients, the method is not restricted in this way.)

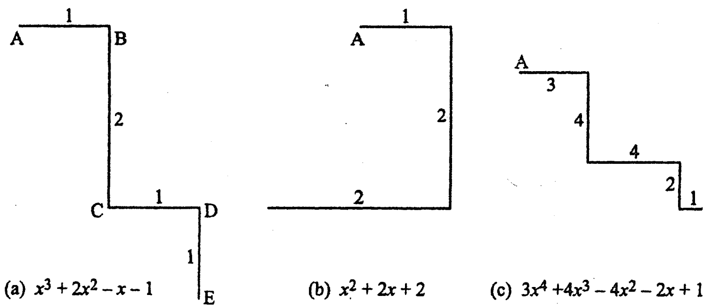


Figure 2

Graph paper or squared paper is obviously an advantage here. We will concentrate on the first example, Figure 2(a). *AB* is drawn a convenient length and, since the leading coefficient of the polynomial is 1, it represents a unit length. Then *BC*, *CD* and *DE* are drawn with lengths 2, 1 and 1 respectively, corresponding to the magnitudes of the other coefficients. The four coefficients of the polynomial have the signs + + − −. Because there is no change of sign between either the first and second or the third and fourth coefficients, both *ABC* and *CDE* involve right-hand corners. The change of sign between the second and the third coefficients however makes *BCD* a left-hand corner.

The line-segments *BC*, *CD* and *DE* are extended as in Figure 3. We now draw another sequence of perpendicular line segments starting from *A*, which we call a *trace*. We can start off in any direction, and each segment finishes when it hits the extensions of *BC*, *CD* and *DE* taken in turn. In this example we have drawn *APQR* where *P* is on *BC* (extended), *Q* is on *CD* and *R* is on *DE*. The aim is to adjust the angle *BAP* so that *R* equals the final point *E*. For the trace shown in Figure 3, as the angle is reduced, *R* moves upward. Figure 4 shows the required angle to make $R = E$.

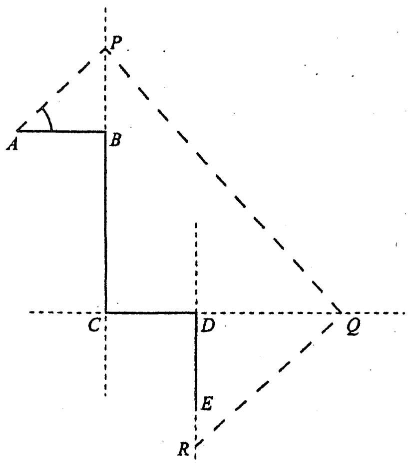


Figure 3

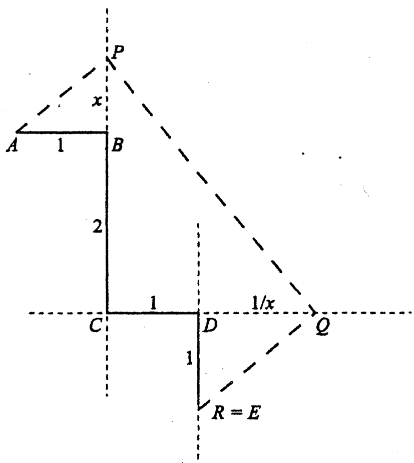


Figure 4

The triangles $ABP$, $PCQ$ and $QDE$ are similar. Consequently if we let $BP = x$, then, using triangles $ABP$, $QDE$,

$$\frac{PB}{AB} = \frac{DE}{DQ} \quad \text{i.e.} \quad \frac{x}{1} = \frac{1}{DQ}$$

So $DQ = 1/x$. Then also we have, using triangles $ABP$, $PCB$,

$$\frac{PB}{AB} = \frac{QC}{PC} \quad \text{i.e.} \quad \frac{x}{1} = \frac{1/x+1}{x+2}$$

Multiplying both sides of this equation by the common denominator $x(x+2)$ gives $x^3 + 2x^2 = 1$, i.e. $x^3 + 2x^2 - 1 = 0$. So this value of $x$ satisfies $p(x) = 0$ and so makes it a *zero of the original polynomial.*

In fact there are two more positions of $P$ on $BC$ that make $R$ identical with $E$. One of these and the corresponding trace are shown in Figure 5. (Can you find the other?) Taking $BP = x$ again, a similar argument to the above, using similar triangles $ABP$, $PCQ$ and $QDE$, finds this root of the equation. This second root is in fact negative (as also is the third) and Figure 5 suggests a way to deal with this. We regard the $BP$ as positive when $P$ is *above* $B$ and negative when it is *below*. For consistency, we measure the angle $BAP$ in an anticlockwise direction from the line $AB$, and let $x = \tan(\angle BAP)$. Because $AB = 1$ in the present example, this agrees with the definition $BP = x$ given above. Note that $x$ is positive for the first trace considered, but negative for the second.

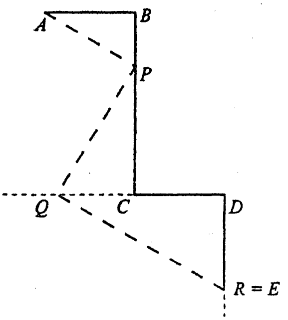


Figure 5

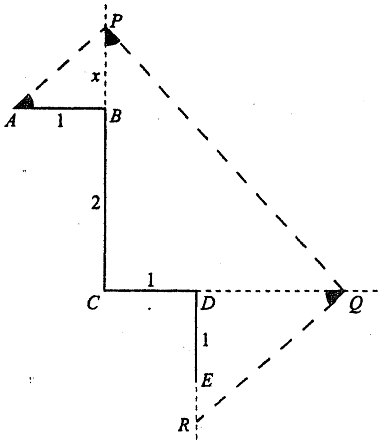


Figure 6

The connection between Lill's method and Horner's nested evaluation algorithm is revealed if we consider Figure 6, which shows a trace for which $R$ is not equal to $E$. Using the fact that $x$ is the tangent of each of the shaded angles, we find

$$x = \frac{BP}{AB} = \frac{CQ}{CP} = \frac{DR}{DQ} \tag{1}$$

and consequently, starting from $AB = 1$, we deduce

| | |
|---|---|
| $BP = AB.x = 1.x$ | from (1) |
| $PC = 1.x + 2$ | from Figure 6 |
| $CQ = (1.x + 2)x$ | because $CQ = CP$, again from (1), |
| $DQ = (1.x + 2)x - 1$ | again from Figure 6 |
| $DR = ((1.x + 2)x - 1)x$ | because $DR = DQ$, again from (1), |
| $ER = ((1.x + 2)x - 1)x - 1$ | again from Figure 6 |

This last expression for $ER$ is just the nested form of $p(x)$. Hence, the distance $ER$ is simply the value of $p(x)$, and it is now clear why we want to find $x$ so that $ER = 0$.

Figure 7 (opposite) shows the pattern of lines $ABCDEF$ corresponding to the quartic polynomial $q(x) = 2x^4 - x^3 + 3x^2 + x - 4$, together with the trace $APQRS$ where the lines are at 45° to the horizontal. Only a very small change in the angle at $A$ is needed to make $S$ coincide with $F$ and so give a root of the equation $q(x) = 0$, which evidently must be close to tan(–45°), in other words. –1. Readers should try to find another trace that succeeds. In this example there are only two such traces, and these correspond to the two real roots of the equation $q(x) = 0$.

Figure 2(b) showed the pattern $ABCD$ for the polynomial $x^2 + 2x + 2$, which has no real zeroes. Readers should experiment with different traces and check that none can succeed. How many zeroes can you find for the polynomial in Figure 2(c)?

In Lill's day, this graphical technique would have been very convenient for finding rough estimates of roots, to be refined later by numerical means such as Newton-Raphson. Compare the effort involved here with that required to evaluate the polynomial at sufficiently many points to get a good picture of its graph. These days we can reach for a computer or graphing calculator and get a rough picture of where the real roots lie, and so Lill's method may be no more than a curiosity. But is it? It is still faster to sketch one of Lill's traces on the back of an envelope than it is to boot up the computer, run the graphing program, enter the polynomial and click on "graph". And it's certainly much more fun!

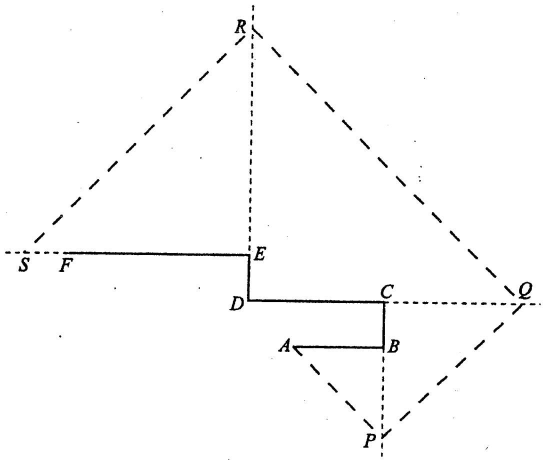


Figure 7

But we can combine the old with the new here and use a computer to construct the traces for us: both the calculation of the required line segments and their graphing can be done very quickly by computer, bringing Lill's method to life with animation. An interactive version of Lill's method can be found at

http://www.maths.otago.ac.nz/java/lillapplet.html

This Java applet copes with polynomials up to degree 7. By clicking and dragging on the image, you can swing the trace around until you hit the target and read off the corresponding root. polynomials up to degree 7. By clicking and dragging on the image, you can swing the trace around until you hit the target and read off the corresponding root. You can also very easily see how many real roots there are.

Finally, the reader is asked to consider the case when one or more of the polynomial's coefficients is zero. This is not really a complication; it merely results in coincident corners. For example, what is the drawing for the polynomial $x^3 - x + 3$? (Hint: It must be very similar to that for $x^3 + 0.01x^2 - x + 3$.) If you are not sure, you can always consult the Java applet for confirmation.

### Further Reading

There is a web-page devoted to Horner at:

http://www.gap-system.org/~history/Mathematicians/Horner.html

Lill's original paper, "Resolution graphique des équations numériques de tous les degrés" appeared Volume 2, Part 6 of the journal *Nouvelles Annales Mathématiques* in 1868.

## Ants on a Stick

One hundred ants are dropped on a metre stick. Each ant is traveling either to the left or the right with constant speed 1 metre per minute. When two ants meet, they bounce off each other and reverse direction. When an ant reaches an end of the stick, it falls off. At some point all the ants will have fallen off. The time at which this happens will depend on the initial configuration of the ants.

Question: over ALL possible initial configurations, what is the longest amount of time that you would need to wait to guarantee that the stick has no more ants?

Continued on p 158

# HISTORY OF MATHEMATICS

## Charles Babbage's Ninth Bridgewater Treatise

**Michael A B Deakin, Monash University**

A couple of years ago, my wife and I found ourselves in a small country town. It was one of those towns that have long ago lost most of their original *raison d'être* and now survive on their tourist potential. We made our way to a rather promising antique shop, where she browsed among the items of furniture while I checked out the second-hand books. One of these was by an author I recognised: Charles Babbage.

Babbage was a mathematician who lived from 1791 to 1871. He is most celebrated today as the inventor of two calculating machines: the *difference engine* and the *analytical engine.* Limitations of funding and of the available technology ensured that no fully working prototype of either was built in Babbage's lifetime, but he is now regarded as a pioneer of many of the ideas behind modern computers. (Quite how much actual influence, if any, he exerted on the development of the computers of today is not, however, clear.) His associate, Ada Lovelace, appeared in my column for October 2000.

So, when I saw this book for sale at the very reasonable price of $6, I bought it, and in due course I took it home and read it. Its very title seemed a conundrum: it is called *The Ninth Bridgewater Treatise: A Fragment.* The first part of this name was explained in the work itself, and I repeat it here for the benefit of readers. There were eight "Bridgewater Treatises" in the official count, and they were quite famous in their day. You can still read about them at various websites, of which perhaps the best (apart from a few minor inaccuracies) is that from *The Catholic Encyclopedia* (although the treatises have no connection with the Roman Catholic Church):

http://www.newadvent.org/cathen/02783b.htm

The background is this. The eighth and last Earl of Bridgewater, the Rev Francis Egerton, died childless in 1829. His will left an amount of £8000 to the Royal Society of London with detailed instructions as to how it was to be used. It was to be invested "in the public funds" and in due course applied (along with any interest it earned) to endow the publication of 1000 copies of a work "On the Power, Wisdom and Goodness of God as manifested in the Creation illustrating such work by all reasonable arguments as, for instance, the variety and formation of God's creatures, in the animal, vegetable and mineral kingdoms; the effect of digestion and thereby of conversion; the construction of the hand of man and an infinite variety of other arguments; as also by discoveries ancient and modern in arts, sciences, and the whole extent of modern literature."

The Royal Society (perhaps reflecting that the reverend Earl seemed to have been very fond of the number eight) nominated eight authors each to produce a volume along the lines indicated. The first four of these were prominent British theologians and the other four were eminent medical men. In due course, their eight treatises appeared. They were (with the correct dates of their first publication in parentheses):

1. *On the Adaptation of External Nature to the Moral and Intellectual Constitution of Man*, by the Rev Thomas Chalmers, DD, Professor of Divinity in the University of Edinburgh (1833),
2. *On Geology and Mineralogy*, by the Rev William Buckland, DD, FRS, Canon of Christ Church and Professor of Geology in the University of Oxford (1837),
3. *On Astronomy and General Physics*, by the Rev William Whewell, MA, FRS, Fellow of Trinity College, Cambridge (1833),
4. *On the History, Habits and Instincts of Animals*, by the Rev William Kirby, MA, FRS (1835),
5. *On the Adaptation of External Nature to the Physical Condition of Man*, by John Kidd, MD, FRS, Regius Professor of Medicine in the University of Oxford (1833),
6. *The Hand: its Mechanism and Vital Endowments, as evincing Design*, by Sir Charles Bell, KH, FRS (1834),
7. *On Animal and Vegetable Physiology*, by Peter Mark Roget, MD, FRS, Secretary to the Royal Society (1834),
8. *On Chemistry, Meteorology and the Function of Digestion*, by William Prout, MD, FRS (1834).

It is apparent from this list that the intentions of the will were carried out conscientiously. The president at the time was Davies Gilbert and he took advice from the Archbishop of Canterbury, the Bishop of London and "a nobleman who had been intimate [i.e. friendly] with the testator". The eight authors were all (as the above list shows) eminent authorities. Nonetheless, *The Catholic Encyclopedia* notes that "the selection of writers was somewhat severely criticized at the time". It would seem that Babbage's entry onto the scene was a part of this criticism. He certainly had some differences with Chalmers and also with Whewell.

The area of discussion was "Natural Theology", which is concerned with those aspects of Theology that are supposed to be amenable to natural reason, without the intervention of divine revelation. The claim is that it is possible, by reason alone, to deduce the existence of God, the immortality of the human soul and the freedom of the human will.

The argument for the existence of God took the form of an "argument from design". This had been popularised by the writer William Paley in his book *Natural Theology* (1802). Paley's argument uses the analogy of someone who finds a watch, and deduces from its intricate design that it is the product of a watchmaker. He then characterises the universe as an even more intricate mechanism than the watch, and so leads to the deduction of an intelligent designer: God.

This view became very influential, but has taken something of a battering in more recent years. The long times available for evolution and the force of natural selection as its governing principle have made it less and less necessary to invoke such external intervention. Indeed, *The Catholic Encyclopedia* retreats from endorsing this position in the course of its discussion of the Bridgewater Treatises. An explicit rejection of Paley's view underlies the title of Richard Dawkins' 1986 book *The Blind Watchmaker.*

The treatises may thus now be seen as part of a response by the Christian church to the rise of the scientific spirit. Buckland's treatise, in particular, was an attempt to find common ground between the biblical accounts of creation and the emerging science of Paleontology. The Bridgewater treatises were published only some few decades before *The*

*Origin of Species*. [There is a good account of all this in Deborah Cadbury's *The Dinosaur Hunters* ($4^{th}$ Estate Press, 2001).] This clearly takes Buckland outside the purview of Natural Theology, and this is one of the grounds that Babbage advances against others of the chosen eight.

But one might well ask what Babbage, a mathematician, could bring to such debate. It is hardly surprising that the Royal Society (of which he was a fellow) overlooked him in its search for authors. Certainly, Babbage was a devout Christian (in the 1950s, a relative, Barton Babbage, served as dean of St Paul's cathedral in Melbourne). Charles Babbage accepted Paley's "argument from design", although he did, in his preface, admit that it was possible that even persons of sound intellect and high morality might not be convinced by it.

So what did Babbage have to say? Well, it's a very strange mixture, and it often strays well beyond the boundaries of Mathematics *per se*. Furthermore, although he wishes to remain within the realm of Natural Theology, as he interprets the Earl of Bridgewater's intentions, he strays here also. And yes! The book is a fragment. It was published in incomplete form in 1837, and ran (still incomplete) to a second edition in 1838 and to an American second edition in 1841. The copy I bought was much more recent. In 1967, the second edition was reprinted, but with the addition of an index. Chapters six, seven and fifteen in particular are clearly unfinished. Whole paragraphs are signalled as missing, replaced by arrays of dashes. Quite what was meant (ultimately) to replace them has never been settled.

Among the main points of interest for the mathematical reader are the descriptions of his "calculating engines". In Chapter 2, he supposes the use of the "engine" to produce lists of numbers that exhibit a straightforward pattern of regularity, but for which the appearance is deceptive. There is a pattern and it is regular, but it is more complicated than we initially thought. With our present-day familiarity with computer programs, it is not at all difficult to appreciate the point. A computer outputs the numbers 1 to 1,000,001 in turn, but then outputs 1,010,0002, 1,030,003, etc, where the excess over and above what we had come to expect is $10{,}000T_n$, where $T_n$ is the $n$th triangular number (in the sequence 1, 3, 6, 10, 15, etc). At some further time, the output might surprise us again, and so on. It would be a simple matter to program one of today's computers to do this and a lot more.

We can well appreciate the suggestion that our world may indeed be more complex than we imagine, but I must confess myself baffled as to what all this has to do with the existence of God. Later however he applies similar thoughts to the freedom of the human will.

He returns to this point with another example. Here he introduces the curve:

$$y^4 - 4y^2 = Q(x),$$

where $Q(x)$ is a quartic ($4^{th}$-degree) polynomial in $x$. The graphs of such functions can take a wide variety of apparently different forms, depending on the nature of the zeroes of $Q(x)$.

Here he does supply a context, and we may perhaps apply it to the earlier example also. He has embarked on a discussion of miracles. Miracles, as the term is commonly understood, are suspensions of the normal laws of nature. Commonly these are seen as instances of divine intervention (perhaps as a result of prayers of intercession), and thus are interpreted as providing evidence of God's existence. Babbage discusses miracles at some length and from all sides. On the one hand, he is concerned to say (as these examples illustrate) that what we interpret as a miracle may merely be a part of a more intricate pattern than we previously allowed for.

[Here we may think of eclipses for example, which are now seen as perfectly natural occurrences, and not in any way miraculous. Another example is the stance taken by the Nobel prize-winner Alexis Carrell, who accepted the alleged miracles of Lourdes as genuine, but regarded them as examples of an unusual, but still natural, psychological power, not as instances of divine intervention.]

Babbage classes miracles, if they are genuinely cases of divine intervention, as examples of revealed Theology, rather than Natural Theology. But on the other hand if they are not really miracles in this sense, but only the marks of some higher pattern in the natural order, then they do constitute material for Natural Theology and so fall under the scope of the Bridgewater bequest.

All the same, Babbage is concerned to show that miracles can occur. The philosopher David Hume had endeavored to show that they do not. As Babbage correctly points out, Hume's use of the term "miraculous" is synonymous with "improbable". With this understanding, his argument is easily put, and Babbage quotes a paraphrase by Hume's compatriot Boswell:

"It is more probable that witnesses should lie, or be mistaken, than that [miracles] should happen".

Babbage now begins his discussion with: "The only sound way of trying the validity of this assertion is to *measure* the numerical value of the two improbabilities". The details are consigned to an appendix (Note E), and they make too trusting use of the Laplace law of succession (see my column for June 2000). However, we may readily fix things up. Suppose that the probability of some unlikely event is $p$, where $p$ is very small. Even if we cannot put a numerical value on $p$ (which is where Babbage uses the Laplace law), we can consider the testimony of witnesses who are generally truthful and reliable. Suppose there are $N$ independent witnesses and each has the chance $P$ of being mistaken or lying (where $P$ is small, but not as small as $p$). Then if all agree in asserting that the unlikely event *did* occur, then the chance that it did not is $P^N$. The point is that we can always find a value of $N$ that will make $P^N < p$. If 11 independent witnesses, each with a reliability of 99%, testify to an event with a probability of less than $10^{-21}$, then we should believe them!

I'm not sure that this helps Babbage's argument as much as he thinks. For one thing, the witnesses must be independent, and this is not always as clear as we might like to believe. Babbage couches his discussion in terms of the Christian doctrine of the Resurrection. But how many independent witnesses do we really have? The apostle Paul (I Corinthians 15) gives an apparently impressive list, but even if we assume all these to be independent of one another, the fact remains that all their testimony is filtered through a single channel: Paul himself. This is *before* we take on board Babbage's own sensible comments on the interpretation of Scripture. The other point to bear in mind is that while it is always possible to find an $N$ that satisfies the required inequality, that value, if $p$ is small enough, may turn out to be unrealistically large.

But although I find Babbage's analysis flawed, and his book a strange mishmash, it has certainly prompted me to look once more into these questions. It was a well-spent \$6 that led me to it, and I do think it fortunate that it was the 1967 edition I acquired. Hunting around on the internet, you can find booksellers offering earlier editions as rare books and at much much higher prices!

# COMPUTERS AND COMPUTING

## A Spreadsheet to assist with Elections

**Paul Grossman, paulgrossman@netspace.net.au.**

### Introduction

The results of elections can be greatly influenced by the electoral system used. For the election a body of members – whether they form a Committee, a Board of Management or a House of Representatives – the most equitable procedure is arguably the *Single Transferable Vote* first proposed by T. Hare in 1859 and later refined by Clark and others. This system, also known as *Quota Preferential Voting,* leads to proportional representation of parties or interest groups and also to the election of independents with adequate voter support. It avoids the frustration of voters who think they have "wasted" their vote on an unsuccessful candidate or on a candidate who would have been elected without their help. The system has been used in Tasmanian elections since 1918, and has also been adopted in the ACT, in Upper Houses of some Australian States and in overseas countries such as Ireland. It forms the basis of our Senate elections where however those who vote "above the line" delegate their choices to the parties.

The single transferable vote could profitably be used for elections in clubs and societies, but its introduction is usually resisted on the grounds that counting the votes is too cumbersome. There are of course computer programs available but their use has drawbacks. Firstly, unless voting is electronic, data from the voting papers must be entered into the computer and their accuracy checked; this is an arduous task. Secondly, voters and scrutineers must accept the program on faith and may well suspect that it may be flawed.

In the following I shall outline the principles of the single-transferable-vote system and then introduce a spreadsheet I have devised to take care of much of the simple but tedious computations. The spreadsheet thus expedites the procedure of vote allocation and allows for easy application of corrections. It displays all figures so that scrutineers can check any step using a calculator, an abacus or pen and paper.

## Principles on which the Single Transferable Vote is based

1. Each voter has a single vote, but this vote may be split during counting to benefit more than one candidate. For that reason voters list candidates in order of preference.

2. Any candidate supported by a sufficiently large proportion of the electorate (the quota) has the right to a seat.

3. When the number of votes in favour of a candidate exceeds the quota, it is considered that only part of each vote has been used and the unused portion is transferred to the next unelected candidate favoured by the voter.

4. Should there still be unfilled seats after transfer of excess votes, the candidate with the lowest aggregate of votes is excluded. Aggregate here means the sum of full votes (first preferences) and vote fractions received by transfer. The aggregate of the excluded candidate is transferred to the next preferences listed.

5. This process of transfers of excesses and of excluded candidates' votes is continued until all seats are filled.

## The Quota

If there are $p$ vacancies and $N$ votes, an obvious interpretation of principle 2 means that winning $N/p$ votes is sufficient for election. However, is such a high quota necessary? We can reduce the quota while still ensuring that no more than the $p$ most favoured candidates will be elected, as long as the quota exceeds $N/(p + 1)$. A quota lower than $N/p$ means that not all $N$ votes or fractions of votes are utilized to determine the winning candidates. Critics have argued that this can deprive some voters of their right to use their later preferences to influence the outcome. On the other hand, unused vote fractions are advantageous, indeed essential in practice, giving us some latitude in the process of transfers of vote fractions. When adding fractions we can limit the number of decimal places. We can also reach a result without distributing the votes of every unsuccessful candidate.

A desirable value for the quota is $Q = N/(p+1) + \delta$, where two conditions apply:

1. $\delta < N/(p\,(p+1))$, and preferably much smaller.
2. $\delta \geq 10^{-k}$, where $k$ is the number of decimal places one wishes to work with.

Readers might verify the reasons for these conditions.

When the number of voters is in the millions, as in a Senate election, an integer quota is acceptable and the formula

$$Q = \mathrm{INT}\,(N/(p+1)+1)$$

fulfils the conditions. For small numbers of voters, such as members of a school committee, an integer quota would be too imprecise; besides, Condition 1 might be violated by the above formula. It is convenient to choose $Q = 10^{-3}\,\mathrm{INT}\,(10^3\,N/(p+1) + 1)$ and to deal with thousandths of a vote. In practice we sometimes omit writing the decimal point (as if each vote had a value of 1000).

## The Transfer Value

Suppose candidate $K$ receives $n_K$ primary votes (first preferences on the voting papers). If $n_K > Q$, he or she has been elected. Fractions $Q/n_K$ of these votes are sufficient for $K$'s election and the unused portions $1 - Q/n_K$ may be transferred to the second preferences listed on the voting papers of $K$'s supporters. The weighting factor $T_K = 1 - Q/n_K$ is called the "transfer value". Clearly the product $T_K\,n_K$ is the number of $K$'s votes in excess over the quota.

Candidate $L$ does not reach the quota on primary votes $n_L$ but only after a transfer of $n_{KL}$ vote fractions from $K$, thus with a total of $n_L + T_K\,n_{KL}$. Let us define a transfer value

$$T_L = 1 - \text{quota/ total vote} = 1 - Q/(\,n_L + T_K\,n_K)$$

and we find again that the number $\Delta$ of votes in excess of the quota equals the product of the transfer value and the total vote,

$$\Delta = T_L\,n_L + T_L\,T_K\,n_K.$$

This shows that on each of the papers favouring $L$ in the primary vote a fraction $T_L$ may be transferred to the next preferences, and on papers listing $K$ first and $L$ second a smaller fraction $T_L T_K$ to the third preference.

The transfer value is a number smaller than unity. However, when we deal with thousandths of votes and omit the decimal point, as mentioned at the end of the section on the Quota, the transfer value is also listed as $10^3$ times its real value.

**The Spreadsheet**

The Proportional Representation Society of Australia (PRSA), which is active in promoting the *single transferable vote*, uses a counting sheet for use by electoral officers. The spreadsheet presented here follows the same layout and enables anyone to check each step by following the PRSA procedure with a pocket calculator.

Figure 1 shows part of the spreadsheet, which can be extended or reduced in both directions as required. The faintly shaded cells (detectable by the termination of the gridlines) show up as yellow on the computer screen. These contain functions and are coloured to warn the operator not to accidentally override them. Results from a ballot count have already been entered on Figure 1. The entries are in script to distinguish them from the figures and letters appearing in the shaded cells.

Column A lists the names of the candidates and is frozen, so that it remains visible when you have to scroll to the right. Columns B-D refer to the primary votes, i.e. the first preferences listed by the voters. To the right, starting from E, are sets of three columns, one of which is hidden to save space. The hidden columns G, J, M etc. are used for auxiliary computations and their results transferred to visible columns. The sets of three (two visible) are used for the transfers of surpluses of elected candidates and of the total votes of excluded candidates.

Suppose you are the returning officer and you proceed as follows.

**Step 1.** You enter details about the poll, in particular the number of vacancies (B5) and the names of the candidates. Having been handed the voting papers you sort the papers according to first preferences and enter the number received by each candidate in column C. (In Figure 1, your entries are printed in script to distinguish them from the figures produced by the computer).

The spreadsheet then displays:

11. The total number of papers (C31), the vote values (D even rows) and their total (D31) and the maximum vote value (D29).

12. The quota (I5) and a capital red E against any candidates who have exceeded the quota, in this case Samantha.

13. The highest vote value (F8) and the number of votes (F9) to be transferred with the integral part of the transfer value (F12). You have to enter Samantha's name yourself in F7.

14. The remainder (E14), i.e. vote fractions unused because only the integer part of the transfer value is being used. (The computer could readily cope with a closer approximation to the transfer value, but integer parts only are used to enable scrutineers to check any of the results using a pocket calculator in accordance with PRSA practice).

**Step 2.** Sort Samantha's 12 papers according to second preferences and enter in column E how many transferred papers each candidate receives. (Had another member been elected in the first round, that member would not be given any more transferred votes and these would go to the third preferences). For Samantha you enter the number of transferred votes with a negative sign.

The spreadsheet then displays:

21. The vote values transferred (i.e. the number of second preferences multiplied by the transfer value) in column E, even rows.

22. The number 0 in E31, which is a check to confirm that you have distributed as many vote fractions as you have removed from Samantha.

23. The cumulative vote values in column F, even rows. (Before Step3 the number in F12 was 7812.)

24. A capital red E for any candidate who has exceeded the quota with the help of transfers, in this case Nanh.

**Step 3.** Because the transfer value used is smaller than the real one, Samantha's cumulative vote value (F18) was higher than the quota by the remainder in E14. You therefore override F18 and enter the quota from I5.

The spreadsheet displays:

31. A change in F17 from a capital to a lower-case e to distinguish elected members whose surpluses have been transferred from those who have yet to be processed.

32. Maximum and minimum cumulative vote value (F29, F30)

33. The total vote value in F31, which must equal that in D31.

34. The vote value of the newly elected member Nanh (I8) and the transfer value for the excess from his primary votes.

**Step 4.** Since Nanh has received votes from two sources, namely the primary votes and the transfers from Samantha, the transfer of his excess requires two sets of columns. You enter Nanh's name in I7 and L7 in precisely the same way. You also enter, in I9 and L9, the number of voting papers received from each source respectively. Since the seven votes in Col. L were of reduced value you override the 1000 in L11 and enter the past transfer value 349.

The spreadsheet displays:

41. The vote value in L8 (Unless the name is copied exactly, the vote value in L8 will not remain the same but will reduce to 7987, the maximum after the excess from Nanh's primary votes has been transferred).

42. The transfer value in L12. Note that the transfer value in L12 is 0.349 times that in I12.

**Step 5.** You proceed as in Step 2, listing the number of papers transferred to the highest yet unelected preference but you do not take Step3, overriding Nanh's total, until you reach L20.

The spreadsheet displays:

51. All the transferred vote values and cumulative vote values as before, as well as the totals, maxima and minima.

52. A change from capital E to lower-case e in row 19 once the quota is entered in L20. Since there is not another elected member, the vote value appearing in the next set of columns (O8) is the minimum 3653. This is no longer shown in the figure here.

**Step 6.** You find that Natasha is the candidate with the minimum. She is excluded and her votes are transferred, the papers from each source separately under the transfer value under which they were received. You therefore enter her name in four sets of columns with three votes with a past transfer value 1000, one with 349, four with 76 and two with 26 and proceed as before. After the transfers her total vote value will have been reduced to zero.

The procedure continues until four members are elected.

## Some of the formulae used

The numbers or letters in shaded cells are the results of formulae, which are displayed when the cells are selected. Many of the formulae are obvious, showing simple operations. The sum in F31 cannot be obtained from column F because some of the cells are not numerical. This is the reason for the hidden columns. Col. G, not shown, lists the numbers from the even-rowed cells F16 to F28 and their sum is copied into F31. The same applies to the maxima and minima.

For the less obvious formulae let us discuss cells in columns K & L. All later pairs of unhidden columns such as N & O etc. are copies of these. Readers familiar with Microsoft Excel will know that the cells referred to in copied formulae will be relabelled unless it is specified by a $ sign that they remain unaltered.

Cell L8 `=IF(L$7=I$7,I$8,IF(J29>$I$5,J29,J30))` displays the vote value of the candidate whose excess or total votes are to be transferred. If the candidate's name is the same as that listed in the earlier set of columns, then obviously votes from several sources are being processed and the vote value remains the same. Otherwise, the vote value is the maximum from the earlier columns provided that maximum exceeds the quota. Or else it is the minimum, which belongs to a candidate about to be excluded.

Cell K10 =I$11*(L$8-$I$5)/L$8 is the expression for the transfer value for the excess of an elected member. Its companion cell L10 =IF(L$8>$I$5,K$10,L$11) specifies that K10 should be used when the vote value exceeds the quota, otherwise the transfer value L11 from previous transfers applies. The numerical values are not shown since the operator is interested only in the integer part, which is listed in I12.

Cell K14 =ROUND((L$10-L$12)*L$9,0) displays the remainder, i.e. the fractions of votes neglected by the use of the integer part of the transfer value. It is rounded to the nearest whole number.

Cell L17 =IF(L16>$I$5,"E",IF(L16=$I$5,"e","")) alerts the operator to the status of a candidate. Those elected with excess yet to be transferred have the letter E displayed, those without an excess have lower case e.

## Conclusion

The spreadsheet has been used successfully in elections comprising some 40 to 50 voters, considerably reducing the time needed to process the results. Readers interested in copies and detailed instructions may contact the author via the email address given at the head of this article.

"It would be a mistake to think that solving problems is a 'purely intellectual' affair; determination and the emotions play an important role."

G Polya, *How to solve it*

| | A | B | C | D | E | F | H | I | K | L |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | **Quota preferential voting** | | | | | |
| 2 | | | | | Copyright Paul Grossman | | | | | |
| 3 | Organization | *High School* | | | | | Date | *5/6/2003* | | |
| 4 | Election of | *Student Committee* | | | | | | | | |
| 5 | Number of vacancies | *4* | | | | | Quota | 7801 | | |
| 6 | | | **First preferences** | | **Surplus or Elimination** | | **Surplus or Elimination** | | **Surplus or Elimination** | |
| 7 | | | | | Name | Samantha | | Nanh | | Nanh |
| 8 | | | | | vote value | 12000 | vote value | 8443 | vote value | 8443 |
| 9 | | | | | no.transfers | 12 | no.transfers | *6* | no.transfers | *7* |
| 10 | | | | | | | | | | |
| 11 | | | | | past tr.value | 1000 | past tr.value | 1000 | past tr.value | 349 |
| 12 | | | | | transfer value | 349 | transfer value | 76 | transfer value | 26 |
| 13 | Candidates | | papers | value | transf.p/cand. | progr.total | transf.p/cand. | progr.total | transf.p/cand. | progr.total |
| 14 | | remainders | | | 11 | 11 | 0 | 11 | 4 | 15 |
| 15 | Jason | papers | 5 | | *3* | | *0* | | *3* | |
| 16 | | value | | 5000 | 1047 | 6047 | 0 | 6047 | 78 | 6125 |
| 17 | Samantha | papers | 12 | E | *-12* | e | | e | | e |
| 18 | | value | | 12000 | -4188 | 7801 | 0 | 7801 | 0 | 7801 |
| 19 | Nanh | papers | 6 | | *7* | E | *-6* | E | *-7* | e |
| 20 | | value | | 6000 | 2443 | 8443 | -456 | 7987 | -182 | 7801 |
| 21 | Natasha | papers | 3 | | *1* | | *4* | | *2* | |
| 22 | | value | | 3000 | 349 | 3349 | 304 | 3653 | 52 | 3705 |
| 23 | Tim | papers | 7 | | | | | | *2* | |
| 24 | | value | | 7000 | 0 | 7000 | 0 | 7000 | 52 | 7052 |
| 25 | Cath | papers | 6 | | *1* | | *2* | | | |
| 26 | | value | | 6000 | 349 | 6349 | 152 | 6501 | 0 | 6501 |
| 27 | exhausted | papers | | | | | | | | |
| 28 | | value | | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 29 | maximum | | 12 | 12000 | | 8443 | | 7987 | | 7801 |
| 30 | minimum | | | | | 3349 | | 3653 | | 3705 |
| 31 | total | | 39 | 39000 | 0 | 39000 | 0 | 39000 | 0 | 39000 |

Fig.1 - Part of spreadsheet applied

## UPDATE ON TWIN PRIMES CONJECTURE

In our June issue, we reported a major advance in the study of the "twin primes conjecture", the suggestion that there are infinitely many prime pairs differing only by 2. The advance was the work of Dan Goldston and Cem Yildrin, and it created a lot of interest. In part this was because it opened a new approach to the question; in other part it was because it stated strong technical improvements in previous results.

Sadly, the new work is not without its problems. While the new approach has continued to create interest and to open new avenues of research, the technical result has been found to contain an error. Even when our issue went to press, the mistake had been found. It was pointed out by Andrew Granville (Montreal) and Kannan Soundararajan (Michigan). A term in an equation had been believed by Goldston and Yildrin to be so small as to be negligible. However, this is not always so: it can in some cases be as large as the main term in the same equation.

So we are not quite as far toward the resolution of the twin primes conjecture as had briefly been thought. Nonetheless, the new approach is thought to offer a very promising line of enquiry. The overall idea is to search for infinite sets of primes that are spaced much more closely together than the overall average.

The place to learn about this is the website

http://aimath.org/primegaps

and this will give links to other websites, both technical and popular.

Goldston, in a recent interview, and other mathematicians also have expressed the hope that the error may somehow be overcome. Although there seems no simple or obvious way to do this, we might bear in mind that Wiles' original proof of Fermat's Last Theorem was also flawed, but the defect was swiftly remedied.

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# OLYMPIAD NEWS

**Hasns Lausch, Monash University**

## The 2003 International Mathematical Olympiad

This year's International Mathematical Olympiad (IMO) took place in Tokyo. On the 13th and 14th of July 2003, 457 secondary students from 82 countries sat the contest, which consisted of two sets, each of three problems.

In past years, Bulgaria has regularly sent teams that proved themselves among the best in the world, but without ever taking the first position. However, this year things were different, and they did indeed top the competition. Each member of the Bulgarian team received a gold medal, and the team as a whole achieved a total of 227 points out of a possible 252. China (211 points) came second and the USA (188 points) third. Australia and Brazil (each with 92 points) shared the 26th and 27th positions, just behind Mongolia (93 points).

Australian students won two silver medals, two bronze and two Honorable mentions:

| | |
|---|---|
| Laurence Field, Year 11, Sydney Grammar School, NSW | SILVER |
| Daniel Nadasi, Year 11, Cranbrook School, NSW | SILVER |
| Ross Atkins, Year 12, Pembroke School, SA | BRONZE |
| Ivan Guo, Year 11, Sydney Boys High School, NSW | BRONZE |
| Zhihong Chen, Year 12, Melbourne High School, Vic | HM |
| Marshall Ma, Year 12, James Ruse Agricultural High School, NSW | HM |

Congratulations to all!

Here are this year's IMO problems. On each day a total time of four and a half hours was allotted. Each question was worth 7 marks.

**First Day**

**Problem 1.** Let $A$ be the subset of $S = \{1, 2, \ldots, 1000000\}$ containing exactly 101 elements. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in $S$ such that the sets

$$A_j = \{x + t_j \mid x \in A\} \text{ for } j = 1, 2, \ldots, 100$$

are pairwise disjoint.

**Problem 2.** Determine all pairs of positive integers $\{a, b\}$ such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

**Problem 3.** A convex hexagon is given in which any two opposite sides have the following property: the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

{A convex hexagon $ABCDEF$ has three pairs of opposite sides: $AB$ and $DE$, $BC$ and $EF$, $CD$ and $FA$.}

**Second Day**

**Problem 4.** Let $ABCD$ be a cyclic quadrilateral. Let $P$, $Q$ and $R$ be the feet of the perpendiculars from $D$ to the lines $BC$, $CA$ and $AB$ respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ meet on $AC$.

**Problem 5.** Let $n$ be a positive integer and $x_1$, $x_2$, ..., $x_n$ be real numbers with $x_1 \le x_2 \le \ldots \le x_n$.

(a) Prove that

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left|x_i - x_j\right|\right)^2 \le \frac{2(n^2-1)}{3}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i - x_j\right)^2.$$

(b) Show that equality holds if and only if $x_1$, $x_2$, ..., $x_n$ is an arithmetic sequence.

**Problem 6.** Let $p$ be a prime number. Prove that there exists a prime number $q$ such that for every integer $n$, the number $n^p - p$ is not divisible by $q$.

## The 2003 Senior Contest of the Australian Mathematical Olympiad Committee (AMOC)

The AMOC Senior Contest is the first hurdle for mathematically talented Australian students who wish to qualify for membership of the team that represents Australia in the following year's IMO. This year 70 students took part in a four-hour competition on August 12.

These are the questions.

1. Prove that there does not exist a natural number which, upon transfer of its leftmost digit to the rightmost position, is doubled.

2. Determine all functions $f$ that satisfy:

   (i) $f(x)$ is a real number for each real number $x$;
   (ii) $yf(2x) - xf(2y) = 8xy\left(x^2 - y^2\right)$ for each pair $x$, $y$ of real numbers.

3. For any three distinct real numbers $x$, $y$, $z$, let

$$E(x,y,z)=\frac{(|x|+|y|+|z|)^3}{|(x-y)(y-z)(z-x)|}$$

Determine the minimum possible value of $E(x,y,z)$.

4. Let $S$ be the set of 2003 points in three-dimensional space such that each of its subsets consisting of 78 points contains at least 2 points that have distance at most 1 from each other.

   Prove that there is a sphere of radius 1 such that at least 27 points of $S$ lie on or inside it.

5. Let $ABC$ be a triangle. Let $P$ be the point on $BC$ and $Q$ the point on $AC$ such that $BP = AB = AQ$. Suppose that angle $ACB = 30^\circ$. Let $O$ and $I$ be the circumcentre and the incentre, respectively, of $ABC$.

   Prove that

   (a) $PQ = OI$;
   (b) $PQ$ and $OI$ (extended) are perpendicular.

Continued from p 138

The answer is 1 minute! While ants bouncing off each other seems difficult to keep track of, one key idea (fun fact!) makes it quite simple: two ants bouncing off each other is *equivalent* to two ants that pass through each other, in the sense that the positions of ants in each case are identical. So, you might as well think of all ants acting with independent motions. Viewed in this way, all ants fall off after traversing the length of the stick, i.e., the longest that you would need to wait to ensure that all ants are off is 1 minute.

Seeking alternate ways to look at a problem can offer useful insights!

From the Funfacts site of Harvey Mudd College

# PROBLEMS AND SOLUTIONS

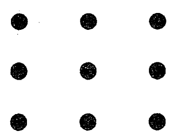
## LATE SUBMISSION

We received solutions to problems in the February issue from Keith Anker and Šefket Arslangić (Bosnia), but too late to include them in our August column.

We move onto the solutions to the problems posed in the April issue.

**SOLUTION TO PROBLEM 27.2.1** (submitted by Peter Grossman).

> In a well-known puzzle, you are challenged to draw four line segments passing through nine points arranged in a 3×3 array (as shown below), without lifting your pen off the paper. Most people know that the solution requires some of the line segments to extend beyond the bounds of the array, but a justification for this claim is rarely given.
>
> Prove that no solution is possible in which all of the line segments lie within the bounds of the array.



We received solutions from the proposer, from Keith Anker and from Julius Guest. Here is the proposer's solution.

Suppose there is such a solution. The solution is a path passing through the nine array points, and consisting of four line segments that may start and end at array points but need not do so. Define an *end point* to be a point that is at one end of a line segment but which is not an array point. Define an *edge* to be a part of a line segment in the solution between two points (either array points or end points) that are adjacent on the path. Since the path passes through each of the nine array points at least once, there must be at least eight edges. (There will be more than eight edges if the path contains end points, or if it passes through any array point more than once.)

Now, on examining the array, we can see that any line segment lying within the bounds of the array can contain at most two edges. It follows that there must be exactly eight edges in any solution, with two edges in each of the four line segments. Further, there can be no end points in the path, and so each line segment must start and end at an array point. Eight such line segments can be drawn, corresponding to the three rows, the three columns, and the two main diagonals of the array. The middle row and middle column can be excluded from consideration, as they cannot be joined to any of the other line segments to form a path. Of the remaining six line segments, the two horizontal and two vertical line segments must be included in the path, in order to ensure that the path passes through the four array points in the middle of the sides of the array. However, none of these four line segments pass through the centre point. Therefore, no solution with four line segments is possible.

**SOLUTION TO PROBLEM 27.2.2** (based in part on a problem in *Mathematical Bafflers*, ed Angela Dunn)

> The game of periwinkle is the same as noughts and crosses (tic-tac-toe) except that the object is not to place three of your symbols in a row, but to *avoid* doing so. Show that the player moving first can always avoid defeat. Is the second player so lucky?

Keith Anker provided a detailed discussion and Julius Guest offered a partial analysis. The first part of the problem is readily answered. We here paraphrase the solution received from Keith Anker and published by Angela Dunn. *The first player seizes the centre, and then counters each subsequent move by taking the diametrically opposite square.* Call the second aspect of this strategy Strategy *S*.

The game is referred to in Dunn's book as Toe-Tac-Tic, and elsewhere other names are also used.

To further the analysis, suppose the game to be played between *A* (for Ada, playing first with O's) and *B* (for Bob, playing second with X's). Now we have seen that Ada need never lose; the question is: can she do better? Can she force a win?

Before we proceed to systematic analysis, we can guess that the answer is "No". Bob has only 4 moves with which to make a line of 3 crosses, while Ada has 5 moves with which to make a line of 3 noughts. As the making of a line of 3 is to *lose* the game, we surmise that the advantage lies with Bob. (Just as in regular noughts and crosses, the first player holds an advantage, although not a winning one.)

The first point to make is that Ada cannot force a win by following the program given above (seizing the centre, followed by Strategy *S*). To follow the progress of different games introduce a notation in which the different squares are numbered, starting from the top left in order: 1, 2, 3, …, 9. (So 2 is the square in the centre of the top row, 7 is the square in the lower left hand corner, 5 is the centre, etc.) Thus Ada begins with 5. Suppose Bob now plays 2 (for reasons that will become clearer later on). Ada, following the program, now plays 8. Bob then plays 4 (again for reasons that will become clearer later on). Ada continues to follow her program and so plays 6. Now Bob can play 9, and he has avoided any possibility of lining up three crosses. The game is a draw.

Look at the pattern of Bob's first three moves.

| | | |
|---|---|---|
| | X | |
| X | | |
| | | X |

No matter how he places his fourth and final cross, he cannot make a line of three. So Bob cannot lose. The same conclusion follows whenever Bob has two midpoints of adjacent sides and the opposite corner.

Another "safe" pattern for Bob is that in which he has the midpoints of three sides (see the diagram overleaf):

| | | |
|---|---|---|
| | X | |
| X | | X |
| | | |

Here the perfectly typical case of 2, 4, 6 is shown, but any one of these numbers could be replaced by an 8. This one is even more favorable to Bob than is the previous pattern. Of course, he must avoid playing 5, but his final move is the 8th (second last) move in the entire game and so there must be *two* vacant squares at this time. If one of these is 5, then he takes the other. If this is 1 (or 3), then Ada has lost; indeed badly, because she has two lines of three. If it is 6 (or 9), again Ada loses, and so she does if it is 8. Thus this configuration is a winning one for Bob.

Ada's only hope of forcing a win therefore is to prevent either of these configurations. She can either: (a) Try to prevent Bob from taking the midpoints of two adjacent sides, or (b) Allow the two adjacent sides but block the opposite corner.

The first strategy is disastrous. It loses. Ada tries to stop Bob from gaining two adjacent midpoints. To this end, she plays 2. Bob now plays 4 (with a view to playing 8 next move). So Ada must now play 8 to keep to her plan, and now Bob wins by playing 6. Ada cannot play 5, and so must take a corner square, say 1. Bob now plays 7, and now, whatever she does, Ada is lost.

The other attempt Ada might make to force a win would still involve playing either on a side or in a corner. All Bob has to do to hold the draw is to follow Strategy *S*, until such time as Ada takes the centre. However the game unfolds, he cannot lose. But in fact he can do better. Anker offers the following summary: *If Ada starts by playing a corner square (1, say), then she loses if Bob plays 2. If she starts on a side (2, say), she also loses if Bob plays 4.*

A complete analysis of the game goes beyond what is given here, but enough has been said to show that, with best play on both sides, the game is a draw.

**SOLUTION TO PROBLEM 27.2.3** (from *Mathematical Bafflers*, ed Angela Dunn)

> *ABC* and *DEF* are two similar triangles, both with integral sides. Two of the sides of *ABC* are equal to two of the sides of *DEF*. The third sides are different from one another and the difference in their lengths is 387. Find the lengths of all the sides of both triangles.

We received solutions from Keith Anker, Julius Guest and David Shaw. Here is Shaw's.

Let *a*, *b*, *c* be the lengths of the sides of the triangle *ABC*, and let $a > b > c$. Then the lengths of the sides of the triangle *DEF* are $c + 387$, *a*, *b*, where now $c + 387 > a > b$. Because of the similarity of the triangles,

$$\frac{c+387}{a} = \frac{a}{b} = \frac{b}{c}$$

Then $a^2 = b(c+387) = b\left(\frac{b^2}{a} + 387\right)$ so that $a^3 - b^3 = 387ab$.

Looking at a list of cubes, we see that $8^3 - 5^3 = 512 - 125 = 387$, so put $a = 8k$ and $b = 5k$. We then have $a^3 - b^3 = 512k^3 - 125k^3 = 387k^3$.

Therefore $ab = 40k^2 = k^3$. So $k = 40$. Then

$a = 8 \times 40 = 320$, $b = 5 \times 40 = 200$ and $c = \frac{200^2}{320} = 125$. In the second triangle, the sides are 512, 320, 200 in length.

[This gives a complete solution for the problem we set, but Mr Shaw went on to comment that similar problems may be constructed by replacing 387 by any difference of two cubes. The simplest example results when the difference is 7. In that case, the first triangle has sides 4, 2, 1 and the second 8, 4, 2.]

**SOLUTION TO PROBLEM 27.2.4** (from *The Australian Mathematics Teacher*, June 2002)

> Hobson is in gaol but is given a chance of escape if he can make the right choice. Three boxes are presented to him, one of which contains the key to his cell, i.e. the means of escape. The three boxes are made of gold, silver and lead, and each carries an inscription. One and only one of the inscriptions is true. They are as follows:
>
> | | |
> |---|---|
> | On the gold box: | The key is in this box |
> | On the silver box: | The key is not in this box |
> | On the leaden box: | The key is not in the gold box. |
>
> Meanwhile Jobson, in another part of the same gaol, is given a similar opportunity. However in his case, at least one inscription is known to be true and at least one false. In his case they read:
>
> | | |
> |---|---|
> | On the gold box: | The key is not in the silver box |
> | On the silver box: | The key is not in this box |
> | On the leaden box: | The key is in this box. |
>
> How should Hobson and Jobson decide?

Keith Anker and Julius Guest sent solutions. Here is a composite.

(1) Hobson's choice: Denote the statements by $A$, $B$, $C$ respectively. If the key is in the gold box, then both $A$ and $B$ are true, contrary to the data; if the key is in the leaden box, then both $B$ and $C$ are true, again contrary to the data. However, if the key is in the silver box, then $A$ and $B$ are false and $C$ is true. This provides the answer.

(2) Jobson's choice: Denote the statements by $D$, $E$, $F$ respectively. If the key is in the silver box, all three statements are false, contrary to the data; if the key is in the leaden box, then all three statements are true, again contrary to the data. However, if the

key is in the gold box, then $D$ is true, while $E$ and $F$ are false. This therefore provides the answer.

We close with four new problems.

**PROBLEM 27.5.1** (Submitted by Willie Yong (Singapore), Jim Boyd (USA) and Richard Palmaccio (USA), jointly)

Evaluate $4\sin 20^\circ + \tan 20^\circ$.

**PROBLEM 27.5.2** (Submitted by Šefket Arslangić (Bosnia))

Prove that

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} < \frac{1}{\sqrt{2}}$$

for all positive integers $n$.

**PROBLEM 27.5.3** (Submitted by Julius Guest)

Let

$$S_n = \frac{1^2}{2\times3\times4\times5} + \frac{2^2}{3\times4\times5\times6} + \ldots \frac{n^2}{(n+1)(n+2)(nn+3)(n+4)}$$

Find an explicit formula for $S_n$ and determine $\lim\limits_{n\to\infty} S_n$.

**PROBLEM 27.5.4** (Submitted by Keith Anker)

Lines $l_1$ and $l_2$ are perpendicular to one another and lie in the plane of a triangle $ABC$. Using only measurements in the directions of $l_1$ and $l_2$, determine the area of $ABC$.

# INDEX: VOLUMES 26 & 27

## Feature Articles



## Olympiad News

**History Columns**



**Computers and Computing**

**Front Covers**



**News Items**

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