Function

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J. Mynde sculp.

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Function is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. Founded in 1977 by Prof G B Preston, *Function* is addressed principally to students in the upper years of secondary schools, but more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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* $14 for *bona fide* secondary or tertiary students.

NOTES FOR CONTRIBUTORS

Function welcomes contributions from its readers, either as:

(1) feature articles
(2) guest columns in the History or the Computing sections
(3) front cover material
(4) problems (preferably, but not necessarily, with solutions)
(5) solutions to published or submitted problems
(6) letters to the editor
(7) news and other miscellaneous items.

In the first three of these categories, all contributions are refereed by a process of peer review. Articles are assessed by an acknowledged expert in the relevant specialty as well as by reference to the editorial board. Problems and solutions are usually handled by the appropriate editor or else (as currently) by the chief editor. (At present, *Function* receives many more problems than it can use. Preference is given to those with some element of originality or unusual point of interest.) Letters to the editor and other miscellaneous items are handled by the chief editor.

Contributors should bear in mind the consideration that *Function* is directed to a target audience of Year 11 & 12 students. It is addressed to the student, not to the teacher. It is dedicated both to *genuine* Mathematics (not Mickey Mouse stuff), and to first-rate expository writing. In addition to its readability at the level of the target audience, we like each *Function* article to exhibit some freshness of viewpoint, approach, expression or application.

The editors reserve the right to make appropriate alterations to the text of submitted material, for reasons of house style, clarity of expression, layout and the like. They also reserve the right to abridge submitted material or to publish excerpts.

As a general rule, *Function* articles should be no longer than 3000 words or the equivalent of this in formulae, diagrams, etc. It assists the editors if submissions can be made in electronic form (MS Word), but this is not a requirement. It would also assist the editors if diagrams could be submitted as embedded Word files or else as camera ready copy, but again this is not a requirement.

THE FRONT COVER

The front cover for this issue reproduces the frontispiece of one of the most influential books in the Theory of Probability. It is taken from the third edition of Abraham de Moivre's *The Doctrine of Chances*. Abraham de Moivre (1667-1754) is an important figure in the history of Mathematics, especially Probability Theory. He was French by birth (and hence the surname is pronounced "duh MWAHvruh"), but spent most of his life in England. His family were Huguenots (i.e. Protestants), whereas France at that time was staunchly Catholic. Between 1598 and 1685, however, a law known as *The Edict of Nantes* gave a large measure of civil liberty and state protection to the Huguenots.

When this law was revoked, the Protestants suffered persecution, de Moivre for one being jailed. On his release, he fled to England as part of a wave of emigration that deprived France of many of its most industrious citizens. In England, de Moivre's talents were recognised in his election to the Royal Society and in his friendships with the leading mathematicians of his adopted country, including Isaac Newton.

However, he did not succeed in gaining the academic employment for which he no doubt hoped, and indeed he led rather a hand-to-mouth existence. To keep the wolf from the door, he gave private tuition to a succession of pupils and offered his services as an adviser to gamblers. This is the context in which he investigated the laws of probability. *The Doctrine of Chances* is his major work. It ran to three editions, the first being published in 1718 and the last appearing in 1755, after his death.

de Moivre was concerned to establish that probabilities were governed by laws, rather than by personal "luck" (as gamblers then and sometimes even now may incline to believe). The opening paragraphs bear quotation.

> "1. The Probability of an Event is greater or less, according to the number of Chances by which it may happen, compared with the whole number of Chances by which it may either happen or fail.
>
> "2. Wherefore, if we constitute a Fraction whereof the Numerator be the number of Chances whereby an Event may happen, and the Denominator the number of all the Chances whereby it may either happen or fail, that Fraction will be a proper designation of the Probability of happening."

Although the language is slightly archaic, the message is perfectly clear.

de Moivre is also remembered by the attachment of his name to the formula $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, which unites the study of Trigonometry with that of Algebra. He also made other important contributions to the Theory of Probability. He was an early contributor to the theory of the Normal Curve, and he also was the first to derive the approximation to the value of $n!$ that is now known as "Stirling's Formula".

Our cover picture is an example of what is known as an Allegory: the scene is not a realistic one and was never intended as such. Allegories were meant to illustrate moral reflections. Quite what this one represents is a matter for conjecture, as it seems not to be explained in any of the standard reference works. So you are free to take the following possible explanation with a very large crystal of salt!

The scene is represented against a classical backdrop, and so perhaps the figure at left is the Greek goddess Athene, or her Roman counterpart Minerva, both of whom represented Wisdom, or in some versions, Common Sense. This figure is drawing aside a cloud (a "Cloud of Unknowing"?) as one would a curtain, to reveal a second female figure. The posture of this figure and the wheel beside her are both reminiscent of the figure of Britannia, a personification of Britain (who appeared on the reverse side of our coins prior to Federation). However, whereas Britannia is normally shown in battle-dress, this figure comes "armed" with a mathematical diagram. Between her feet is placed a "Horn of Plenty", which may represent the riches that await gamblers who correctly calculate their odds.

To the right, we see four gamblers involved with a game of dice, and two children, similarly occupied, at their feet. The adults seem to be engaged in some form of altercation, and possibly we are meant to assume that the children are likewise quarrelling. The overall message would seem to be that gambling should only be undertaken by those informed with a true knowledge of "The Doctrine of Chances"!

To lower right is an inscription telling us that the scene was engraved by one J Mynde, but who he was or whether he was (after a common fashion of the day) reproducing an earlier work, we are not told.

ASTROLOGY

K C Westfold

[Professor Kevin Westfold was the foundation professor of Mathematics at Monash University, where he had the honour of delivering the very first lecture. He later became Dean of Science, and also served several times as Acting Vice-Chancellor, before rejoining the department of Mathematics as Professor of Astronomy in 1977. He retired in 1982, and died in October 2001. Many of his mathematical notes have been given to the School of Mathematical Sciences, as the former department of Mathematics is now called. Among the items in this collection are texts of various popular lectures that Professor Westfold gave from time to time. It is our intention to publish versions of some of these in *Function*, and this article is the first such. It is an edited version, combining various drafts and notes compiled for a number of different occasions. Eds]

Astrology is perhaps the oldest and most widespread of current superstitions. Although it is regarded as superstition by the greater part of the educated world, it still has great vogue. Here I want to give a brief description of its history and the basis and the theory on which it rests. Nowadays, Astrology is regarded by scientists with contempt, whereas Astronomy is regarded as a prestigious activity. Yet, until the end of the 17th century, these two terms were almost synonymous. Etymologically, Astrology refers to the science of the stars and Astronomy to their arrangement. Modern terminology however is different.

Astrology has been traced back to the Mesopotamian basin, the home of the ancient Babylonians and the cradle of civilisation, around 3000BC. The Babylonians observed the heavens, and noticed the movement of seven particular heavenly bodies against the background of the fixed stars. These seven special bodies later were called "planets" by the Greeks, from their word for "wanderers". The seven planets were: the Sun, the Moon, Mercury, Venus, Mars, Jupiter and Saturn. These are the later Latin names: for the Babylonians, Mercury was Nebo, Venus Ishtar, Mars Nergel, Jupiter Marduk, and Saturn Nineb.

The planets gave their names to various Greek and later Roman gods. We preserve references to the seven planets and their associated gods in the names of the seven days in our week. Sunday is the Sun's day, Monday the Moon's day, Tuesday is dedicated to Mars, Wednesday to Mercury, Thursday to Jupiter, Friday to Venus, Saturday to Saturn. (In English, these names are closest to the Scandinavian names for these same gods, but in French, say, the Roman names are more evident.)

The effect of the Sun on human activity was plain and easy to grasp. It regulated the pattern of day and night as well as the progress of the seasons. Other natural phenomena were associated with the other planets. The good (benign) things that happened: warmth, fine weather, abundant harvest, etc were attributed to the benign gods, Sun, Moon, Jupiter and Venus. Bad (malignant) things like floods and other disasters were attributed to the malignant gods Saturn and Mars.

As the benign influences like warmth and the malignant influences like floods both came from heaven, it was natural to suppose that this was the abode of the gods, and it was also natural to identify the gods with the planets. The motions of these were predictable, so it was also natural to suppose that the decrees of the individual gods were likewise predictable, as indeed could be argued very plausibly in the case of the Sun.

The motions of the planets all occur in a narrow belt of sky, some 7° wide and centred on the *ecliptic*, the apparent path of the sun through the fixed stars over the course of a year. The ecliptic is a circle comprising 360°, and this was divided into 12 "houses", or "signs", each named after a constellation prominent in that part of the sky. Thus each of these signs took turns at hosting the sun in its course through the heavens. In order, these were: Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricorn, Aquarius and Pisces. These are collectively known as "the Zodiac". The point of passage of the Sun from one house to the next is termed the "cusp".

To each of the 12 signs was assigned a 30° sector of the sky, to give a 12-month calendar with each month being a little over 30 days long. In the Northern Hemisphere, where the Babylonians lived, the Spring equinox, the date on which the length of the day began to exceed that of the night, coincided with the entry of the Sun into the constellation Aries. That happens on the 22nd of April, the date on which the Babylonian calendar began. This is also identified as the point at which the celestial equator (the projection of earth's equator into the heavens) crosses the ecliptic. This point, the "first point of Aries", is occupied by the sun on that date.

The determination of these patterns was one of the great achievements of ancient Astronomy, and we still make use of these insights today. The pattern of the different configurations remains valid. It is summarised in Figure 1.

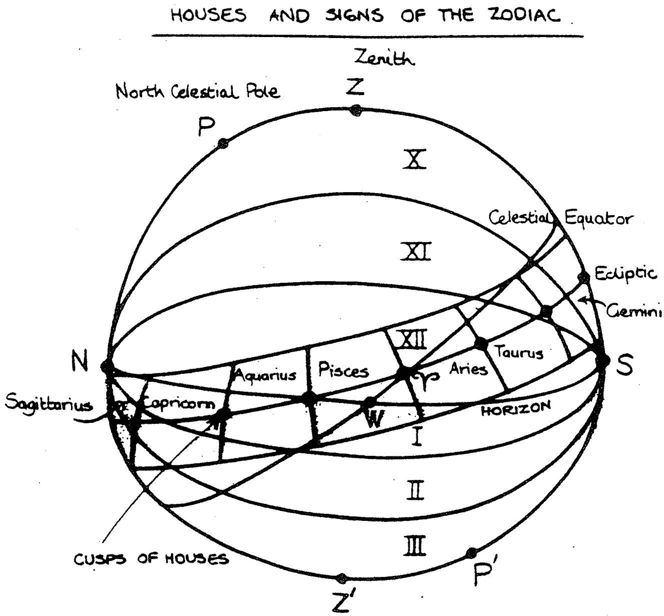

Figure 1

The astrological tradition of the Babylonians was greatly developed by the ancient Greeks, who learned of it via their contacts with the Egyptians.

In that elaboration, a further layer of theory was added to the observational basis and this layer related to the influence of the planets on human affairs. This aspect was developed under the Greeks and in Medieval Europe. In this way of looking at things, for example, the different "elements" of the Greeks were identified with different signs of the Zodiac. Thus Aries corresponded with Fire, Taurus with Earth, Gemini with Air and Cancer with Water. After this, the four elements

Fire, Earth, Air and Water repeated twice more to fill the entire 12 months.

But things went much further than this; colours, metals, plants, particular stars, medicinal herbs and drugs, all manner of things were brought into the overall scheme. Horoscopes were cast for individuals. The sign that was in the ascendant (i.e. in which the Sun rose) at the time of that person's birth was seen as affecting the life of the person. The signs of the succeeding houses concerned respectively: riches, brothers and sisters, parents, children, health, marriage, death, religion, dignitaries, friends, and lastly, enemies. Then too, behaviour and illnesses were seen as determined; the signs ruled different parts of the body. Each planet had its specific effect on bodily function. For example, a horoscope showing Mercury in Aries indicated a susceptibility to headaches.

These later elaborations of astronomical theory are what we today refer to as Astrology. An earlier distinction called it "Judicial Astrology" to distinguish it from "Natural Astrology", which is what we nowadays call "Positional Astronomy".

However, it is important to recognise that even the elaborated system was not irrational. It was rational just as Aristotelian Science was rational. However, like Aristotelian Science, it bore very little relation to what actually takes place in the world of experience. Astrology lost its credibility when Newton and his successors demonstrated that the motions of the heavenly bodies are governed by the same laws of motion and of gravitation that apply on earth. It suffered a further blow when the advent of powerful telescopes displayed the vastness of the Universe. If we take as the scale for the Solar System the distance from the Sun to Pluto, 5.5 light-hours, the distance to the nearest star (α Centauri) is 7×10^3 times this. Other distances, such as that to the galaxy Andromeda or to the edge of the known Universe, are very much greater still.

The only known effect of the planets upon us is the gravitational one, and this is minute except in the case of the Sun and the Moon. If there are to be effects of a non-gravitational character, what are these and how are they mediated? What laws describe them? What is their speed of transmission? What are the effects of the more recently discovered planets: Uranus, Neptune and Pluto? How can these be incorporated into a neat 7-day cycle?

Until some hypotheses are advanced to address such questions, Astrology must be dismissed as nothing but an activity practised by charlatans or credulous cranks.

It is interesting to note that the Jews have never had any truck with Astrology. They would have been exposed to it at the time of the Babylonian captivity, but they would have nothing to do with it, because it involved the gods of their captors. They heeded the call: "Hear, Oh Israel, the Lord thy God is one God. Thou shalt have no other gods but Me."

This same strong monotheism also prevented the Mohammedans from taking to Astrology. Christians likewise eschewed it for the same reason until, during the Middle Ages, it lost its connection with the ancient pantheon. In this new guise, it attracted adherents, and reached considerable popularity with the predictions of the celebrated Nostradamus. What spell it continues to exert, however, must nowadays be attributed to its apparently "scientific" character. But as genuine scientific knowledge advanced, Astrology moved to its present place well outside the scope of reputable Science.

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Quadratic Equations in Ancient Babylon

[In a previous issue (August, 2001), we reprinted an excerpt from one of the classics of Mathematics, George Boole's *Laws of Thought*. This time we present a slightly edited excerpt from another, Florian Cajori's *History of Mathematics*. It comes from the 5th Edition, which was substantially revised, especially in this aspect, by a later editor, who is identified only by the initials A G. In this edition, our passage is to be found on pp 4-6, with some further notes on p 487. The notes refer the reader to work by Otto Neugebauer and A Sachs. Until recently these researchers had written the last word on Babylonian Mathematics. However, a later generation of scholars has taken matters even further. The work of Jens Høyrup and Eleanor Robson, in particular, has added more to our understanding of the work of those ancient mathematicians. Eleanor Robson provided the background for *Function's* History columns for August and October, 1996. For more notes, see the comments following the excerpt given here. Eds]

Although the Babylonians never developed an algebraic symbolism and, in this sense, never created the subject we call algebra, they did deal successfully with a number of topics that today fall under the heading of elementary algebra. Such topics were dealt with in two classes of texts, which we shall call problem texts and solution texts. Let us consider quadratic equations, a subject that they cultivated extensively. Let us follow the solution of one problem, as given in a solution text: the area of a rectangle is unity, and the sum of the length and the width is the number 2 1 12 12. In modern notation,

$$xy = 1, \qquad x + y = b, \qquad (b = 2\ 1\ 12\ 12 = 2.02\ldots).$$

[*The number* 2 1 12 12 *is to be read in sexagesimal notation* (*base* 60) *as*

$$2 + \frac{1}{60} + \frac{12}{3600} + \frac{12}{216000} = 2.0200555. \ \textit{Eds}]$$

The text instructs the student somewhat as follows: Take one-half the sum of the length and the width and square it, obtaining

$$(b/2)^2 = 1\ 1\ 12\ 33\ 43\ 12\ 36.$$

[*The reader may care to check this and later calculations as an exercise. Eds*]

[What we now have is $x^2 + xy/2 + y^2/4$.] Subtract 1, the product of the length and the width, and take the square root, obtaining

$$0\ 8\ 31\ 6.$$

[What we have computed is the square root of $x^2/4 - xy/2 + y^2/4$, or $x/2 - y/2$.]

We check our last result by squaring. We then add it to, and also subtract it from one-half the sum of the length and the width [i.e., $x/2 + y/2 = b/2$], obtaining

$$x = b/2 + \sqrt{\ } = 1\ 0\ 36\ 6 + 0\ 8\ 31\ 6 = 1\ 9\ 7\ 12,$$

$$x = b/2 - \sqrt{\ } = 1\ 0\ 36\ 6 - 0\ 8\ 31\ 6 = 0\ 52\ 5.$$

This result is exactly equivalent to our standard algebraic formula for the solution of a quadratic equation.

We note several typical features. First, the generality of the method is made clear throughout, even though it is a particular numerical problem that is under consideration. In our example, although 1 is to be subtracted at one stage, it is described as the number 1, "the product of the length and the width". Similarly, if a multiplication is called for in the general case and the multiplier in the example under consideration happened to be 1, the multiplication by 1 is actually carried out. And the solution text often ends with the words "such is the procedure". Second, although the process concerns the particular numbers that arise in the course of solution, the verbal description of the numbers – "the sum of the length and the width", for example – is the exact equivalent of our own $x + y$, indeed the verbal description is often directly translatable into algebraic symbols. Third, there is no hesitation in making use of numbers having many sexagesimal digits. Fourth (although this may be an artifact of pedagogy), the square root usually comes out "even" – the number whose square root is taken turns out to be a perfect square. The typical problem text gives a large number of problems in which one condition, say $xy = 7\ 30$, is kept fixed and the second condition is given, at first, in normal form and then in more and more complicated forms, all reducible to the normal form. All the problems in one problem text turn out to have the same numerical solution, so that it obviously was of no concern to the teacher that the student already knew the answer; only the method counted.

In our illustrative example, (already in normal form) the numbers represent lengths. However, in general they may represent disparate quantities – number of days and number of workmen, area and volume, and so forth. Nor need the number of workmen prove to be an integer. The technique of solving such problems was all that mattered.

[In a recently published book, *Using History to teach Mathematics* (Ed V Katz), there are two articles relevant to this material. The first is an overview of Babylonian Mathematics by Eleanor Robson, and the second an account of precisely the Babylonian derivation of "the formula" for the solution of the quadratic. This latter article is by Luis Radford and Georges Guerette. Both articles are highly recommended. Eds]

LETTERS TO THE EDITOR

A Dreadful Error!

I read *Function* with interest and have found it a source of information, stimulation and challenge. However, I was aghast to find a crass blunder slip through in a journal published by mathematicians. The author was obviously not Dr Fwls ap Rhyll who is more subtle. It was someone who would endorse the argument: "the last two tosses were heads and since three heads in a row are unlikely I opt for tails next time."

In an article entitled A Hat of a Different Colour (*Vol* **25**, *Part* 4, pp 113-114), one of the aspiring prize-winners, Ada, is reported as having seen blue hats on each of the others. She then argues: "If my hat is blue, then there will be three blue hats, and so the hats will all match; the likelihood of this is 1/4."

While I agree with the strategy proposed in the article which is decided *a priori*, I disagree with Ada's statement made after having obtained the additional knowledge that two hats are blue.

Paul U A Grossman
(by email)

[Dr Grossman is quite correct, and the chief editor is to blame for the lapse! The reasoning attributed to Ada in the news-item is in fact not only faulty, but also irrelevant. The strategy is predetermined by the meeting of the three participants prior to the actual trial; so Ada has no business arguing the matter at all. All she has to do is follow instructions. Those instructions follow from the pre-determined strategy, which may be summarised in the table below.

Ada	Bet	Col	Response	Outcome
Red	Red	Red	All three say "Blue"	Loss
Red	Red	Blue	A, B pass; C says "Blue"	Win
Red	Blue	Red	A, C pass; B says "Blue"	Win
Blue	Red	Red	B, C pass; A says "Blue"	Win
Red	Blue	Blue	B, C pass; A says "Red"	Win
Blue	Red	Blue	A, C pass; B says "Red"	Win
Blue	Blue	Red	A, B pass; C says "Red"	Win
Blue	Blue	Blue	All three say "Red"	Loss

This table makes it quite clear that, using the pre-determined strategy, the three participants have a chance of $\frac{3}{4}$ of winning the prize. As was (correctly) said in the article: "It is important to notice that Ada has a chance of $\frac{1}{2}$ of getting her own hat-colour right. When she guesses she does not know what responses Bet and Col are entering simultaneously. The pre-arranged strategy means that *if she is right* then the others will not destroy her good work. Of course if she is wrong the three had no chance anyway. The laws of probability are not violated; it is the 'rules of the game' that make the existence of such a winning strategy possible." A strange feature of this "game" is that the incorrect reasoning attributed to Ada in fact leads to the right answer! Still, this is no excuse! MD]

Generalising a Result

I was impressed by the formidable powers of anti-differentiation displayed by your readers in solving Problem 25.2.2, and was inspired to look at this problem:

For how many values of $m = 1, 2, 3, \ldots.$ *can you evaluate the infinite sum*

$$S_m = \frac{1}{1.2.3} + \frac{1}{(m+1)(m+2)(m+3)} + \frac{1}{(2m+1)(2m+2)(2m+3)} + \ldots\ldots\ldots?$$

The general term is $\dfrac{1}{(km+1)(km+2)(km+3)}$, where $k = 0, 1, 2, \ldots\ldots$.

But $$\frac{1}{(km+1)(km+2)(km+3)} = \frac{1}{2}\left(\frac{1}{(km+1)(km+2)} - \frac{1}{(km+2)(km+3)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{km+1} - \frac{2}{km+2} + \frac{1}{km+3}\right)$$

Now set $f(x)$ equal to the sum of the infinite series whose general term is $\frac{1}{2}\left(\frac{x^{km+1}}{km+1}-\frac{2x^{km+2}}{km+2}+\frac{x^{km+3}}{km+3}\right)$. Then the derivative $f'(x)$ will be the sum of the infinite series whose general term is $\frac{1}{2}\left(x^{km}-2x^{km+1}+x^{km+2}\right)$, which is $\frac{1}{2}x^{km}\left(1-2x+x^2\right)$. But this sum is just the sum of a geometric series and so we have $f'(x)=\frac{1}{2}\left(1-2x+x^2\right)\frac{1}{1-x^m}$. Assuming that $x<1$, we may write $f'(x)=\frac{1}{2}\left(\frac{1-x}{1+x+x^2+.....+x^{m-1}}\right)$.

Now $f(0)=0$, and so we get the formula

$$S_m=\frac{1}{2}\int_0^1\frac{1-x}{1+x+x^2+....+x^{m-1}}\,dx.$$

Clearly these integrals become progressively more difficult as m increases, but the first four are not too bad. We have:

m	S_m
1	$\frac{1}{4}=0.25$
2	$\ln 2=0.1931471805....$
3	$\frac{\pi}{12}\sqrt{3}-\frac{1}{4}\ln 3=0.17879676889...$
4	$\frac{1}{4}\ln 2=0.17328679514...$
5	$?=0.1706890021....$

The values of S_m form a decreasing sequence, tending to a limit of 1/6 as $m\to\infty$. This is not difficult to prove, but the labour required for the exact calculation of S_m for large m becomes very formidable. I looked at the cases of $m = 24$ and $m = 7$. The first of these is extremely tedious and the second essentially impossible.

However, there is a different approach, a surprise solution by an alternative method, that works well for $m = 4$.

$$2S_4 = \left(\frac{1}{1}-\frac{2}{2}+\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{2}{6}+\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{2}{8}+\frac{1}{9}\right)+\ldots\ldots$$

$$=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{6}\right)+\left(\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{10}\right)+\ldots\ldots$$

$$=\left[\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right)-\left(\frac{1}{2}-\frac{1}{4}\right)\right]+\left[\left(\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}\right)-\left(\frac{1}{6}-\frac{1}{8}\right)\right]+\ldots\ldots$$

$$=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}\right)+\ldots\ldots-\frac{1}{2}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots\right]$$

$$=\ln 2-\frac{1}{2}\ln 2=\frac{1}{2}\ln 2\,.$$

This is the answer quoted in the table above.

Joe Kupka
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Two Corrections

Several errors crept into our last issue. Two are important enough to warrant correction here. On p 151, the final formula is misprinted. The plus sign should be a minus. On p 160 in the third last line, the word "on" got left out. So the relevant sentence should read "Thus the circumcentre of the triangle ABC is *on* the 9-point circle ...".

We thank Keith Anker for pointing these out to us.

HISTORY OF MATHEMATICS

The Locus of Mathematical Reality

Michael A B Deakin

In this issue I wish to continue the discussion I raised in the second part of my column for October 1999. The question is the status of mathematical truth: in what sense are mathematical truths true?

In answering this question, most mathematicians, if they ever consider the matter at all, fall back on some form of naïve platonism. This is to say that, without actually formalising their position into so many words, they would tend to agree that mathematical truths exist, in some sense, "out there". They are not concrete things; one doesn't run into Pythagoras's Theorem on a trip down the street to do the shopping. But, in some rather more remote sense, they are real.

Mathematicians speak of the "discovery of Pythagoras's Theorem", for example, as if Pythagoras's Theorem had some sort of existence independent of ourselves. It is as if Pythagoras's Theorem was there, awaiting discovery, like Australia was before people reached it. Now, of course, the cases are not exactly comparable. Australia, although we may think of it as a mental construct, especially in the political realm, say, is also a physical reality. We can run into it, not in the course of our weekly shopping trip, but by sailing in certain waters, as Captain Cook and others did.

However when it comes to Pythagoras's Theorem, we are talking about a purely mental construct, and this meaning thus becomes the primary one. Although Pythagoras's Theorem refers to right-angled triangles and we do in daily life encounter objects with this shape, these objects are not themselves right-angled triangles. From the time of Euclid, at very least, we have distinguished the *ideal* right-angled triangle, whose sides have zero thickness and are perfectly straight, from its embodiments in (say) buildings, areas of land or the like.

In rough terms, platonism sees these *ideal* objects as instances of a higher reality, beyond the reality apparent to our senses. For a platonist, we use our facility for abstraction to transcend the instances of objects shaped like right-angled triangles and to form the idea of a right-angled triangle itself. A thorough-going platonist would probably also hold that the triangle that we reach this way is actually "more real" than the specific instances that gave rise to our concept of it: that we have of necessity to proceed as we do, but the underlying reality is the other way around.

Other such cases are also adduced: Truth, Beauty, etc, but for many explanations the mathematical example is the clearest. We may argue about what things are beautiful, or even about the nature of Truth, but in the main we do not argue about Mathematics. This aspect of things has led many platonists to give a special place of honour to Mathematics: it provided the best illustration of their view of the world. It is even said of Plato that the inscription above the entrance to his academy read, "Let no one ignorant of Mathematics enter here".

The platonic view of Mathematics often remains unarticulated, but it was explicitly defended by G H Hardy in *A Mathematician's Apology*. [Hardy was the subject of this column for June 1995.] *Inter alia*, Hardy wrote: " 317 is prime, not because we think so, or because our minds are shaped one way rather than another, but *because it is so*, because mathematical reality is built that way." In more sombre vein, he wrote: "... the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply the notes of our observations''.

More recently, this same view has found an ardent champion in Martin Gardner, the columnist and writer of popular Mathematics. (Gardner was in fact a Philosophy major in his student days.) In his recent anthology *The Night is Dark*, Gardner writes "mathematical objects seem to have a peculiar existence of their own, independent of both the outside world and the human mind".

This view has been challenged. In this discussion, I will draw in particular on two sources. The first is a paper published 30 years ago, but still influential today: Philip J Davis's "Fidelity in Mathematical Discourse: Is One and One really Two?". (Davis later expounded his views at greater length in a book, *The Mathematical Experience*, co-authored with Reuben Hersh.) The other is a very recent book co-authored by a professor of Linguistics and a research psychologist: Lakoff and Núñez's *Where Mathematics comes from: How the Embodied Mind brings Mathematics into Being*. A few months ago, I was asked by the editors of the bibliographic journal *Mathematical Reviews* to review this latter work, and that was what provided the impetus for this column.

Davis published his paper in *American Mathematical Monthly* back in 1972. The work is in part an attack on the platonic conception of Mathematics, but it does, in the course of that attack, also succeed in documenting very well that this is the underlying philosophy of most other working mathematicians. Davis is a very considerable mathematician, and the author of a number of influential textbooks. He was a pioneer in machine proof and computer algebra, and his paper is a direct outgrowth from his work in this area.

Because Davis is himself a mathematician (whereas Lakoff and Núñez are not) his account rings truer to the mathematically informed reader. However, in their critiques of the platonist account of Mathematics, there is no real disagreement between them. I will tend here to follow Davis, however; I think he better makes the relevant points.

Davis lists as hallmarks of the platonic mindset a number of naïve and usually unstated assumptions:

1. The belief in the existence of certain ideal mathematical entities such as the real number system,
2. The belief in certain modes of deduction,
3. The belief that if a certain mathematical statement makes sense, then it can be proven true or false,
4. The belief that fundamentally, mathematics exists apart from the human beings that do mathematics.

In regard to this final "belief", Davis could not resist a not-so-subtle dig: "Pi is in the sky".

It should perhaps be emphasised that the four "beliefs" are not equivalent; rather they tend to coexist in the minds of mathematicians. However, no one of them implies the others. In particular, the third has taken something of a beating with incompleteness theorems by Gödel and others, so that nowadays, many mathematicians would disavow explicit belief in this statement. This is a big area in itself, and I won't go into it here – it is worthy of at least a column in its own right.

Rather let us get back to more general considerations. The usual argument for platonism is well put by Davis, even though his purpose is to question it. It goes like this: The symbol 2, the word "two", the Roman numeral II, the foreign words "duo" (Latin), "deux" (French), "zwei" (German), "rua" (Motu), "ukasar" (Gumulgal) and a host of other such names all designate the number we call "two". The concept of "two", it is thus argued, must exist independently of the symbol we use to denote it. Somewhere "out there" is an ideal disembodied "two", even though we can only express it by means of whatever specific symbol we use to embody it.

Davis's critique proceeds by denying a crucial assumption underlying such an argument. Although it is true that many different means are employed to express the concept "two", we cannot begin to discuss that concept without using *some symbol or other*. It might be a word, it could be a numerical symbol, a Chinese character, or a gesture in Auslan, or whatever; it's nature and form are completely arbitrary, but we cannot do without it.

This symbol enters the brain via (say) our visual or aural pathway, and once inside the brain, leaves some sort of physical trace, and ultimately further symbols are generated, by handwriting, by computer manipulation or as speech, etc. This is Davis's account of how Mathematics gets done. It is similar to the one espoused by Lakoff and Núñez, who emphasise two points:

1. Mathematics is a product of the human mind;
2. The human mind is an embodied mind.

These authors subject platonism to a trenchant critique of, which proceeds by demolishing the idea that we humans could access platonic numbers (say), unless through the medium of our minds, in which case, we would not have any guarantee of the independent existence of these platonic ideals, outside of our human minds.

For Lakoff and Núñez, the correspondence between the Mathematics generated by one mathematician and that generated by another is simply explained. Human evolution has produced in each of us very similar minds to those of our fellow humans, just as all human languages present basic similarities despite their differences of detail.

I would find it hard to disagree with either of the basic premises Lakoff and Núñez propose, and similar views can be found in the work of other authors. What has been called "the locus of mathematical reality" has been removed from an "out there" to a firmly established place within the human mind. We see this change of emphasis in the title of H R Jacobs's work *Mathematics: A Human Endeavor* or in the title given by Hadamard to his study *The Psychology of Invention in the Mathematical Field*. (Notice how Hadamard refers to "invention", not to "discovery".)

To place Mathematics firmly within human minds, rather than outside them, is a shift of emphasis with profound consequences. To my own mind, these are particularly important in their implications for Mathematical Education. (I was astounded that Lakoff and Núñez almost completely ignore these!)

But think of a simple example. Suppose I know of a number x that it satisfies the quadratic equation $x^2 - 3x + 2 = 0$. Is this the same thing as knowing that $x = 1$ or 2? From the purely logical point of view, it is, for each of these statements implies the other. But *psychologically*, matters are very different. In the one case, the possible values of x are known *explicitly*; in the other they are merely *implicit* in the data. Until we have solved the equation, we do not really "know" the values of x at all. This becomes even more pointed an objection for more complicated examples. When a problem is set, the answer (error apart) is implicit in the data, but the problem is not *solved* until we can enunciate "the answer".

A similar thing happens at the research level. The reality has never been better expressed than by John Henry (Cardinal) Newman in his essay *The Grammar of Assent*. Once one leaves the realm of "short and lucid demonstrations" and enters that of "long and intricate mathematical investigations", sources of doubt creep in. "[In] that case, though every step may be indisputable, it still requires a specially sustained attention and an effort of memory to have in mind all at once all the steps of the proof, with their bearings on each other, and the antecedents which they severally involve; and these conditions of the inference may interfere with the promptness of our assent."

He goes on to imagine a mathematician who believes he (it is a male he writes of) has proved a new result. Rather than immediately accepting it, he is stricken with doubts and checks his work, over and over again, has colleagues assess it, seeks independent and simpler proofs, etc; all against the possibility of some error's having crept in.

Newman's point is that mere formal proof need not carry conviction. The converse of this observation is the point of one of Hadamard's analyses. Here "common sense" often tells us that something is true, even in cases where formal proof is lacking, difficult to come by, or else lacking in motivation.

The case he discusses is that of a particle undergoing projectile motion, whose path lies in the plane defined by the initial conditions. We easily accept this as there is "no reason why the movable point should go to the right rather than to the left of the aforesaid plane". He goes on to note that the "mathematical proof, such as is classically given, proceeds in an utterly different way". However, he does note that the line of naïve reasoning can be transformed into a fully rigorous proof.

Much of the sophistication of many modern proofs derives from the perceived need to guard against the most subtle and contrived possible counterexamples. At the elementary level, these may perhaps glossed over. The more pressing need is to guard against sources of error. Davis, whose pioneering work in computer proof led him to these considerations, lists various sources of error in computer calculations: hardware limitations, software limitations, program error, faulty operation, etc. He makes the point that all these possible sources of error have their human counterparts.

Nowadays, we have much more experience with the routine use of computers. I would now trust a reputable package, such as MAPLE, to expand a difficult algebraic expression, more than I would trust myself. When it comes to computation, there are many situations where we routinely trust our computers, and indeed have no choice, because it is quite beyond our own powers to do the relevant calculations by hand.

This does not, however, mean that we must trust blindly to computer-generated results. We check them, much as Newman's mathematician checked his results. But do we trust them absolutely? Davis thinks not. He makes the suggestion that Mathematics is fast becoming an experimental science. We accumulate evidence for some proposition or other much as a physicist accumulates evidence for a proposed new law. When the evidence is overwhelming, we adopt the new law, but always with the proviso that it might be improved upon (as Einstein's Mechanics improved upon Newton's).

It is certainly true, as Davis is at some pains to point out, that error is a constant possibility. At the more complicated end of the spectrum, we need to be especially vigilant. Checking is essential. I tend to part company with Davis however when it comes to "short and lucid demonstrations" as Newman called them. To me it is inconceivable that 7 + 5 could possibly be anything other than 12. Perhaps the situation may be compared to the concept of "actual impossibility" advanced by Borel (for events where the probability of their occurrence is minute); the probability of 7 + 5 not being 12 has such a small probability attached to it that we discount it.

I first encountered the notion that Mathematics might not be platonic, but rather depended upon a consensus of the learned, in a somewhat unlikely place. In one of my own school textbooks, Hall & Knight's *Higher Algebra*, I read: "The general algebraical solution of a degree higher than the fourth has not been obtained, and Abel's demonstration of the impossibility of such a solution is generally accepted by Mathematicians." Even this mild statement would nowadays be dismissed as overly cautious. We *know* that the quintic equation and its higher degree analogues cannot be so solved in general.

To my mind, we need not place the platonic view at odds with that of Davis and others. Rather, we can say that Mathematics has *aspects* that may be viewed as platonic and others that are best looked at otherwise. Particularly important for the learning of Mathematics is the view that acquiring new mathematical knowledge is a matter of active engagement. We have to *work* to understand a proof or an example. We have to use *our* minds to absorb and appreciate the communication that others have provided to help us to grasp a piece of Mathematics. Here it is the *psychological* aspect of mathematical activity that is most to the fore. Once *we* have grasped a concept, or seen the gist of a proof, once we have made it our own, then we can accept it as true, and assign to it the status of an independent fact.

The object of our study of Mathematics is to take each piece of Mathematics we scrutinise and make it our own. Once we have done this, we can accept the truth of the theorem or derivation as established and not merely rely on authority.

Finally, it should be said that neither Davis nor Lakoff and Núñez nor any of the other authors I have quoted espouse the notion that Mathematics depends on culture in the sense that different human cultures could disagree about the truth of a mathematical result, that 7 + 5 could equal 12 in some cultures and 13 in others. Oddly this crackpot notion has been entertained – mostly by innumerate anthropologists and their like; never (as far as I know) by mathematicians.

Gardner in particular argues against such a notion. It was once espoused by an Australian philosopher, Douglas Gasking, but fell into disrepute. I had thought it dead, but recently it has resurfaced as part of the complex of beliefs known as "post-modernism". I doubt that all those who would claim the post-modernist label would take such a position, but it *is* held. It is good to see Lakoff and Núñez argue against this view.

News Items

The Abel Prize

Recently it was announced that the Norwegian government has created a 22-million (US) dollar fund for a yearly prize in mathematics. The first prize will be awarded in 2003, the 200th anniversary of the birth of Norwegian mathematician Niels Abel, and is named after him. The Abel Prize will be worth about $US500,000. More details can be found in a short article in the September 7th, 2001 issue of the journal *Science* (see p 1761).

Niels Henrik Abel (1802-1829) was one of the very greatest of mathematicians. His early death (from TB) was a great loss not only to Norway but also to the world. Of his many advances, perhaps his best-known is the proof of the insolubility of general polynomial equations of degree higher than 4, by means of standard algebraic operations.

The new prize is seen my many as the equivalent of a Nobel Prize in Mathematics. The lack of such a prize has bothered mathematicians for over 100 years. For more background on the missing Nobel Prize, see *Function,* April, 1987 and April, 1992.

The Play *Proof*

"Proof" is the title of a recent play that had a successful run on Broadway, won its author (David Auburn) the 2001 Pulitzer Prize for Drama (distinguished play by an American author) and went on to win three Tony Awards: Best Play, Best Actress (Mary-Louise Parker) and Best Director (Daniel Sullivan). It has attracted especial interest in the mathematical community because its theme is Mathematics. The following synopsis of the plot is taken from the play's website.

"*Proof* is the story of an enigmatic young woman, Catherine, her manipulative sister, their brilliant father, and an unexpected suitor. They are all pieces of the puzzle in the search for the truth behind a mysterious mathematical proof.

"On the eve of her twenty-fifth birthday, Catherine, a young woman who has spent years caring for her brilliant but unstable father, Robert, must deal with the arrival of her estranged sister, Claire, and with the attentions of Hal, a former student of her father's who hopes to find valuable work in the 103 notebooks that Robert left behind.

"What Hal discovers in an old notebook is a brilliant mathematical result that tests the sisters' kinship as well as the romantic feelings growing between Catherine and Hal.

"As Catherine confronts Hal's affections and Claire's plans for her life, she struggles to solve the most perplexing problem of all: How much of her father's madness – or genius – will she inherit?

"This poignant drama about love and reconciliation unfolds on the back porch of a house settled in Chicago's Hyde Park."

Hyde Park, incidentally, is the neighbourhood of the University of Chicago, and is one of the best-known intellectual hothouses in the USA.

Clay Olympiad Scholar

The Clay Mathematics Institute conferred its prestigious American Olympiad Scholar award on June 5 to Michael Hamburg, an 11th grader from South Bend, Indiana, USA. Michael beat out 350,000 other students nationwide to win the award, which commends the student with the most ingenious and elegant solution to a problem on the USA Mathematical Olympiad Competition's exam. Michael was one of only 9 out of 270 finalists to answer Problem 6 on the exam correctly, and his inventive "proof without words" won unanimous praise from the judges. (The Clay Institute is the same organisation that has put up a total of $US7million for the solution of seven major outstanding mathematical problems, see *Function*, April 2001.)

Problem 6 of the last Mathematical Olympiad was particularly difficult. It read (we here correct a small but important error in our earlier account in *Volume* **25**, *Part* **4**):

Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

Quite a challenge!

COMPUTERS AND COMPUTING

MathWorld is back on line

Cristina Varsavsky

In the first issue of the previous volume of *Function* we published the sad news of an on-line mathematics treasure – Eric Weisstein's *World of Mathematics* – which had been shut down by court order because of a copyright dispute. We are glad to inform our readers that the conflict has been resolved and that the website

http://mathworld.wolfram.com/

is now back and running.

In a letter to his "gentle readers", Eric Weisstein explains the year-long ordeal to get his site back on line, and the devastating experience of a lawsuit which could have destroyed a significant part of his life's work.

If you would like to know the details, these are published on the website. The account also includes the history of the *MathWorld* concept as being a dynamic encyclopaedia with detailed mathematical information built up by contributors from all around world, and accessible to lay people.

A new feature of *MathWorld* is the headline "Mathematical News" intended to keep readers abreast with what is happening in mathematics. On the day of writing this column, I found a headline announcing the discovery of the new largest known Mersenne prime number. Mersenne numbers are integers of the form $2^n - 1$; the search for numbers of this form that are also prime is a computationally challenging exercise which requires the fastest computers. The largest Mersenne prime number discovered last November has 4,053,946 digits; you can even download a 4 MB text file which shows them all!

The headlines also announce the availability of another resource for teachers, students, researchers and all lovers of mathematics – the world most encyclopedic collection of mathematical functions and their inter relationships. Today, I have access to 37,710 functions – just a few mouse-clicks away – and this number grows day by day thanks to the many contributors to this site, maintained by Wolfram Research.

Browsing around I also learned about *panmagic squares* – squares with all the diagonals, including those obtained by wrapping around the edges, all add up to the magic constant. Here is an example:

8	17	1	15	24
11	25	9	18	2
19	3	12	21	10
22	6	20	4	13
5	14	23	7	16

The magic constant of this magic square is 65: all rows, columns and the two main diagonals add up to 65. But if you join the top and the bottom edges, the diagonals also add up to 65. For example, $8+14+20+21+2=65$, and $11+17+23+4+10=65$, and so on ... .

In the recreational mathematics section I also found the *Goat Problem*, which goes like this: Let a circular field of unit radius be fenced in, and tie a goat in its interior to a point on the fence with a chain of length *r*. What length of chain must be used in order to allow the goat to graze exactly one half the area of the field? If you cannot work out the answer to this problem, you will find solution on this site.

There is also a collection of illusions, some very well known, such as the Necker cube, the goblet illusion and the tribar, and impossible figures such as the Penrose stairway. If you'd like to amuse your friends, there is a long list of *printer errors*, that is, typesetting "errors" in which exponents or multiplication signs are omitted but the resulting expression is equivalent to the original one. For example,

$$31^2325 = 312325$$

or

$$2^5\frac{25}{31} = 25\frac{25}{31}$$

where a whole number followed by a fraction is interpreted as a mixed fraction.

I could go on and on. *MathWorld* is a magic world – once you start browsing you cannot stop; the wealth of resources is amazing.

I invite you to discover this world by yourself, and as the author says in his letter to the web site visitors, to "plunge in, enjoy, learn, and help share and spread the wonder of mathematics".

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ANOTHER LETTER TO THE EDITOR

Devilish Business

The cover diagram for October 2001 is described as a picture of an electric machine, and indeed it looks mighty like one. However, this same picture (almost) is to be found at the website

http://www-groups.dcs.st-and.ac.uk/~history/Curves/Devils.html

as one of the (many) Famous Curves listed by the group at St Andrews University. But there it is called the "Devil's Curve". They give the curve as having the equation

$$y^4 - x^4 + ay^2 + bx^2 = 0\,.$$

The case they show is $a = -24$, $b = 25$, whereas yours was $a = -96$, $b = 100$. This merely scales the diagram by a factor of 2 in each direction.

The website gives the information that this curve was first studied by Cramer (after whom Cramer's rule of determinants is named), but does not say why the curve is named after the Devil!

Bernard Anderson
Portland College

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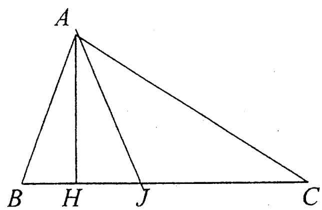
PROBLEMS AND SOLUTIONS

We begin by revisiting some of the previous problems and solutions.

More on Problem 25.2.3

This problem was submitted by Claudio Arconcher and concerned the construction of a triangle from various pieces of data. We published a solution by Julius Guest, who solved the problem without using one of the givens. In an editorial note, we stated that in certain cases, it would not be possible to solve the problem at all.

We have had further correspondence from the proposer and also from Keith Anker. Both suggest that the wording of the problem outlaws the cases of impossibility raised in the note. The proposer sent in a solution that did not satisfy the editors, and Anker sent in a critique of Guest's solution pointing to serious flaws in that. Part of the difficulty lies in the interpretation of the problem. It is clear that the proposer did not intend the problem to be as difficult as it turned out to be.



The diagram above shows some of the problem's features. The challenge is to construct the triangle *ABC* given three points *H*, *J* and *M* (this last not shown). As was said in *Function*, *Vol* **25**, *Part* **4**, the diagram represents the *result* of the construction and so serves to fix ideas. But we are not allowed to use it except to help us understand the data. We need to interpret the problem. It was taken by the proposer that his wording implied that the given points were necessarily collinear, i.e.

H, J and M all lie on a straight line. We may assume without loss of generality that H lies to the left of J.

Our published solution ignored one piece of data, but took the wording to imply another point to be given. The same can be said of most of the other solutions sent in. However, on the interpretation now adopted, we are to use the three points H, J and M *and nothing else* to construct the triangle ABC in such a way that $\angle ABC$ is a right angle, that J is the mid-point of BC, that AH is perpendicular to BC and that AM bisects $\angle ABC$.

A full analysis will appear in our next issue.

More on Problem 25.3.3

A solution by Marcello de Souza reached us too late for acknowledgement in the previous issue.

We continue by giving the solutions to the problems posed in *Volume* 25, *Part* 4 (August 2001).

Solution to Problem 25.4.1

Problem 25.4.1 read: A man dies and leaves behind one son and N daughters to share his estate. This is to be divided up according to the rabbinical law of inheritance described in the History of Mathematics section. Each of the daughters receives a fraction d of the total, while the residue s goes to the son. Can it happen that $d > s$, and if so, in what circumstances?

Keith Anker sent us a solution to this problem, which can be attacked fairly easily from the formulae given in the article. Here is a solution, essentially following Anker's argument.

We have

$$d = \frac{1}{N}\left(1 - \left(\frac{9}{10}\right)^N\right) \text{ and } s = \left(\frac{9}{10}\right)^N .$$

The condition that $d > s$ may then be reduced to the condition that $(N+1)\left(\frac{9}{10}\right)^N < 1$. Call the left-hand side of this expression $f(N)$. Then the following properties may readily be proved: $f(1) = 1.8$, $f(N)$ continues to increase from this point until $f(8) = f(9) = 3.874204....$, thereafter $f(N)$ decreases and as N tends to infinity, $f(N)$ tends to zero. Thus we should find a unique point at which $f(N)$ is first less then 1. Calculation shows that this point is reached when $N = 34$. Thus if the man had 34 or more daughters and one son, then the daughters would each receive more than the son.

(It should perhaps be said that none of them would get very much in this case; the son would receive 0.0278 of the property and each daughter 0.0286 in the case of $N = 34$, and these numbers would be even smaller if N were larger!)

Solution to Problem 25.4.2

Problem 25.4.2 (submitted by Julius Guest) read: In one of his letters to his friend and fellow mathematician Mersenne, Fermat stated the following theorem: "No prime number of the form $10s + 1$ (where s is a positive integer) is a divisor of any number of the form $5^n + 1$ (where n is a positive integer)". Was Fermat right?

The problem was solved by Keith Anker, J C Barton, Carlos Victor and the proposer. All produced the same answer, which is "No!", and did so by means of the same counterexample.

For consider the case $n = 5$. Then $5^n + 1 = 3126 = 6 \times 521$, and 521 is a prime of the form $10s + 1$. The next value of n also produces a counterexample also, but $n = 7$ does not. It is interesting that Fermat should have stated a result that is so easily refuted. He was normally more careful than this!

Solution to Problem 25.4.3

Problem 25.4.3 (from *Math-Jeunes*) read: A number, written in decimal notation, consists of 2001 6's followed by a 7. What is its square?

Solutions were received from Keith Anker, J C Barton, Julius Guest, Marcel de Souza and Carlos Victor. All were essentially the same. We here follow Barton's account.

Begin by noting that $67^2 = 4489$ and $667^2 = 444889$. Now consider the number s whose decimal expansion comprises a string of $m-1$ 6's followed by a 7. We now calculate s^2. The two simplest cases suggest that the decimal expansion of this number will consist of a string consisting of m 4's followed by $m-1$ 8's and finishing with a 9. This statement is true; the proof is by mathematical induction.

We assume that the result holds for some particular value of m. This is with a view to demonstrating that in this case, it will also hold for the next value of m also. To this end consider the number t comprising, in its decimal expansion, a string of m 6's followed by a 7. Then

$$t = 10s - 3\text{, and so } t^2 = (10s-3)^2 = 100s^2 + 9 - 60s.$$

This last expression may be expressed in terms of the decimal expansion of s^2. The result is found by noting that $100s^2 + 9$ will have a decimal expansion comprising m 4's followed by $m-1$ 8's, then a 0 and at last a 9. The number $60s$ will have a decimal expansion comprising a 4 followed by $m-1$ 0's after which comes a 2 and then a further zero. Imagine subtracting these two numbers:

$$\begin{array}{r} 44444....4888....8909 \\ -\ 400....0020 \\ \hline 44444....4488....8889 \end{array}$$

The difference will comprise $m+1$ 4's, followed by m 8's and finally a 9. Thus the pattern persists and if the result holds for some value of m, then it will also hold for the next, and then the next, and so on. As it holds for m = 2, it therefore holds for all subsequent m, including the case m = 2002. The required answer is thus a number whose decimal expansion consists of 2002 4's, followed by 2001 8's and finally a 9.

Guest indicated a somewhat different method of argument, based on the sum of a geometric progression. Anker and de Souza produced the same argument but each in a rather different form.

Solution to Problem 25.4.4

The problem set (by J A Deakin) was to find the primitive (indefinite integral) of the function $\dfrac{x+1}{x\left(1+xe^{x}\right)}$.

Solutions were received from Keith Anker, J C Barton, Julius Guest, Carlos Victor and the proposer.

Guest considered the substitution $t = 1 + xe^{x}$, from which we find $dt = (x+1)e^{x}dx$. Then we easily reach:

$$\int \frac{(x+1)dx}{x\left(1+xe^{x}\right)} = \int \frac{dt}{t(t+1)} = \int\left(\frac{dt}{t-1} - \frac{dt}{t}\right) = \ln\left|\frac{t-1}{t}\right| + c = \ln\left|\frac{xe^{x}}{1+xe^{x}}\right| + c,$$

where c is an arbitrary constant.

Barton reached an equivalent form by a slightly different argument. He noted that "it is worth remarking that there can be no systematic method of finding this integral". Both the above and the variant he used in his own account rely on *ad hoc* substitutions that happen to work. In his Cambridge Tract on Integration, G H Hardy noted that "such integrals as $\int f\left(x, e^{x}\right)dx$ .... are generally new transcendentals", i.e. not evaluable by elementary means.

Anker and Victor used the substitution $u = xe^{x}$ to reach the same solution, but Anker reached a form that incorporated the arbitrary constant into the argument of the logarithm.

Now for some new problems.

Problem 26.1.1 (From I Todhunter's *Algebra*, a once much-used school textbook)

For $abc \neq 0$, prove that

$$\left(\frac{b}{c}+\frac{c}{b}\right)^{2} + \left(\frac{c}{a}+\frac{a}{c}\right)^{2} + \left(\frac{a}{b}+\frac{b}{a}\right)^{2} = 4 + \left(\frac{b}{a}+\frac{a}{b}\right)\left(\frac{a}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{c}\right)$$

[The copy of Todhunter's *Algebra* in the Mathematics library at the University of Melbourne once belonged to the late Professor Sir Thomas Cherry, and was presented to them by Lady Cherry on her husband's death. From some scribbles on the first few pages, it would appear that it was one of the young Thomas's own school texts. This particular problem is one of three marked with a pencilled cross; it is not clear what this signified.]

Problem 26.1.2 (Translated from *Volume* 1 of Fibonacci's *Scritti*)

Six men each have some coins; leaving out the first man's share, there are 75 coins; leaving out the second man's share, there are 70; leaving out the third man's share, there are 67; leaving out the fourth man's share, there are 64; leaving out the fifth man's share, there are 54; leaving out the sixth man's share, there are 50. How many coins does each man have?

Problem 26.1.3 (Submitted by Julius Guest)

Ada and Bert each carpeted their living-rooms with the same type of carpet. Ada's living-room is 50 cm longer than it is wide; Bert's is 10 cm longer than Ada's, but also 10 cm narrower. Ada ended up paying \$2.40 more than Bert. What is the price of carpet per square metre?

Problem 26.1.4 (Submitted by Julius Guest)

Anderson, Brown, Chester, Driver and Eagle all live in the same street; three are teachers and two are engineers. A detective trying to determine Chester's profession is told:

(1) Neither Anderson nor Brown is an engineer,
(2) Neither Driver nor Eagle is an engineer,
(3) Both Driver and Chester are engineers.

However, all these pieces of information turn out to be false. What is Chester's profession?

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