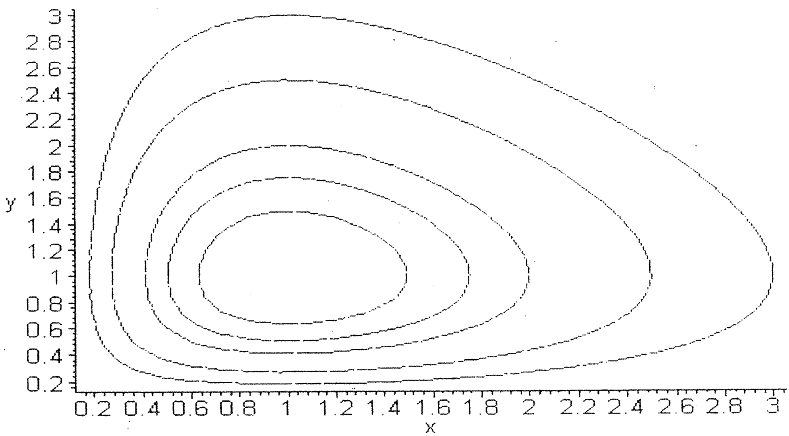
Function

A School Mathematics Journal

Volume 27 Part 4 August 2003



School of Mathematical Sciences – Monash University

Reg. by Aust. Post Publ. No. PP338685/0015

Function is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. It was founded in 1977 by Prof G B Preston, and is addressed principally to students in the upper years of secondary schools, but also more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Function is published five times a year, appearing in February, April, June, August, and October. Price for five issues (including postage and GST): $32.50* ; single issues $7. Payments should be sent to: The Business Manager, Function, Department of Mathematics & Statistics, PO Box 28M, Monash University VIC 3800, AUSTRALIA; cheques and money orders should be made payable to Monash University.

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THE FRONT COVER

The History Column for this issue introduces the simultaneous differential equations describing the interaction of a predator and a prey species:

$$\frac{dN_1}{dt} = rN_1 - kN_1N_2,$$

$$\frac{dN_2}{dt} = KN_1N_2 - DN_2,$$

where N_1 , N_2 are the numbers of prey and predators respectively, t is the time and the other letters represent positive constants. These equations cannot be solved in terms of the functions that *Function's* readers will know. They may however be simplified, and reduced to a standard form. To reach this, put:

$$N_1 = \frac{Dx}{K} \qquad N_2 = \frac{ry}{k} \qquad t = \frac{s}{r} \qquad a = \frac{D}{r}$$

and so obtain:

$$\frac{dx}{ds} = x(1-y)$$
$$\frac{dy}{ds} = ay(x-1)$$

which is the standard form.

It will be seen immediately that this is considerably simpler than the previous form. However, it is still not possible to solve it fully. But we can nonetheless make progress. Divide the two equations to reach:

$$\frac{dy}{dx} = \frac{ay(x-1)}{x(1-y)}$$

an equation that *can* be integrated.

The integral may be presented in a number of different forms. Here we use:

$$xy^a = Ae^{ax+y}, \tag{*}$$

where A is a constant. (Readers may care to check this result by differentiating.)

Our front cover shows several such curves, graphs of Equation (*), each corresponding to a different value of A. The curves appear as closed loops and this property may be proved to hold exactly. The curves may be used to follow the behaviour of the system. Take as an example, the outermost curve, and follow it round in an anticlockwise direction from the topmost point. Here, y, and hence the number of predators, is at its highest, and the number of prey is insufficient to sustain them. They therefore begin to die off, but not sufficiently quickly to avoid decimating the prey. Once we reach the leftmost point of the curve, the prey have become very scarce indeed (x is very small) and the predators continue to die off in large numbers. Down in the bottom left corner of the diagram, both prey and predators are scarce. This situation is good for the prey, who can multiply while the predators are still few in number. Along the bottom leg of the curve, x increases and y is relatively constant. The prey are having a good run and predators are still relatively scarce. But the good times do not last. The conditions have now become favorable to the predators (there are relatively few of them and food is plentiful!). The numbers of predators begin to increase and the stocks of prey are depleted. And so the entire cycle begins again.

The reader will note that we eliminated s, equivalently t, from the analysis when we divided the two equations. However from the full equations, we see that when $y > 1$ (i.e. predators are plentiful) x is decreasing. This observation tells us to follow the curves around in an anticlockwise direction.

It is possible to solve the equations numerically, and this produces periodic functions of s for both x and y. However, we do not go into these further details here.

THE USEFULNESS OF A SEEMINGLY USELESS FORMULA

G J Troup, Monash University

The seemingly useless formula of my title concerns spectral lines. For the purposes of this article, a spectral line may be looked upon as a plot arising in a physical context and having a somewhat simple shape and a simple mathematical description. It appears as a graph with a single maximum and, for present purposes, the graph will be taken to be symmetric about this maximum. This is to oversimplify, but it will suffice for the purposes of this article. A typical spectral line is shown in Figure 1.

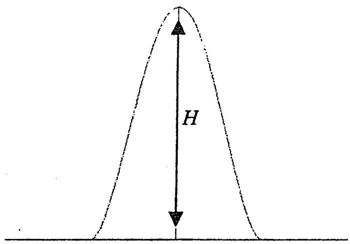


Figure 1

The graph shows a single maximum point with two points of inflection, one on either side of it. The horizontal distance between these two points is called w (for width, as it provides a measure of the width of the spectral line).

Some colleagues and I were using spectral line analysis in the course of a study for the wine industry (comparing the number of tannin molecules in the grapeseeds of red wine grapes as those grapes ripened). We were asked to participate in this research, using a technique called *magnetic resonance spectroscopy*, but the same problem was also being investigated by another team using a different method.

The "total absorption" of the spectral line is defined as the area A under the line. This is a figure we often need and so we need a way to calculate the area under a graph of the type illustrated in Figure 1.

This can be done via the use of analogue-to-digital converters suitably hooked up to the spectroscope and to an integration package in a computer. However, when (as all too often is the case) funding limitations apply, they can make this route difficult or even impossible to follow. Nevertheless, it is possible to process the graphical output by hand, making use of a seemingly useless formula!

We can in fact give a formula for the area in very simple terms. If we denote the height to the maximum in Figure 1 by H and continue to use w to stand for the width, then we may expect a formula for A to look like

$$A = kHw, \tag{1}$$

where k is a constant independent of the unit of length that we are using. (It is simply a number; however, its *value* will depend on the shape of the spectral line.) This is because we may use different units to measure both the horizontal and the vertical distances on our graph, and the formula must retain the same form whichever units we employ.[1]

However, there is a further complication. In *magnetic resonance spectroscopy*, what is recorded is not the spectral line itself, but its derivative. Figure 2 shows the graph of the derivative of the function graphed in Figure 1. The zero corresponds to the maximum in Figure 1, while the peak and trough points of Figure 2 correspond to points of inflection in Figure 1.

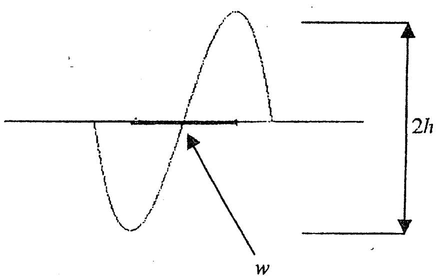


Figure 2

[1] The type of argument used here is the subject of an extended discussion in *Function*, *Vol* **10**, *Part* **1**. Eds.

A graph such as Figure 2 is characterised by the parameters w, now clearly evident as the horizontal separation of the maximum and minimum points, and h, which is their vertical separation. h is twice the maximum value of the slope of the curve, and we may estimate this as being proportional to the ratio H/w by means of arguments similar to those that gave rise to Equation (1), i.e. $H \propto hw$. Now we may combine this information with that of Equation (1) to produce a new equation:

$$A = Khw^2, \tag{2}$$

where K is a new constant of proportionality.

Equation (2) is our formula, but it involves the unknown number K, and so it would appear to tell us very little. Indeed, further investigation would seem to limit its usefulness even more. The value of K depends on the actual shape of the spectral line, and it is not especially difficult to show that for different shapes, its value may range from 0 all the way to ∞. In other words, it may take any positive value whatsoever! This is why I describe the formula as "seemingly useless".

However, all is not lost! If the set of samples being analysed produce spectral lines *all of the same shape*, then the unknown constant will be the same for all of them. It is in fact a reasonable physical assumption that the absorption lines of particular absorbers in a solid will all have the same shape. This argument applies quite well in a wide range of practical situations.

The upshot of this is that in comparing two spectral lines, all we need in order to obtain the relative areas beneath the curves, are the values of w and h. In the study that involved me and my co-workers, we were able to assure ourselves that all the curves we encountered did have the same shape, and so we could proceed using very simple instruments: ruler, pencil and calculator.

I provide an illustrative example in the Appendix to this paper. It shows the detailed working out of one special case.

It is pleasing to report that the two teams working on the problem obtained results that agreed well with one another, and also with the underlying theory, according to which the number of tannin molecules rises to a maximum prior to maturation, and then decreases.[2]

Although we were able to use simple considerations to allow very simple modes of processing our data, it is sobering to reflect that although the wine industry is worth millions of dollars in export earnings to Australia, this is not reflected in the priorities accorded to funding research in the associated Physics and Mathematics! Although in this instance we were able to work our way around funding limitations, other studies might not have such luck come their way!

Appendix

The common forms of naturally occurring spectral lines are not very well suited to checking this material. However, the following simple example will serve to illustrate the points made above.

Take the family of curves:

$$y = \begin{cases} \left(x^2 - a^2\right)^2 & \text{if } -a < x < a \\ 0 & \text{otherwise.} \end{cases}$$

These functions all exhibit a single maximum at $x = 0$, where $y = H = a^4$. There are points of inflection at $x = \pm\frac{a}{\sqrt{3}}$, and so $w = \frac{2a}{\sqrt{3}}$. The values of y' at the points of inflection are $\mp\frac{8a^3}{3\sqrt{3}}$ so that the distance between them, h, is $\frac{16\ a}{3\ \sqrt{3}}$. Equation (2) thus gives for the area A the value $K\left(\frac{16a^3}{3\sqrt{3}}\right)\left(\frac{2a}{\sqrt{3}}\right)^2$ which simplifies to $\frac{64Ka^5}{9\sqrt{3}}$. Readers may like to check by integration that the exact value of A is $\frac{16a^5}{15}$

Concluded on p 128

[2] The technical report on this work may be found in *The Australian Journal of Grape and Wine Research*, *Vol* **6** (2000), pp 244-254.

THREE RECENT REVIEWS OF OLD RESULTS

A New Look at Cubic Equations

Function has several times in the past looked at the solution of cubic equations. See for example the account in our issue for April 1992. Recently a new approach has appeared. It is the work of the English mathematician A J B Ward, and it was published in the *International Journal of Mathematical Education in Science and Technology*, February 2003, pp 153-158. Our account here derives from this work, but presents the material in a somewhat different order.

Let us begin by rehashing the theory of the more familiar quadratic equation, which is solved in general by "completing the square". If we adopt the standard form for the quadratic as

$$x^2 + ax + b = 0,$$

then we may compare this with another standard form

$$(x - p)^2 = q^2.$$

The virtue of this latter form is that it can be solved immediately by taking square roots of both sides. The result is that $x = p \pm q$, where the term $\pm q$ represents the fact that q^2 has these *two* square roots. If we now substitute this solution back into the first standard form we find, after a little rearrangement,

$$\left(p^2 + q^2 + ap + b\right) \pm q(2p + a) = 0.$$

This is equivalent to two equations:

$$2p + a = 0 \qquad \text{and} \qquad p^2 + q^2 + ap + b = 0.$$

Solving these produces the results $p = -\dfrac{a}{2}$ and $q = \dfrac{1}{2}\sqrt{a^2 - 4b}$, and now the reader can complete the rest, producing a version of the familiar "formula".

Ward's idea, which he calls "completing the cube", is to approach the cubic in an analogous way, beginning with the standard form:

$$x^3 + ax + b = 0.$$

(Every cubic may be reduced to this form; see the *Function* article referred to above. We may also assume that $b \neq 0$, as otherwise one of the roots would be zero, and the cubic would reduce to a quadratic.)

By analogy with what has just been done for the quadratic, write the equation as

$$(x - p)^3 = q^3,$$

and its solution as $x = p + \omega q$, where the term ωq is one of the *three* cube roots of q^3. Now back-substitute as before, to reach

$$(p^3 + q^3 + b) + (3\omega pq + a)(p + \omega q) = 0.$$

By analogy with the quadratic case, Ward reduced this to two equations. The first of these is

$$p^3 + q^3 + b = 0.$$

Putting this into the previous equation, leads to the further equation

$$3\omega pq + a = 0.$$

(Because $x = p + \omega q \neq 0$, as $b \neq 0$.)

We can now rewrite these two equations as:

$$p^3 + q^3 = -b \qquad \text{and} \qquad p^3 q^3 = -\frac{a^3}{27}.$$

Thus p^3 and q^3 are the roots of a quadratic

$$y^2 + by - \frac{a^3}{27} = 0.$$

This equation has the solutions p^3 and q^3 and thus p and q may be found. This enables us to complete the solution of the given cubic. The last few steps, those from here on, are the same as those arising from the more usual approach.

A Fresh Look at *n*th Roots

We are familiar with the fact that the square root of 2 is irrational. (Several proofs of this result were presented in *Function* in April 1999.) But what about the cube root of 2, the fourth root and all the rest? They too are all irrational, and a remarkable new proof of this result has just appeared. It was discovered by William Henry Schultz, an undergraduate at the Charlotte campus of the University of North Carolina. He showed it to his instructor who, in his turn, was also impressed, and forwarded it to the *American Mathematical Monthly*, where it appeared in their issue for May 2003.

To prove the result, Schultz supposed the opposite and looked at the possibility that $\sqrt[n]{2} = \frac{p}{q}$ or in other words that it was rational. He considered the cases $n \geq 3$, taking it as read that the case $n = 2$ was already settled.

But now $2q^n = q^n + q^n = p^n$.

This makes the number triple (q, q, p) a solution of the Fermat problem, and it is now established (see *Function*, April 1994) that this is impossible. Thus the assumption of rationality leads to a contradiction and must be false.

Because the result is so neat and so surprising that it seems a pity, downright churlish in fact, to criticise it, but there are two grounds on which it can be challenged. The first point to make is that it is not established conclusively that it does not assume what it sets out to prove. We would need to know that nowhere in Wiles' proof of the impossibility of the Fermat problem does he use the irrationality of $\sqrt[n]{2}$, and this would be quite a difficult task, and possibly even a mistaken one.

The second point to make is that there are better proofs of the result. Here is one. Suppose that $\sqrt[n]{2} = \frac{p}{q}$ and that the fraction $\frac{p}{q}$ is in its lowest terms, which is to say that no cancellation is possible between numerator and denominator. Then $2 = \frac{p^n}{q^n}$ and here the left-hand side is an integer and the left a fraction in which no cancellation is possible between numerator and denominator. This can only happen if $q = 1$, which in turn implies that $\sqrt[n]{2}$ is integral, which is wrong!

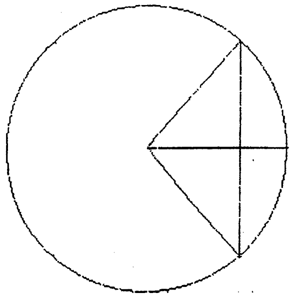
Indeed this latter proof readily generalises to show that $\sqrt[n]{m}$ (where m is an integer) is either itself integral or else irrational, so that the alternative proof tells us more.

Schultz probably knew of this proof, but nevertheless produced his new proof. Despite the criticism, it is very elegant, and we thought you might like to share it!

Columella's Formula

The Roman author Lucius Columella (1st Century AD) wrote extensively on rural life, and two of his works, *De re rustica* and *De arboribus*, have survived. They have been issued together in English translation as *Of Husbandry*. For the most part they are concerned with farming, but among the details he includes a formula for estimating the area of a segment of a circle. This has been the subject of a recent article in *The Mathematical Intelligencer*, authored by J-M Lévy-Leblond, a mathematical physicist at the Université de Nice.

Look at the diagram opposite. The *segment* of the circle shown is the portion of the circle's interior cut off at the extreme right of the diagram. It subtends an angle, θ say, at the centre of the circle. As is standard in modern trigonometric discussions, the radius has been set equal to 1.



Columella was interested in finding the area of a paddock shaped like this. (Lévy-Leblond comments that this is an unlikely shape for a paddock. However, if the paddock lay on a bend in a river or else was situated beside a curved coastline, then perhaps it is not quite such a strange idea.)

Two lengths are easily measured. They are the length of the straight boundary of the region and the width of the segment at its widest extent. In an older usage, these were called the *chord* of θ and the *versed sine* of θ. Here we will use this terminology, although Lévy-Leblond does not. Write chθ for the chord and vsθ for the versed sine. In our more familiar notation

$$\mathrm{ch}\,\theta = 2\sin\frac{\theta}{2} \quad \text{and} \quad \mathrm{vs}\,\theta = 1-\cos\frac{\theta}{2}$$

Long experience has taught us to use the more familiar sine and cosine functions in place of these more basic alternatives. Try writing out some of the familiar trig formulae in terms of chθ and vsθ, and you will soon see the benefits of our modern viewpoint.

There is a simple (exact) formula for the area of the segment. It reads:

$$A = \frac{1}{2}(\theta - \sin\theta),$$

where A is the area, and θ is measured in *radians*.

However Lévy-Leblond notes that this involves finding the centre of the circle and determining the angle θ, which is less direct than the simple measurement of chθ and vsθ. It is terms of these quantities that Columella gives an approximate expression for A.

In the notation used here, it reads

$$A(\theta) = (\mathrm{vs}\,\theta + \mathrm{ch}\,\theta)\mathrm{vs}\,\theta + \frac{1}{56}\mathrm{vs}^2\,\theta,$$

where now $A(\theta)$ is written for the area. The formula is quite accurate. Lévy-Leblond plots it against the exact result, and notes that its worst errors occur for small values of θ, which probably makes for an unrealistic paddock anyway. But then he goes on to speculate about how the formula was derived. Here is his suggestion.

If we measure two lengths, and wish to compute an area, then there are three possible areas we can form from the two lengths. Call the lengths chθ and vsθ and then note that we can have $\mathrm{ch}^2\,\theta$, $\mathrm{ch}\,\theta\,\mathrm{vs}\,\theta$ and $\mathrm{vs}^2\,\theta$ or some sum of these all as possible areas.

Therefore suppose that we ask that the area have the form

$$A(\theta) = \alpha\,\mathrm{vs}^2\,\theta + \beta\,\mathrm{vs}\,\theta\,\mathrm{ch}\,\theta + \gamma\,\mathrm{ch}^2\,\theta, \tag{*}$$

and that it give *exact results* for $\theta = \pi$ and $\theta = \frac{\pi}{2}$, two values for which the calculation is relatively straightforward. In fact we have:

$$A(\pi) = \frac{\pi}{2} \qquad \mathrm{ch}\,\pi = 2 \qquad \mathrm{vs}\,\pi = 1$$

$$A\left(\frac{\pi}{2}\right) = \frac{\pi}{4} - \frac{1}{2} \qquad \mathrm{ch}\frac{\pi}{2} = \sqrt{2} \qquad \mathrm{vs}\frac{\pi}{2} = 1 - \frac{\sqrt{2}}{2}$$

Feeding these values into the assumed form (*) gives two equations in the three unknowns α, β, γ, which, after a little work, may be written:

$$\alpha + 2\beta + 4\gamma = \frac{\pi}{2}$$

$$\frac{3}{2}\alpha + \sqrt{2}(\beta - \alpha) - \beta + 2\gamma = \frac{\pi}{4} - \frac{1}{2}$$

At this point, Lévy-Leblond becomes somewhat speculative. He suggests that the term in $\sqrt{2}$ was abolished by setting $\alpha = \beta$. This gives two simpler equations in the two remaining unknowns β, γ:

$$3\beta + 4\gamma = \frac{\pi}{2}$$

$$\beta + 4\gamma = \frac{\pi}{2} - 1.$$

We may now solve to find that $\alpha = \beta = \frac{1}{2}$ $\gamma = \frac{\pi - 3}{8}$ and if we use the approximation $\pi \approx \frac{22}{7}$ Columella's formula follows. Lévy-Leblond notes that it is often asserted that the Romans used the cruder approximation $\pi \approx 3$, but this would not give Columella's formula as he has it.

As an exercise, try using a similar derivation to find an approximation for the perimeter of a segment of a circle. The result is surprisingly good; at its very worst it is still under 10% off. $\sqrt{2}$ vanishes from the calculation of its own accord, and this time the fit is exact at $\theta = 0$.

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HISTORY OF MATHEMATICS

Vito Volterra

Michael A B Deakin, Monash University

I had long wanted to devote one of my columns to this extraordinary man, and last February I promised to do so. Vito Volterra is one of the most important mathematicians from the late 19th to the early 20th centuries. He was born in 1860 in what was then known as the Papal States (before the incorporation of that principality into today's Italy), and he lived till 1940. His family background was humble, but by dint of ability and hard work he achieved a tertiary education and a subsequent academic career.

His early work lay in Mathematical Physics, and he was appointed to a chair in (essentially) this discipline at the University of Pisa in 1883. In 1892, he moved to a similar appointment at the university of Turin and by 1900 he had moved again to the University of Rome. Although by the outbreak of World War I he was past the age of military service, he volunteered for the Italian Air force and worked on airships (dirigibles) rather than heavier-than-air machines (the Italians were among the last to move over to what we now regard as the norm[1]). In this capacity he introduced the use of helium in place of the more dangerous hydrogen.

His interest in Mathematics sprung initially from the demands of Theoretical Physics. In the course of a short article for a general readership, it is not feasible to describe either in depth or in full breadth the many contributions he made to Mathematics, especially over a fifty-year period extending from about 1880 to 1930 or thereabouts.

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[1] The Italian air force general Umberto Nobile achieved one of the first major successes of this form of flight, when with the Norwegian explorer Roald Amundsen and the American aviator Lincoln Ellsworth he took an airship across the Arctic Ocean in 1926 (the first ever such crossing). However, Nobile afterwards fell from grace when a later (1928) airship expedition, an attempt to reach the North Pole, came to grief with the loss of 17 lives. Amundsen was another victim, for he perished in the search for survivors. Dirigibles finally lost all favour with the burning of the Hindenburg in 1936. Had helium been used in that vessel, the tragedy would have been averted.

Rather I will concentrate on certain major themes, and so hope to give a flavour of the Mathematics he developed. It is summed up by the *Encyclopedia Britannica* in the words "[he] strongly influenced the modern development of calculus". This is no exaggeration. To substantiate it, let me describe one line of his research.

Much of Mathematical Physics takes the form of Differential Equations, such as were discussed briefly in my February column. Think of the simplest case introduced there: the vibration of a taut string (as on a guitar, for example). If u is the displacement of the string from its equilibrium resting position, and x is a co-ordinate that measures the position of a point of the string, when that point is at rest, then the string takes up a position described by the differential equation: $\frac{d^2u}{dx^2} = n^2x$, where n is a constant to be determined. As well as the differential equation, there are two "boundary conditions" specifying what happens at the ends of the string. In the case of the guitar string, the ends are held fixed and so $u = 0$, when $x = 0$ or l (l being the length of the string). These boundary conditions determine the possible values of the constant n.

Volterra discovered that the information included in the differential equation, together with the further information provided by the boundary conditions, could all be included in a single equation of another type: an *integral equation*. Integral equations are equations in which an unknown function occurs as part of the integrand in a definite integral. He was one of the first researchers to undertake a systematic study of such equations, and two of the basic forms now bear his name.

He was able to show that these equations could be looked on as a limiting form of a large array of simultaneous linear algebraic equations, where the number of equations tends to infinity. Indeed, the computer solution of such equations often involves approximating them by a large array of simultaneous equations and then inverting the matrix involved.

Among the uses to which such equations were put is the study of *seiches* ("tides" in lakes) as outlined in my column for last February. As I pointed out then, the analogy between the theory of seiches and of the vibrating string is a close one.

However, there were other such applications. One important one is the study of a phenomenon known as *hysteresis*. This occurs in many contexts. For example, when a deformable solid is subjected to stress, its response depends not only on the stress applied at the precise instant of observation, but also on what has gone on before: on the history of the stresses to which the solid has been subjected. This later led to fruitful extensions of the classical theory of elasticity. Other contexts also benefit from this generalised approach; hysteretic effects also occur in connection with electromagnetism, in biological studies and elsewhere.

This last remark brings me to one of Volterra's best-known contributions: his formalisation of some of the basic ideas in Ecology. He began this work in 1926, and already in his first paper on the subject, he laid down the fundamental equations that inform much discussion even today. The first section of this paper considers the interaction of two species. Two cases are presented. The first is that in which the two compete with one another for a common resource. In many such cases, one species survives and the other is eliminated. The case in which the two co-exist is less likely.

But Volterra is much better remembered for his discussion of predator-prey interaction. Suppose there are two species: Species 1, which is the prey of Species 2, which is the predator. Let there be N_1 individuals of Species 1 and N_2 of Species 2. Then the prey, if unchecked will tend to increase in proportion to their numbers. (This is a law of growth first expounded by the economist Thomas Malthus in 1798.) However, the population of Species 1 will be depleted in proportion to the number of encounters between an individual of Species 1 and one of Species 2. This number in turn is jointly proportional to the numbers of individuals in the two species. Thus Volterra wrote

$$\frac{dN_1}{dt} = rN_1 - kN_1N_2, \qquad (1)$$

where r and k are (positive) constants and t stands for time.

If we now consider the variation in the number of predators (i.e. Species 2), we can see their numbers growing as a result of their eating members of Species 1, but decreasing as a result of their dying off (from, e.g., old age). This leads to a similar equation

$$\frac{dN_2}{dt} = KN_1N_2 - DN_2, \tag{2}$$

where K and D are (positive) constants.

Equations (1, 2) constitute what is now termed the Volterra oscillator. Its behaviour is the subject of the cover story for this issue. Biologically the conclusion is that the populations of both species will fluctuate, but that both species will survive.

Later, Volterra collaborated with the biologist Umberto D'Ancona in a study of fisheries in the Adriatic Sea. This led to the conclusion that if *two* prey species are involved then one of these will go to extinction. The logic can easily be put into words: If the fisherman (the predator) hunts for fish of Type 1 for preference, but will also take fish of Type 2 if need be, then, when Type 1 becomes scarce, he will remain in business by taking Type 2. Type 1 (the preferred species) is now doomed, for the fisherman will not be averse to taking whatever Type 1 fish he can when the opportunity presents itself. (Had Type 2 not existed, then the fishery would have become uneconomic, and the fish-stock would have had time to recover.)

[We see this effect very strongly in the case of the whaling industry. Only the force of very powerful sanctions prevents the larger (and preferred) species (sperm whales and blue whales) from being hunted to extinction. Those whaleries that remain in business do so by relying on their catch of Minke whales, which the early whalers would have scorned to bother with!]

Later, Volterra learned that his work in this area had, to some extent, been anticipated by the US mathematician Alfred J Lotka, who initiated a correspondence between them. In it, there is no sign of the animosity that sometimes besets such disputes: rather their tone is friendly throughout.

However, already in that first paper, Volterra was concerned to go beyond the two species case. He considered the more general situation with larger numbers of predator and prey species. Later he was able to set up even more general models. This matter is somewhat controversial. Volterra saw his models as important special cases, idealisations that simplified the complications of the natural world, but which nonetheless remained relevant to it.

He considered that he was proceeding in the same spirit as that of classical Physics. To give an analogy that will show the flavour of his thought, consider the fall of a stone under the influence of the earth's gravity. Initially, it has no velocity, but it has potential energy by dint of its position above the earth's surface. As it falls, it gathers speed and this potential energy is converted into kinetic energy. Just before it hits the ground, its potential energy is all gone and the entire energy it possesses is kinetic. In an ideal case, all the original potential energy would be converted into kinetic energy, but in real life there is a loss caused by the resistance of the air. This energy turns up as heat. However, for many purposes, the small loss may be disregarded.

The science of Mechanics (from which this example derives) is one of the most successful aspects of Applied Mathematics, especially as it relates to the motions of the heavenly bodies, such as planets and satellites. It works particularly well in this context because of the relative unimportance of frictional effects in space.

The Volterra oscillator has been criticised because of an unrealistic feature. The closed loops like the ones shown on the cover correspond to *exactly* periodic motion. Consider the analogy of a pendulum. Ideally, it oscillates periodically forever, and a diagram analogous to that of the cover diagram shows similar closed loops. However, in real life, it runs down. The ideal situation is now referred to as "structurally unstable", and this term is also applied to the Volterra oscillator. This feature was in fact first demonstrated by Lotka, but Volterra was concerned to investigate the ideal case, as a first step to understanding the more complicated behaviours of more realistic systems.

In this endeavour, he was guided by the extensive elaboration of Classical Mechanics. He showed that a system representing many predator and prey species, subject to special conditions that he likened to the idealised treatments of that discipline, displayed remarkable similarities to those other ideal results. Volterra believed that these investigations constituted a major contribution to the understanding of ecosystems. Nowadays, however, we are more cautious. Even back in 1941, when Sir Edmund Whittaker's obituary of Volterra was published, it included the evaluation: "It would be rash to say whether the analogies with physical science which he unearthed will remain what they appear to be at first, and certainly are, *at least*, a clever and remarkable *tour de force* – or whether they will eventually be seen as the germs of a profound biodynamics, essential to the theoretical and economical biology of the future: what is beyond dispute is that his contributions to pure mathematics will be in demand more and more inescapably as mathematical biology develops."

What has happened is that these elaborations are now seen as not saying anything profound about biological associations, but rather as saying *very profound* things about systems of differential equations. The systems of Classical Mechanics and of Volterra's idealised biological models are now seen as two examples of an entire class of such systems.

Even in Whittaker's day, biologists treated the matter with caution. As he also wrote: "Biologists have been apt to criticise Volterra for pre-occupying himself so elaborately with abstract mathematical models based on simplifying assumptions remote from the complexities of nature." Some of Volterra's results now seem unrealistic; for example he believed that the number of species in a population that had reached equilibrium was necessarily even. Furthermore the recent realisation of the importance of chaotic solutions has undermined some of his more ambitious conclusions. Nonetheless the simpler conclusions of his less ambitious models have made a lasting impact on ecological thinking.

In 1978, the more influential ecological writings of Volterra, Lotka and others were collected together (and where necessary translated into English) and published as a volume, *The Golden Age of Theoretical Ecology, 1923-1940*, published by Springer as *Volume* **22** of their *Lecture Notes in Biomathematics* series. More recently (2002), a volume of Volterra's correspondence on mathematical biology was also published, this time by Birkhäuser, under the title *The Biology of Numbers*. I was fortunate to receive a review copy of this work from the journal *Mathematical Reviews*, and I treasure it greatly. It is clear that Volterra was in close contact with all the leading figures of this remarkable movement.

The last years of his life were overshadowed by the political turmoil that ultimately led to World War II. The Fascists came to power in Italy in 1922, and by 1930, much of Italy's democratic heritage had been dismantled. In 1931, Volterra refused to swear an oath of allegiance to Mussolini, and was in consequence forced out of the University of Rome; in 1932, he was compelled to resign from all the Italian scientific societies of which he was a member. As a Jew, he came to suffer more and more from the resurgent anti-Semitism of the period. From this time, he spent most of each year in voluntary exile, and valued his membership of foreign scientific bodies all the more highly.

In 1936, Pope Pius XI revived the Pontifical Academy of Sciences and gave it a new constitution based on trans-national and non-sectarian values. He invited Volterra to become a member, and this was an honour that it was not in Mussolini's power to revoke. Volterra's very last scientific paper (on hysteresis in elastic media) was published by the Pontifical Academy. The second-last (on hysteresis in biological associations) was published by the Edinburgh Mathematical Society, of which he was also a member, and which likewise lay outside the reach of the Fascist government.

He became seriously ill in 1938 and eventually died in 1940, working on Mathematics almost to the very end. The Pontifical Academy concluded their announcement of his death by saying "we believe him to have passed from this life to the earnest and industrious advancement of the sciences throughout an eternity of wisdom".

Further Reading

The most accessible source for most readers will be the on-line biography at the St Andrews site:

http://www-history.mcs.st-and.ac.uk/history/Mathematicians/Volterra.html

although this is rather a clumsily compiled piece of work. The article in the *Dictionary of Scientific Biography* is much better, and the Obituary Notice by Whittaker is an excellent summary. This is reprinted in the Dover edition of the English translation of one of Volterra's books: *Theory of Functionals and of Integral and Integro-Differential Equations*. The work cited in the body of my account above, *The Biology of Numbers*, contains a very careful and detailed discussion of the significance of Volterra's biological studies, and the references given by the authors of the St Andrews biography are also useful.

COMPUTERS AND COMPUTING

Solving Non-Linear Equations: Part 6, The Secant Method

J C Lattanzio, Monash University

The Secant Method is a modification of the Newton-Raphson. It is suitable for equations $f(x)=0$ where $f(x)$ is not easily differentiable. For example we might have a horrendously complicated function whose derivative could only be evaluated using large amounts of computer time. Such a case occurs with interstellar gas clouds, whose cooling is due to molecular vibrations. The cooling function is known analytically, but solving using Newton-Raphson is not practical because of the complexity of the derivative. Also, often we do not have an actual *function*, but rather the result of many experiments. These return a measured value (*f*) as we vary some input (*x*). We thus define a function $f(x)$, without knowing its form. We can rerun the experiment many times and so determine the value of $f(x)$ for any *x*, but we cannot determine the derivative of such a function.

The idea of the Secant Method is to replace the analytic form $f'(x)$ in the Newton-Raphson formula with the approximation:

$$f'(x_i) \approx \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}$$

Recall that the Newton-Raphson formula was

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

and now replace the derivative by the approximation. This gives

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i)-f(x_{i-1})}$$

This formula simplifies to

$$x_{i+1} = \frac{x_i f(x_{i-1}) - x_{i-1} f(x_i)}{f(x_{i-1}) - f(x_i)}$$

and this is what we use in our iterative scheme.

Consider as an example the solution of the equation

$$f(x) = x^x - 2\cos x = 0\,.$$

The somewhat unusual function whose zero we seek *can* in fact be differentiated, but we will proceed without doing this. If we explore the behaviour of the functions x^x and $2\cos x$, we find that when x = 1, $x^x = 1$ and $2\cos x$ = 1.08... . (Remember that x is measured in *radians*!) For the slightly larger value x = 1.1, $x^x = 1.11...$, and $2\cos x$ = 0.907... . We thus know that there is a root near these values, in fact between them.

So choose $x_0 = 1$, $x_1 = 1.1$. Applying the formula derived above yields the following table (where, as in previous articles, the final column gives the relative change in the estimate for the root):

i	x_i	ε_i
0	1.00000	—
1	1.10000	0.09091
2	1.02839	–0.06964
3	1.02950	0.00108
4	1.02954	0.00004

So we have the value of the root as 1.0295 to four decimal places.

Concluded on p 128

PROBLEMS AND SOLUTIONS

More on Problem 26.5.4

Bernard Anderson writes to say that John Barton's method of solution to this problem may be extended to cover more cases. We stated in our last issue that Barton's argument may be used whenever the set of points S involved was the same after it was rotated by 180° about a central point. Anderson points out that almost the same argument may be used with 120° rotations, etc. Thus the original problem could be solved for an equilateral triangle by the same means, merely replacing the grouping of the points into pairs by a grouping into threes.

By extension, Anderson's generalisation may also be applied to the case of 5-fold, 7-fold symmetry, etc. Thus, whenever there is what we would clearly recognise as a centre, then that point supplies the required minimum. (The case of, e.g., 6-fold symmetry may be viewed as a subcase of either the 2-fold or the 3-fold case, etc.)

However, this still leaves a lot of asymmetric cases not accounted for!

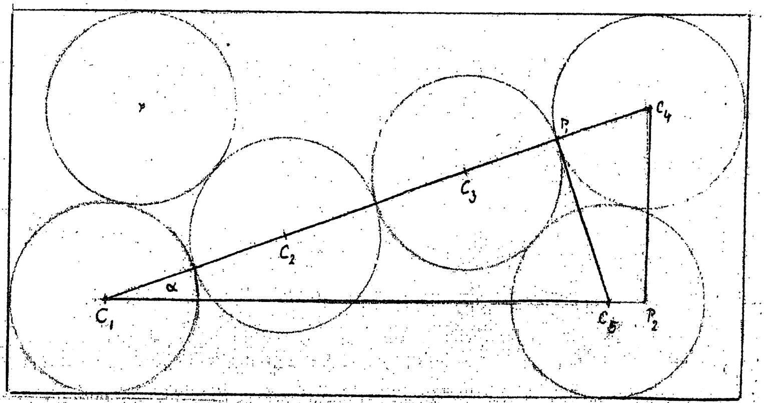
Solution to Problem 27.1.1 (from the Wasan, traditional Japanese Mathematics, reproduced in *History in Mathematics Education*, ed J Fauvel and J van Maanen)

The problem read:

The diagram overleaf shows six circles packed as arranged inside a rectangle. The circles are all equal and the radius of each is 1. Find the dimensions of the rectangle.

We received solutions from John Barton, Paul Grossman and Carlos Victor (Brazil). All followed essentially the same lines, and here we use the notation and diagram supplied by Grossman.

Draw a straight line through the centres C_i of the four consecutive circles. At P_1, where circles 3 and 4 touch, draw a line normal to the first to the centre of circle 5. Since C_3, C_4 and C_5 are the corners of an equilateral triangle, the distance P_1C_5 is $\sqrt{3}$.



Join C_1 and C_5 by a straight line and you find from the triangle $C_1C_5P_1$ that $\tan\alpha = \sqrt{3}/5$. From C_4 drop a line normal to the extension of C_1C_5 to point P_2. Write $C_4P_2 = H$ and $C_1P_2 = W$. Since the distance $C_1C_4 = 6$, it follows that $C_4P_2 = H = 6\sin\alpha$, and $C_1P_2 = W = 6\cos\alpha$. We now have two equations for H and W,

$$H/W = \sqrt{3}/5$$
$$H^2 + W^2 = 36,$$

with the solutions $W = 15/\sqrt{7} = 5.669$ and $H = 3\sqrt{3}/\sqrt{7} = 1.964$.

The rectangle enclosing the circles has dimensions $(W+2) \times (H+2)$, or 7.669×3.964.

Solution to Problem 27.1.2

The problem (after correction of an error) read:

Volume 2 of Arthur Mee's *Children's Encyclopaedia* shows a set of six cards, each containing 30 numbers between 1 and 60 (inclusive). The idea is to ask a friend to choose a number in this range and to identify those cards on which it appears. From this information it is possible to identify the number the friend chose.

How are such puzzles constructed and what is so special about the number 60?

We received solutions from John Barton, Paul Grossman, Julius Guest, Garnet J Greenbury and Carlos Victor (Brazil). Again all followed similar lines, but Guest gave the most thorough discussion.

The key to the trick is to express the numbers in binary notation. Six binary digits allow the representation of all numbers between 0 and 63. If the binary representation contains a 1 at position n, then the number appears on card number n, otherwise not. Thus (for example) 37 has the binary representation 100111. which means that it is to be found on the 1st, 2nd, 3rd and 6th cards, but not on the 4th or the 5th. The number of entries on each card will be exactly half of the total number posed at the outset. Card number 1 will contain all the odd numbers and no others. Card number 2 will contain just the numbers: 2, 3; 6, 7; 10, 11; and so on in pairs spaced 4 apart. This pattern continues until the 6th card which lists all the numbers greater than 31. (The number 0 would appear on none of the cards, but the "rules of the game" as usually presented outlaw it in any case.)

The puzzle, as printed in the encyclopaedia gives only the numbers up to 60, not up to 63. Almost certainly this is because 60 can be thought of as a "round number". It is also "divisor rich" which makes it more likely that the number of entries on each card (i.e. 30) can be presented in a satisfyingly compact tableau. However, there are a few anomalies that must be hidden. Barton checked the reference (p 874 of *Vol* **2** of the work) and found that the 3rd card had the number 13 repeated, as also did the 4th; the 5th repeated the number 31 and the last repeated the number 46. Of course it makes no difference which numbers are repeated on these cards, but otherwise the pattern is uniquely determined.

The trick depends on the binary representation and so works best for numbers like 63 which are one less than a power of 2. 64 is a power of 2 that is satisfyingly close to the "round number" 60. The only other viable candidate would be to choose the "round number" 30 and use five cards each with 15 entries in a 3×5 pattern. This would entail less duplication, but would perhaps be a little less spectacular in use.

Solution to Problem 27.1.3 (based on a problem in *Mathematical Bafflers*, ed Angela Dunn)

The problem read:

x, y, z are positive integers such that $x + y + z = xyz$. Find all solutions of this equation.

We received solutions from John Barton, Paul Grossman, Julius Guest and Carlos Victor (Brazil), who sent two different solutions. Here is one of Victor's.

Let $xy = a$, $yx = b$, $zx = c$ and suppose that $a \geq b \geq c$. The given equation may be rewritten as $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. But also $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{3}{c}$. It follows therefore that $c \leq 3$. Therefore c = 1, 2 or 3. Clearly we cannot have $c = 1$, but $c = 2$ allows either $a = b = 4$ or $a = 6$, $b = 3$. If $c = 3$, then we may have either $a = b = 3$ or $a = 6$, $b = 2$. These possibilities give us the solutions $(x, y, z) = (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$: in other words the six possible permutations of $(1, 2, 3)$.

The solution demonstrates the symmetric nature of the problem. The other solutions all made use of this although in somewhat different ways.

Solution to Problem 27.1.4 (from the same source)

The problem read:

A rower is moving upstream when his cap falls into the water. He does not realize this until 10 minutes later. Then he instantly reverses direction, and chases the cap as it floats downstream. He finally retrieves it one kilometre downstream from the point where it entered the water. What is the speed of the stream?

We received solutions from John Barton, Paul Grossman and Carlos Victor (Brazil). Grossman's had a nice "lateral thinking" flavour to it.

"Consider another reference frame. Imagine an observer sitting in a boat that floats with the water. She had seen the rower approaching from the right and dropping his cap as he was passing her. The cap for her remains stationary, but an adjacent tree on the shore is moving slowly to the left. The rower is also moving to the left, faster than the tree. She follows his progress with a telescope and her radar gun and notices after ten minutes that he is coming back. To her the rower's absolute speed is the same in either direction and ten minutes later he arrives back. "Here is my cap", he says and she replies: "I have been keeping an eye on it". She then picks up her telescope to look for the tree on the shore. The tree is now one kilometer away, so she concludes that the speed of the tree is 3km/hr."

And this, of course, is the speed of the stream.

Here are four new problems for eager puzzlists.

Problem 27.4.1 (from *Australian Senior Mathematics Journal* **15(2)**, 2001)

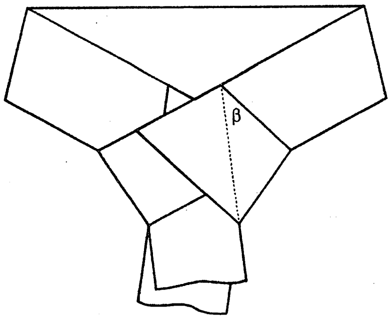
Through the vertices of a triangle *ABC*, draw its circumcircle. Form three other circles by reflecting this circumcircle in each of *AB*, *BC*, *CA*. Show that these three circles all pass through a common point.

Problem 27.4.2 (based on an article by T Eisenberg in *International Journal for Mathematical Education in Science and Technology* **34(1)**, 2003)

a, b and N are non-negative real numbers with $b \neq 0$. Let $a_0 = a$, $b_0 = b$ and then form sequences $\{a_n\}$, $\{b_n\}$ according to the rules $a_{n+1} = a_n + Nb_n$, $b_{n+1} = a_n + b_n$. Prove that as $n \to \infty$, $\dfrac{a_n}{b_n}$ tends to a limit and determine the value of that limit.

Problem 27.4.3 (proposed by Avni Pllana, Austria)

A rectangular strip of paper is folded into a "Tie-knot" as shown overleaf. Disregarding the thickness of the paper, determine the angle β.



Problem 27.4.4 (from the Australian Mathematics Competition, 1988; further discussed in *American Mathematical Monthly*, May 2003)

Simplify $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$.

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Continued from p 106

Thus the result is verified and the relevant value of K in this particular case is $\frac{3\sqrt{3}}{20}$ or about 0.26. This value holds for all members of the family, irrespective of the value of a.

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Continued from p 122

This demonstrates fairly rapid convergence, and in fact it can be shown that the convergence is less rapid than for the Newton-Raphson Method, but more rapid than Fixed-Point Iteration. Recall that the relative errors declined approximately as a sequence of squares for Newton-Raphson ($\varepsilon_{i+1} \approx k\varepsilon_i^2$) and approximately as a geometric sequence ($\varepsilon_{i+1} \approx k\varepsilon_i$) for Fixed-Point Iteration. In the case of the Secant Method, it may be shown that $\varepsilon_{i+1} \approx k\varepsilon_i^{\phi}$, where ϕ is the golden ratio $(1+\sqrt{5})/2$ (about 1.6).

Published by the School of Mathematical Sciences, Monash University