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Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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* \$13 for *bona fide* secondary or tertiary students.

EDITORIAL

In this issue of *Function* you will find quite an interesting collection of articles and contributions.

The front cover shows a special magic square in which all rows, columns and diagonals add up to 264. It also has the rather amazing property that if you turn it upside down, you will see another magic square with rows, columns and diagonals also adding up to 264—a so called a magic square ambigram.

There is much to read about biased coins and loaded dice in this issue. J Kupka gives us an original way of making a coin fair without actually changing the physical appearance of it, but by using a mathematical procedure. On the other hand, the History of Mathematics column looks at the use of statistical methods to decide whether a coin is biased, and to determine the probability of getting a tail when tossing the coin. Both articles make very interesting reading, regardless of your need to apply their results when making decisions by flipping a coin.

Have you ever wondered what is your surface area? Although it is not as straight forward as measuring your weight or your height, there are readily available formulae that could give you a reasonable estimate. Michael Deakin looks at three formulae, which only need your weight and your height, and analyses their differences and similarities. He also gives the web site where you can calculate your surface area, just in case you needed to know it.

We include in this issue quite a few Olympiad problems, corresponding to the 2000 Australian Mathematical Olympiad and the Twelfth Asian Pacific Mathematics Olympiad. With these problems, and the ones given in the problem section, you should have more than enough to keep your mind busy until the next issue.

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ON BIASED COINS AND LOADED DICE

Joseph Kupka

Whenever I toss a coin I always think that heads come up more often than tails. Maybe I am fooling myself. Maybe there *are* more heads but it's just happenstance. Or maybe the slightly greater roundedness on the head side of most coins does make them a tiny bit less likely to land with that side down.

Many gambling games are based on the toss of a fair coin or the roll of a fair die. Their probabilistic outcomes, upon which sums of money are often wagered, depend critically on the fairness of the coin or die. Imagine that you are playing such a game and are consistently losing. You are starting to feel annoyed. Is it just back luck, or could the coin be biased or the dice loaded? If you become upset enough or suspicious enough that you are willing to go to a bit of extra trouble, here is a little device you can use to effectively convert a biased coin into a fair coin or even into a fair die.

Any old coin will do, but you have to believe the following:

(i) There is never changing probability p, $0 < p < 1$, that the coin will land heads (H). Let $q = 1 - p$ be the probability that it will land tails (T).

(ii) The results of different tosses of the coin are completely independent of one another.

These are perhaps the most universally believable and believed assumptions in all of probability theory.

If you prefer the act of rolling a die to flipping a coin, you can associate H and T with outcomes of the die, say evens (2, 4 or 6) to be counted as H and odds (1, 3 or 5) as T.

Making a Fair Coin

O.K. So toss your any-old-coin once. Say it comes up *H*. Call this an "unofficial" head. To make it official, it must be "validated" with a second toss of the coin. If the second toss gives *T*, then the first *H* is validated and becomes an "official" head ***H***. If the second toss gives *H*, then the first *H* is invalidated and you have to start over.

Likewise a *T* on the first toss is validated by *H* on the second and then becomes an "official" tail ***T***. It is invalidated by a *T* on the second toss.

Here is an example of how an *H-T* sequence from our original coin would be converted into an ***H-T*** sequence.

HHTH	*HT*	*HHTTHHTTTTTH*	*TH*	*TTHT*
T	***H***	***T***	***T***	***H***

Upon occasion a large number of tosses is required to get ***H*** or ***T*** to occur. But this is unlikely if your coin is nearly fair to begin with. It becomes more likely if the coin is wildly biased, say if $p = .95$, but then the bias of the coin is clear and you can either accept the extra trouble of this technique or demand a fairer coin!

Let's verify that the probability $P(\boldsymbol{H})$ of an "official" head is exactly 1/2. Likewise the probability $P(\boldsymbol{T})$ of an "official" tail is also 1/2. Thus our ***H-T*** sequence is probabilistically identical to an *H-T* sequence coming from a genuine fair coin.

If ***H*** is validated on the first two tosses, they must give *HT*, which has probability pq. If ***H*** is validated on later tosses, the first two tosses must give *HH* or *TT*, and this has probability $p^2 + q^2$. Tosses 3, 4, 5, ... are independent of tosses 1 and 2, and the probability that they will give ***H*** is just $P(\boldsymbol{H})$ again. (After failing to validate ***H*** on the first two tosses, we are in effect starting over from scratch.) The *additivity* of probability implies that the probability of ***H*** is the probability of ***H*** validated on the first two tosses *plus* the probability of ***H*** validated on later tosses. This gives the equation

$$P(\boldsymbol{H}) = pq + (p^2 + q^2)\,P(\boldsymbol{H}).$$

The solution of the equation is

$$P(\boldsymbol{H}) = \frac{pq}{1-p^2-q^2}.$$

The denominator can be simplified by observing that

$$1 = (p+q)^2 = p^2+q^2+2pq,$$

so

$$P(\boldsymbol{H}) = pq/2pq = 1/2.$$

A very similar argument also shows that $P(\boldsymbol{T}) = 1/2$.

Making a Fair Die

This is a little more complicated.

Let's call what we did in the last section the "1/2" procedure. It produced one of two mutually exclusive events $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ (otherwise known as $\boldsymbol{H}$ and $\boldsymbol{T}$), each with probability 1/2.

We now describe a "1/3" procedure, which produces one of three mutually exclusive events $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$, each with probability 1/3.

This time we subdivide the tosses of the any-old-coin into groups of three (instead of groups of two). If the first three tosses produce *HHH* or *TTT*, the result is "invalid" and we must start again. Otherwise there will either be two *H*'s and a *T* (in which case *T* is the odd one out) or two *T*'s and an *H* (in which case *H* is the odd one out). The *position* of the odd one out determines the event: $\boldsymbol{E}_1$ if the odd one out appears on the first of the three tosses, $\boldsymbol{E}_2$ if on the second, and $\boldsymbol{E}_3$ if on the third. Here is an example:

HTH	*HHHTTTTTH*	*HTT*	*TTTTHH*
$\boldsymbol{E}_2$	$\boldsymbol{E}_3$	$\boldsymbol{E}_1$	$\boldsymbol{E}_1$

We determine $P(\boldsymbol{E}_1)$ in the same general way as $P(\boldsymbol{H})$ earlier. If $\boldsymbol{E}_1$ is determined by the first three tosses, then they must produce either *THH* or *HTT*, and the probability of this is $p^2q + pq^2$. Otherwise the first three tosses are invalid (with probability $p^3 + q^3$) and we start over. Hence

$$P(E_1) = p^2q + pq^2 + (p^3 + q^3)P(E_1).$$

The solution is

$$p(E_1) = \frac{p^2q + pq^2}{1 - p^3 - q^3}.$$

But $1 = (p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$, and so $P(E_1) = 1/3$. Likewise $P(E_2) = P(E_3) = 1/3$.

We make the equivalent of a fair die by combining the "1/2" and "1/3" procedures. Do the "1/2" procedure to produce, say, $\boldsymbol{H}$. Then, in a separate sequence of tosses, do the "1/3" procedure to produce, say, $\boldsymbol{E}_2$. Because the tosses are separate, the events $\boldsymbol{H}$ and $\boldsymbol{E}_2$ are independent, so the probability that they occur together is the product of their individual probabilities. Thus,

$$P(\boldsymbol{H} \cap \boldsymbol{E}_2) = P(\boldsymbol{H})\, P(\boldsymbol{E}_2) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

In this way we get six mutually exclusive events

$$\boldsymbol{H} \cap \boldsymbol{E}_1,\ \boldsymbol{H} \cap \boldsymbol{E}_2,\ \boldsymbol{H} \cap \boldsymbol{E}_3,$$

$$\boldsymbol{T} \cap \boldsymbol{E}_1,\ \boldsymbol{T} \cap \boldsymbol{E}_2,\ \boldsymbol{T} \cap \boldsymbol{E}_3,$$

each with probability 1/6. After identifying these in any preagreed way with the six faces of a die we see that this combined procedure is probabilistically just the same as rolling a fair die once.

For any $n = 3, 4, 5, 6 \ldots$, one could speak of a "$1/n$" procedure: divide the tosses of the coin into groups of n; anything other than $n - 1$ H's and a T or n $- 1$ T's and an H is "invalid" and we start over; with a valid sequence, the position of the odd one out determines one of n mutually exclusive events, each with probability $1/n$.

In particular, we could make a fair die by doing the "1/6" procedure. But this is not a good idea!

The Question of Efficiency

It can get tiresome tossing a coin a large number of times just to determine a single "fair" event. The average number of tosses required for the "1/2" procedure can be shown to be

$$\mu_2 = \frac{1}{pq}.$$

For $n \geq 3$, the average for the "1/n" procedure is

$$\mu_n = \frac{1}{p^{n-1}q + pq^{n-1}}.$$

Our combined procedure for making a fair die therefore averages $\mu_2 + \mu_3$ tosses. When the coin is fair ($p = 1/2$), this amounts to $4 + 4 = 8$ tosses, as contrasted with $\mu_6 = 32$ tosses. As we move away from $p = 1/2$ to a progressively more biased coin, $\mu_2 + \mu_3$ gets worse (i.e. larger), but μ_6 improves for a time. If, by the merest chance, $p = .83223$, then $\mu_6 = 14.90569$, and this is as good as it gets for the "1/6" procedure. However, we still have $\mu_2 + \mu_3 < \mu_6$ for this p-value, and in fact whenever $.159375 < p < .840625$. Only if the coin is more biased than this does the "1/6" procedure become less slow, on average, than the combined procedure.

Biased Coins in Two-up

In the traditional outback game of two-up, two coins are tossed at the same time, and bets are placed on whether the outcome will be "odds" (head and tail) or "evens" (two heads or two tails). Does the use of two coins instead of one somehow make the game fairer? In a certain sense, the answer is yes.

Let x be the probability of heads for one of the coins and y the probability of heads for the other. Then

$$P_E = P(\text{"evens"}) = xy + (1-x)(1-y) \quad \text{and}$$
$$P_0 = P(\text{"odds"}) = x(1-y) + (1-x)y.$$

It is not hard to work out the (x, y)-values for which $p_E > p_0$, or $p_E = p_0$, or $p_E < p_0$, if you treat the cases $x < 1/2$, $x = 1/2$, $x > 1/2$ separately. The answer gives two facts about the game which might not be apparent to the unscientific observer.

First, the game is fair $(p_E = p_0 = 1/2)$ if and only if at least one (but not necessarily both!) of the coins is fair. Therefore, if you are supplying one of the coins for the game and your opponent the other, and you know your coin is fair, then the game is fair, no matter how your opponent might try to doctor his own coin to gain an unfair advantage. (His coin could even be two-headed!)

Second, we have $p_E > p_0$ if and only if both coins are biased in the same direction (that is, x and y are both $> 1/2$ or are both $< 1/2$). Therefore, anyone with a prejudice that all coins are slightly biased in favour of heads should always bet on "evens".

* * * * *

For those who need some extra cash ...

> If someone tells you that the number 13,717,421 can be written as the product of two smaller numbers, you might not know whether to believe him, but if he tells you that it can be factored as 3607 times 3803 then you can easily check that it is true using a hand calculator. The problem of deciding whether an answer that can be quickly checked with insider knowledge, may without such help require much longer to solve, no matter how clever a program we write. This is considered one of the outstanding problems in logic and computer science. [This problem] was formulated by Stephen Cook in 1971.

This is one of the Millennium Prize Problems announced on May 24, 2000 by the Clay Mathematics Institute. All problems are published on www.claymath.org. There is a pot with $12 million to reward those who solve the problems. It is worth a try ...

WHAT'S YOUR SURFACE AREA?

Michael A B Deakin

I recently learned that for certain medical uses, an estimate of a person's surface area is required. One such use is the determination of the dosage of "antineoplasic drugs", which are chemotherapeutic agents used in the fight against cancer. However, studies of heat exchange and its effect on things like heart-rate and respiration also require this information. It becomes important too in the treatment of patients who have suffered major burns.

But a moment's thought will convince you that the direct measurement of a person's surface area is quite a difficult thing to attempt.

It can be done, of course. One way is to draw small triangles and rectangles all over the person's skin, carefully measure each of them and add up all their areas. Another is to paint the person all over with an even layer of paint and to compare their weight before and after this is done, or else to measure by some other means how much paint was applied. These are both quite difficult exercises to perform (as well as being somewhat embarrassing to undergo), and this has led researchers to seek other less time-consuming and less intrusive ways to discover this information.

The formula I learned about was one called "Dubois formula" and it went like this:

$$\sigma = 0.007184\, h^{0.725} w^{0.425} ,$$

where σ is the required surface area in m^2
h is the person's height in cm,
w is the person's weight in kilograms.

The second of these measurements is not in SI units, but the centimetre is widely used in measuring people's heights. If the height were to be measured in metres, then this could be allowed for by putting the constant at the front equal to 0.2025 instead of 0.007184. Can you see why?

To get some further information about this formula, I turned to the internet and found an interesting site at

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http://perso.club-internet.fr/alaffont/compute/bsa/bsa.htm
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This gives three formulas for computing σ, which is called there bsa (short for "body surface area"). For each formula it allows you to compute your own surface area by entering your own height and weight. We get the interesting information that Dubois' formula was proposed by D and E F Dubois in 1916, and was based on a sample of 10 people: 2 children and 8 adults. The heights of the experimental subjects varied between 73 and 184 cm and the weights between 6 and 93 kg.

The other two formulas are called "Gehan's formula" and "Haycock's formula". Gehan's formula was proposed in 1970 by E A Gehan and S L Georges, and was based on the measurement of 401 different people across a very wide range of sizes. The individual measurements they took were never published, but Gehan and Georges claimed that their formula was accurate for heights between 50 and 220 cm and weights between 4 and 132 kg. With the same symbols as listed above, their formula is

$$\sigma = 0.02350 h^{0.42246} w^{0.51456} .$$

Haycock's formula was based on the measurement of 81 people (again across a wide range of sizes) and it was the subject of a medical report by G B Haycock and several others issued in 1978. Again, the original data were not disclosed, but a range of accuracy was given which had heights between 30 and 200 cm, with weights between 1 and 120 kg. Again with the same symbols, the formula reads

$$\sigma = 0.024265 h^{0.3964} w^{0.5378} .$$

The three formulas look very different, but this appearance is somewhat deceptive. Take first the case of the very smallest person, with a height of 30 cm and a weight of 1 kg (obviously a very premature baby). Haycock's formula (the only one that claims to apply to such small people) gives a bsa of $0.093\ \mathrm{m}^2$, while Gehan's formula gives $0.085\ \mathrm{m}^2$. (Both these formulas are being used outside their claimed domain of validity.)

At the other end of the scale, we have the very largest person, with a height of 220 cm and a weight of 132 kg. This is clearly a very large person, rough the size of the basketball player Luke Longley. Only Gehan's formula claims to work in such an extreme case, but all three formulas actually agree very well. Gehan's formula gives $2.83\ \mathrm{m}^2$, Haycock's $2.84\ \mathrm{m}^2$, and Dubois' $2.86\ \mathrm{m}^2$.

This is probably as good an agreement as we have a right to expect. This is a matter I will return to later.

Next I fed in my own data and got these figures: $1.90\ \mathrm{m}^2$ (Gehan), $1.89\ \mathrm{m}^2$ (Haycock), and $1.86\ \mathrm{m}^2$ (Dubois). Again the agreement is probably as good as we can expect. Then I supposed that I had suddenly put on a lot of weight (25 kg of it!) and recalculated. Now I got these figures: $2.20\ \mathrm{m}^2$ (Gehan), $2.21\ \mathrm{m}^2$ (Haycock), and $2.11\ \mathrm{m}^2$ (Dubois). This last figure is a bit of an outlier, but Dubois' formula is outside its range of validity here.

All three of these formulas have some doubtful aspects. Usually weight is given only to two decimal places and height to at most three. (Because the first, the hundreds, digit is either 0, 1 or at most 2, the accuracy is somewhat less than 3-place). Those four and five place constants that occur in all the formulas are thus rather overexact. (It is far from clear to me from the descriptions on the web quite how accurately the various groups of researchers claimed to be able to measure σ. I have not had access to any of the original reports: they are none of them particularly easy to come by.)

To investigate this point of overexactness, I modified Haycock's formula to read:

$$\sigma = 0.025h^{0.4}w^{0.55} ,$$

using 2 significant figures to round numbers near the numbers actually given. For the premature baby, I got $0.097\ \mathrm{m}^2$ (compare $0.093\ \mathrm{m}^2$); for the "basketballer", I got $2.90\ \mathrm{m}^2$ (compare $2.84\ \mathrm{m}^2$); for myself, I got $2.10\ \mathrm{m}^2$ (compare the earlier $1.89\ \mathrm{m}^2$); for the obese version of myself, I got $2.46\ \mathrm{m}^2$ (compare the earlier $2.21\ \mathrm{m}^2$). Even though the approximation I made is quite drastic, the maximum error it causes is just over 11%.

The other problem is that only one of the formulas can be given a good theoretical basis. This is Dubois', the earliest of them, but still the most widely used. The only real criticism that has ever been validly levelled at this formula arises from the fact that its foundation rests on that very small database from a mere ten individuals.

However it has a property that the other do not, and this gives it a certain pre-eminence. To see this, notice that all the formulas have the form

$$\sigma = kh^{\alpha} w^{\beta} ,$$

where k, α, β are constants, whose values we want to determine.

To do this we measure σ, h, w for a number of people (the more the better). We then write $S = \log \sigma$, $K = \log k$, $H = \log h$, $W = \log w$, where we may take the logarithms to any base (just as long as it's the same for all of them). We next write the formula as:

$$S = k + \alpha H + \beta W.$$

and using standard statistical techniques, estimate K (and thus k), α, β. (A somewhat similar problem was discussed in *Function, Vol* 13, *Part* 2 p 44). This is undoubtedly what was done in finding Gehan's formula and Haycock's formula.

However, there is a theoretical consideration that this approach ignores. It is this. Suppose that we had two people of exactly the same shape, but with one of them larger than the other. The second person would have every length of his or her body multiplied by a fixed factor L (say) compared with the first person. So if the first person has a height h, the second will have a height Lh. But now, the first person's bsa σ will correspond to a bsa $L^2\sigma$ for the second person. This is because the bsa is measured in units of length-squared. Similarly, the volume of the first person (and his weight, w) will be multiplied by a factor L^3, because volume is measured in units of length-cubed. (Volume can be measured directly by immersing the person in water, but it is less traumatic and much easier to deduce it from the weight, or else (as here) simpler to use the weight directly.)

Thus if the first person's bsa is given by $\sigma = kh^{\alpha} w^{\beta}$, then the second person's will be given by $L^2\sigma = k(Lh)^{\alpha} (L^3w)^{\beta}$. So, if both of these equations are to hold, we have to have:

$$\alpha + 3\beta = 2.$$

This relation holds exactly for Dubois' formula, but only approximately for the other two.

Haycock's formula has 2.0098 on the right instead of 2, and had Haycock and his collaborators slightly modified their statistical technique to force the sum to be exactly 2, then in view of the remarks above, it probably would not have made much difference to the effectiveness of their formula.

Gehan's formula is a little more problematical. Instead of 2, we find 1.96614, and although this is still close to 2, it is not as nearly equal to 2 as is 2.0098. It is a pity that neither of the sets of researchers made their original data available. If we had it, it would be possible to check whether the forcing of the exact equation had a significant effect on the accuracy of the estimation of σ.

Finally I should mention that when less accurate formulae are enough, then there are simpler ways to estimate σ. A medical friend has a rough guide, which boils down to

$$\sigma = 5gh/4.$$

where g is the girth and h the height, but this time both in metres. I wasn't able to check the results for three of the four cases considered above: the girth data for the premature baby and the "basketballer" weren't available, and the "obese me" doesn't exist! However, for the "real me", this rough formula gives $\sigma = 1.9m^2$, which is remarkably close, when compared with the figures given by the more elaborate formulas. However, this test is inconclusive as it is based on a single example.

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LETTERS TO THE EDITOR

On the problem discussed by Dr. Fwls in the Letter to Editor, *Function Vol* 24, *Part* 2

Dear Editor

The sextic with the 6 real and different solutions was tampered with in the coefficient of the 5th degree term, 21, by the addition of an increment $n = 0.015$, where n is a parameter that can be varied across a range to study the changing morphology of the curve.

Because of Dr Fwls choosing $n = 0.015$, the turning point, from which the arcs cross the x axis at $x = 4$ and $x = 5$, has been brought down below the x axis to such an extent that it has been absorbed by the running arclength from the previous turning point, resulting in only four different real solutions and one conjugate pair.

The graphs below depict this transition. Let n_1 be the value of n for which the sextic has a double root x_*.

$$F(x_*) = x_*^6 - (21+n)\,x_*^5 + 175x_*^4 - 735x_*^3 + 1624x_*^2 - 1764x_* + 720 = 0 \qquad (1)$$

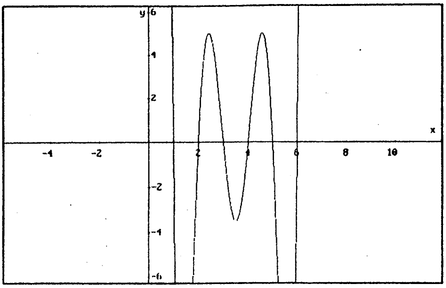
$$F'(x_5) = 6x_*^5 - 5(21+n)x_*^4 + 700x_*^3 - 2205x_*^2 + 3248x_* - 1764 = 0 \qquad (2)$$

Eliminating n leaves us with:

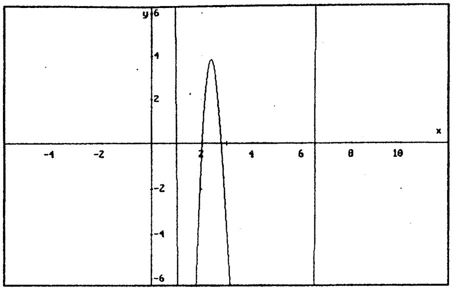
$$x_*^6 - 175x_*^4 + 1470x_*^3 - 4872x_*^2 + 7056x_* - 3600 = 0. \qquad (3)$$

Here x_* is the particular solution of equation (1), that is the repeated root, and it is therefore also a solution of equation (2) since it is a turning point. The solution for $n = n_1$ is found by substituting x_* into equation (3). Note that equation (3) will produce 5 values and it is only the one in the range between 4 and 5 that is the correct one.

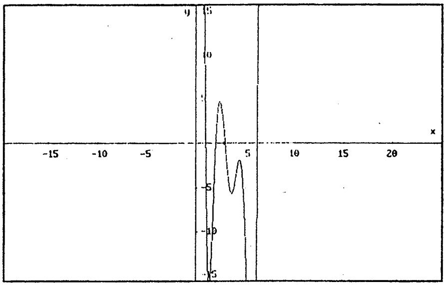
This brought out that unique case, where for a singular value of $n = n_1 = 0.0026945$ the turning point is tangential to the x axis resulting in two of the six real solutions being identical. This occurs at $x = 4.45253$. This is the transition value, where, for $n > n_1$, there are only 4 real different solutions to the sextic and for $0 < n < n_1$, there are 6 real different solutions.



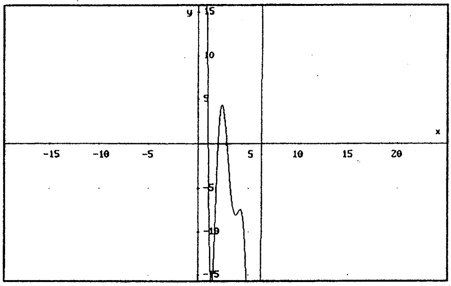
$n = 0$



$n = 0.015$



$n = n_2 > n_1$



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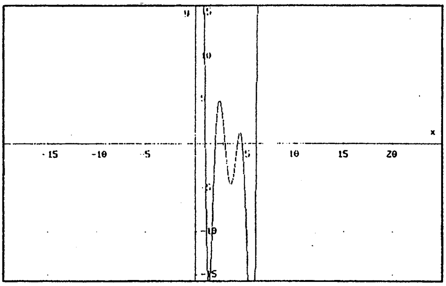
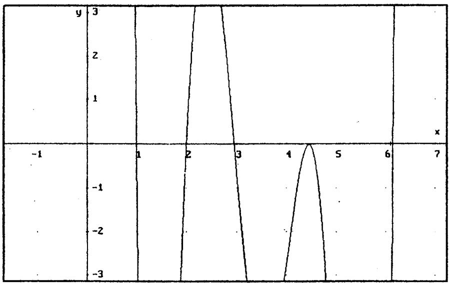


$n < n_1$



$n = n_1$

Yours sincerely

David Halprin

Dear Editor

It is always a pleasure to read Dr Fwl's new adventures in algebra. This time he didn't attack algebra, thank goodness! Just the same he gave the reader (and me) plenty to think about.

As Dr Fwls' question was very short I shall reply in kind:

The six roots of the sextic

$$f(x) = x^6 - 21.015x^5 + 175x^4 - 735x^3 + 1624x^2 - 1764x + 720 \tag{1}$$

are

$$\begin{aligned} x_1 &= 0.999876, & (2)\\ x_2 &= 2.021616, & (3)\\ x_3 &= 2.790832, & (4)\\ x_4 &= 6.523542, & (5)\\ x_5 &= 4.339567 + i\,0.841209, & (6)\\ x_6 &= 4.339567 - i\,0.841209, & (7) \end{aligned}$$

where all numbers are correct to 6 decimal places.

How did I get that? Well, this may take a little longer.

First of all Kim Dean gave us two good hints. He presented us with a sextic which was almost identical with Fwls' equation except that its coefficient in x^5 differed very slightly from his. So we naturally expected similar answers. But Kim's sextic has six real roots—namely 1, 2, 3, 4, 5 and 6—while the roots of the doctor's sextic are nothing like it. Thus the doctor's sextic is very ill-conditioned indeed! Kim's second hint was implied: the real roots of our sextic will probably lie in (0, 10).

As a first move let us find where we should look for these roots. We note that for any negative $x, f(x)$ is invariably positive. Hence a lower bound for our real roots should be $x = 0$, in other words all our real roots must be positive. For an upper bound we need to use (1) together with its first five derivatives:

$$f'(x) = 6x^5 - 105.075x^4 + 700x^3 - 2205x^2 + 3248x - 1764 \tag{8}$$

$$f''(x) = 30x^4 - 420.3x^3 + 2100x^2 - 4410x + 3248 \tag{9}$$

$$f'''(x) = 120x^3 - 1260.9x^2 + 4200x - 4410 \tag{10}$$

$$f^{iv}(x) = 360x^2 - 2521.8x + 4200 \tag{11}$$

$$f^{v}(x) = 720x - 2521.8 \tag{12}$$

Now for an upper bound we need all of

$f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$, $f^{iv}(x)$, $f^{v}(x)$ to be positive.

For $x = 6$ we obtain

$$f(6) < 0,\ f'(6) > 0,\ f''(6) > 0,\ f'''(6) > 0,\ f^{iv}(6) > 0,\ f^{v}(6) > 0 \tag{13}$$

so we are almost there. And for $x = 7$, $f(7)$ together with all its first five derivatives are indeed positive. So our naughty sextic has all its real roots in the interval $(0, 7)$.

Next, what method should we use? Lobachevsky's root-squaring method springs to mind, but after some reflection I settled on a computer-assisted Horner method. Horner's method is essentially one where we improve a selected root by one decimal place for each run and we look for a sign change. I wrote a little Qbasic program and let my PC do the donkey work for me. Here it is:

```
REM  Solving K. Dean's ill-conditioned sextic via
Horner's method
REM  The doctor's latest problem.
SCREEN 9:  COLOR 14,  1:  CLS: LOCATE  10,  1
FOR X  =  2  TO  3.  1  STEP  .1
 F  =  X  ^  6  -  21.015 *  X  ^ 5 +  175 *  X ^ 4
 F  =  F  -735 * X ^ 3  +1624 * X ^ 2 - 1764 * X + 720
 PRINT  TAB(15);  X;  F
NEXT  X
END
```

Let me now illustrate how I use this program.

When we run the above program we note that there are two sign changes in the interval $(2,3)$. One in the interval $(2,2.1)$ and another one in $(2.7,2.8)$. Suppose we wish to improve the root in (2.7,2.8). We then modify the present `FOR TO STEP` loop line as

```
FOR X = 2.7 TO 2.8 STEP.01
```

And leave everything else as before.

Running this modified program we detect there is a sign change in the interval (2.79, 2.80). So in our next run we need to change the last for-loop to

```
FOR X = 2.79 TO 2.80 STEP.001
```

Running this new program we detect a sign change in (2.790, 2.791). Carrying on this way we eventually find that this root is 2.790932 correct to 6 decimal places.

After we have run through (0,7) we realise that our sextic has only four real roots and the remaining two roots must be complex. We now form the quartic $g(x)$ which consists of the four real roots found before. We then obtain that

$$g(x) = x^4 - 12.335866x^3 + 48.395901x^2 - 73.913275x + 36.852001 = 0 \tag{14}$$

Lastly, we divide $f(x)$ by $g(x)$ (ignoring its small remainder) to arrive at:

$$x^2 - 8.679133\,x + 19.539470 = 0 \tag{15}$$

Solving (15) provides

$$x_5 = 4.339567 + i\,0.841209 \quad \text{and} \quad x_6 = 4.339567 - i\,0.841209 \tag{16}$$

As a check it is wise to add all six roots and obtain 21.015, but as this is precisely the coefficient of x^5 in $f(x)$ all is well.

Yours sincerely

Julius Guest

On the Sine Sum Identity

Dear Editor

A recent article by Lun on the Sine Sum Identity makes very interesting and absorbing reading. While Lun presented the elementary proof of the sine sum identity our readers might also be interested to look at a simple analytic proof for the same.

We start with a simple variant of De Moivre's identity

$$\cos x + i \sin x = e^{ix} \tag{1}$$

which holds for all real x. Taking the complex conjugate of (1) yields

$$\cos x - i \sin x = e^{-ix}, \tag{2}$$

and adding (1) and (2) provides

$$\cos x = \frac{(e^{ix} + e^{-ix})}{2}. \tag{3}$$

Subtracting (2) from (1) gives

$$\sin x = \frac{(e^{ix} - e^{-ix})}{2i} \tag{4}$$

Next, using (3) and (4) we find that

$$\sin x \ \cos y = \frac{(e^{ix} - e^{-ix})\,(e^{iy} + e^{-iy})}{4i} \tag{5}$$

Also

$$\cos x \sin y = \frac{(e^{ix} + e^{-ix})\,(e^{iy} - e^{-iy})}{4i} \tag{6}$$

Now, adding (5) and (6) gives

$$\sin x \cos y + \cos x \sin y = \frac{(e^{ix} - e^{-ix})(e^{iy} + e^{-iy}) + (e^{ix} + e^{-ix})(e^{iy} - e^{-iy})}{4i} \tag{7}$$

Expanding the right-hand side of (7) leads to

$$\frac{(e^{i(x+y)} - e^{-i(x-y)} + e^{i(x-y)} - e^{-i(x+y)}) + (e^{i(x+y)} + e^{-i(x-y)} - e^{i(x-y)} - e^{-i(x+y)})}{4i}$$

which in turn simplifies to

$$\frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} \tag{8}$$

but by using (4) we realise that (8) is equal to $\sin(x+y)$ so the sine sum identity is established for all real x and y.

Yours sincerely

Julius Guest

* * * * *

The biologist can push it back to the original protist, and the chemist can push it back to the crystal, but none of them touch the real question of why or how the thing began at all. The astronomer goes back untold millions of years and ends in gas and emptiness, and then the mathematician sweeps the whole cosmos into unreality and leaves one with mind as the only thing of which we have any immediate apprehension. Cogito ergo sum, ergo omnia esse videntur. All this bother, and we are no further than Descartes. Have you noticed that the astronomers and mathematicians are much the most cheerful people of the lot? I suppose that perpetually contemplating things on so vast a scale makes them feel either that it doesn't matter a hoot anyway, or that anything so large and elaborate must have some sense in it somewhere.

—Dorothy L Sayers in (with R Eustace) *The Documents in the Case*,
New York: Harper and Row, 1930, p 54.

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.

—L N Tolstoy
in H Eves *Return to Mathematical Circles*,
Boston: Prindle, Weber and Schmidt, 1989.

HISTORY OF MATHEMATICS

Why Statistics can be Difficult

Michael A B Deakin

How often have you seen a question in statistics that begins: "A fair coin is tossed ..." or "A fair die is rolled ..."? If you have been studying mathematics into the higher years of secondary school, then you will almost certainly have met such problems. They form the starting point of much probability theory, on which statistics is based. Yet they remind me of questions we used to see in elementary mechanics, which would start out something like this: "A rope passes over a frictionless pulley ... ". Now a mere moment's reflection suffices to tell us that we are most unlikely ever to encounter a frictionless pulley, but generations of students happily solved such problems. If they ever thought at all of the implausibility of the underlying hypothesis, it would have been explained away by means of the assumption that the frictional forces involved, while not exactly zero, were understood to be very small in relation to the other forces acting.

But what of our fair coin? Isn't it something like a frictionless pulley? How do we know that any such thing exists? Well, actually, we don't! However, long experience, the experience of many people, leads us to think that the toss of a coin produces one or the other of two outcomes ("heads" or "tails") with nearly equal frequency. Indeed this experience is so ingrained that we tend to assume automatically that all coins are fair in this sense.

However, this assumption is by no means always justified. Shortly after I arrived in Chicago as a graduate student in the mid'60s, I was shown a quite striking example of this. If you take a US 1c coin and spin it on a tabletop, it will almost always come to rest tails up. The precise proportion of "tails" in such an experiment depends somewhat on the surface of the tabletop and on the state of the coin, but a call of "tails" will succeed in about 95% of cases. Such a coin is said to be biased, and we tend to regard biased coins as unusual curiosities. But as the above example shows, they can occur in quite unexpected contexts.

Perhaps our natural prejudice comes about because a coin tends to look symmetrical, and symmetry would seem to require that the odds of "heads" and of "tails" are equal, thus ensuring that the coin is fair. When it comes to asymmetrical objects, like drawing pins for example, we are quite happy with the possibility that the two outcomes "point up" and "point down" might occur with different frequencies if the pins were tossed. [For an article on the tossing of drawing pins, see *Function, Vol* 3, *Part* 1. That article uses a very simple dynamical model, but one which works surprisingly well. A somewhat similar model could be applied to the US 1c coin, on which the image of Lincoln's head on one side greatly outweighs the picture of Monticello on the other.]

But now let's look at the matter from another point of view. The usual approach is to carry out an experiment and to compare the results with what we would expect from the so-called "null hypothesis": that nothing untoward is going on. The null hypothesis is the position of "doubting Thomas" in Chapter 20 of John's Gospel: "If you want me to believe you, then you'd better produce some pretty strong evidence."

So if our coin was tossed or spun 20 times and came up tails on 19 of those times, the null hypothesis would be that this was merely a fluke; "it could have just happened that way". However in that case we would have witnessed an event that happens in only 20 ways out of a total of 2^{20} possibilities. The probability of this happening by pure chance is thus $20/2^{20}$ or about 0.00019. Because this is a very small number, we tend to discount the situation it represents and say instead that we have very strong evidence that the coin is actually biased.

How small would such a probability need to be before we could argue in this way? The answer to this question is somewhat arbitrary, and there are those who don't like it on this account. But we generally pick in advance a (small) value of the probability below which we are prepared to be convinced. Values of 0.05, 0.01 and 0.001 are commonly used for this purpose (they correspond respectively to odds of 19:1, 99:1 and 999:1 against). These values (which I will rather irreverently call "magic numbers") are convenient, are sanctioned by custom and also have other more arcane virtues, but their acceptance is mere convention. No theorem tells us that these numbers are any better than other numbers we might use.

But notice now that all we have established is that the coin is very unlikely to be fair. We have not learned what the probability is that the coin will actually land tails up. To find this is a much more difficult problem. And notice also that it is of a different type from the more familiar in which the examiner, the textbook-writer or some other figure of authority tells us what the probability is. Here we have to determine it for ourselves and do so from limited data!

This is the classic case of a problem in statistics, as opposed to probability theory, and it is easy to recognise that it is much more difficult. One approach would be to guess that the true probability is 0.95 (i.e. 19/20) and to compute the odds for that value. Such a coin would land "tails" on 19 out of 20 occasions, 0.377 of the time, with other outcomes spanning the rest. If the probability of tails (on any given toss) were 0.7162, then the probability of seeing exactly 19 tails from 20 tosses would be 0.01, and the same is true if the probability of a tail in any individual toss was 0.9994. So if we use the middle one of our three "magic numbers", then we could assert that we had strong evidence that the probability of tails for this situation was somewhere between 0.7162 and 0.9994.

However, there have been other approaches to this problem and these are what I want to share with you in this column. There is continuing controversy in this area of statistics, and this controversy greatly affects the judgements we make today about this early work. I hope to show you some of the reasons for this, but requirements of space and of clarity have imposed restrictions that will necessarily make my account incomplete. I have drawn in particular from four sources:

1. Victor Katz's *A History of Mathematics*, especially his Chapter 14.1,
2. Andrew Dale's English translation of Laplace's *Philosophical Essay on Probabilities*,
3. Section V.2(*e*) of Feller's *An Introduction to Probability Theory and its Applications*, *Volume* 1,
4. Example 4G of Parzen's *Modern Probability Theory and its Applications*.

To simplify matters, I take a seemingly straightforward situation to analyse. Let us say that we toss a coin and that it falls tails. This would not strike us as particularly remarkable, nor would we be overly surprised to see a second "tails"; nor perhaps even a third. But if the coin is fair and we see a run of 5 consecutive tails, then we have witnessed an event with a probability of 0.03125, and this value is below the largest of our three "magic numbers". A run of 7 successive tails takes us below the next "magic number" and a run of 10 tails takes us (just) below the third and smallest. Even if we adopt this most stringent of criteria, we would reject the null hypothesis that the coin was fair. We say to our doubting Thomas, "You can go on believing the coin is fair if you like, but the odds are 999 to 1 against you"!

However, we would like to say more about the coin than simply that it is unlikely to be fair. This is the matter that interested two early statisticians, Thomas Bayes (1702–1761) and Pierre-Simon de Laplace (1749–1827).

Bayes, towards the end of his life, wrote *An Essay towards solving a Problem in the Doctrine of Chances,* a work that was not published until after his death. Bayes' name is commemorated in the formula we know today as "Bayes' Theorem". This result is not difficult to state nor even to prove. Let $P(A)$ be the probability that Event A will occur and let $P(B)$ be the corresponding probability for Event B; $P(AB)$ is the probability that both these events happen. Write $P(A \mid B)$ for the conditional probability that Event A will occur when it is already known that Event B has occurred, and similarly define $P(B \mid A)$. Then

$$P(AB) = P(A)P(B \mid A) = P(B)P(A \mid B)$$

and so

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)} = \frac{P(AB)}{P(A)} \tag{1}$$

This is Bayes' Theorem (or Bayes' Formula, as it is also known).

Now let Event A be the event that n tosses of a coin produce n tails, and Event B be the event that the probability of tails on any given toss exceeds $1/2$. Now we don't know the probability in a single toss, so call it x. The probability of Event A will then be x^n. The probability of Events A and B both occurring is thus the sum over all the possible x in the relevant range, so Bayes (remembering that an integral is a form of sum) took it to be

$$\int_{1/2}^{1} x^n \, dx$$

Furthermore, if we knew x, then the probability of Event A would again be x^n and again Bayes used an integral to represent a sum over all the possible values of x. He thus reached

$$P(B \mid A) = \frac{\int_{1/2}^{1} x^n dx}{\int_{0}^{1} x^n dx} = 1 - \left(\frac{1}{2}\right)^{n+1}$$

Notice that this is the probability that there is *some* bias towards tails in the coin. There is no estimate given as to the extent of that bias.

Laplace considered a somewhat different problem. He took Event A to be the event that the first n tosses yielded tails and Event B to be the event that the first $n+1$ tosses yielded tails. Because Laplace (like Bayes) did not assume the actual probability x of tails on a single toss to be known, he assumed it was equally likely to be one of the $N+1$ numbers $0/N$, $1/N$, $2/N$, ..., 1, where N is some large unknown number. Hence Laplace had

$$P(A) = \frac{0^n + 1^n + 2^n + \ldots + N^n}{N^n\,(N+1)}$$
$$= \frac{1^n + 2^n + \ldots + N^n}{N^n\,(N+1)}$$

Similarly,

$$P(B) = \frac{1^{n+1} + 2^{n+1} + \ldots + N^{n+1}}{N^{n+1}\,(N+1)}$$

Since in this case $P(AB) = P(B)$, the required conditional probability is

$$P(B\,|\,A) = \frac{P(B)}{P(A)} = \frac{1^{n+1} + 2^{n+1} + \cdots + N^{n+1}}{N\left(1^n + 2^n + \ldots + N^n\right)}$$

Under Laplace's assumptions, this is the conditional probability that, following n successive tails, the next toss will also result in a tail.

These sums are actually very hard to evaluate in general, so Laplace approximated them by means of integrals. The answer then came out to be the approximation (valid if N is large)

$$P(B\,|\,A) = \frac{n+1}{n+2} \qquad (2)$$

This is known as Laplace's *Law of Succession.*

In describing this, I have followed Parzen's account, but Parzen expresses reservations about it. He draws explicit attention to the controversial nature of Laplace's assumptions along the way, but allows a limited use to the theory.

> "Consider a tourist in a foreign city who scarcely understands the language. With trepidation, he selects a restaurant in which to eat. After ten meals taken there he has felt no ill effects. Consequently, he goes quite confidently to the restaurant the eleventh time in the knowledge that, according to the rule of succession, the probability is 11/12 that he will not be poisoned by his next meal."

However, he also waves a flag of warning.

> "A boy is 10 years old today. The rule says that, having lived ten years, he has a probability of 11/12 of living one more year. On the other hand, his 80-year old grandfather has probability 81/82 of living one more year! Yet, in fact, the boy has the greater probability of living one more year."

Feller in his account bypasses Laplace's dubious assumptions and totally alters the basic situation into one involving "urns" containing red and white balls. He thus produces an account of equation (2) that is entirely above reproach, but which, as he himself remarks, is "somewhat artificial".

Now again, I have reservations. Not about Feller's approach, which solves quite correctly and completely the problem Feller poses, but rather about Laplace's original derivation. Again, consider the case $n = 1$. Laplace, seeing a tail, would conclude that the probability of getting tails on the next toss was 2/3. This appears to be inconsistent with the original analysis, which had the unconditional probability of "tails" as one or other of $0/N$, $1/N$, $2/N$, ..., 1, for some unknown (possibly large) number. Of these, the first possibility now appears to be ruled out. However, the derivation of the conditional probability (2) is still the same.

Here's how I would look at this problem. Pick one of our "magic numbers"; call it m. Then if the true probability of "tails" is x, we know that $x \leqslant 1$. We also seek a lower bound on x; call it p. So we want to find p such that $p \le x \le 1$. Our "doubting Thomas", once we nominate a value for p, will say, "No, your evidence isn't strong enough for me; I still think the true value of x could lie below that value". He won't be convinced until we make the probability of *this* occurrence less than m, whichever value of m we agree upon.

Thus the lower bound p satisfies the equation: $p^n = m$, and so we have

$$p = m^{1/n}, \qquad (3)$$

from which we find

$$m^{1/n} \leq x \leq 1. \tag{4}$$

At first sight, inequality (4) looks quite different from equation (2), but in part this appearance is deceptive. If we concentrate on the value of p, that is to say the lower bound on x, then if n is large, then we can write

$$m^{1/n} = e^{\frac{\ln m}{n}} \approx \left(1 + \frac{\ln m}{n}\right)$$

where we have made use of the known formula

$$e^{a} \approx \left(1 + \frac{a}{n}\right)^{n}$$

valid when n is large.

So if we set $k = \ln(1/m)$, then we have

$$x \geq 1 - \frac{k}{n} \tag{5}$$

while equation (2) gives

$$x = 1 - \frac{1}{n+2} \approx 1 - \frac{1}{n}$$

again taking n to be large.

My formula is more cautious than Laplace's. If we choose the first of our "magic numbers" $m =$ 0.05, then we say if 10 tails turn up in 10 tosses, that $x > 7.4$, or on the approximate formula (5), $x > 0.7$. If odds of 19:1 against aren't strong enough for our "doubting Thomas", then we could go to $m = 0.01$ to find $x > 0.63$, and finally if $m = 0.001$, then $x > 0.501$. Note that all of these numbers are greater than 1/2, so that even on the strictest of the criteria, "doubting Thomas" would be sticking his neck out to claim that the coin could still be fair. This is in line with the earlier analysis of this point.

Laplace would have it that if 10 tosses produced 10 tails, then the posterior probability of tails is 11/12, or 0.917. This is a somewhat stronger statement than I would care to make, even on the very weakest criterion in common use (m = 0.05).

But let me close with some words of caution. We have been talking of assessing probabilities when coins are tossed. Our results apply well to this, but not to Parzen's example of the boy and his grandfather. Parzen has this to say:

> "It is to be emphasized that Bayes' formula and Laplace's rule of succession [in its careful, if artificial, statement] are true theorems, of mathematical probability theory. [Such examples as the boy and his grandfather] do not in any way cast doubt on the validity of these theorems. Rather they serve to illustrate what may be called the *fundamental principle of applied probability theory*: before applying a theorem, one must carefully ponder whether the hypotheses of the theorem may be assumed to be satisfied."

Laplace applied his law of succession to the rising of the sun:

"... if we place the dawn of history at 5000 years before the present date, we have 1,826,213 days on which the sun has constantly risen in each 24 hour period. We may therefore lay the odds of 1,826,214 to 1 that it will rise again tomorrow."

This "application" attracted trenchant criticism from both Parzen and Feller. Here is Feller:

> "In fact [Laplace's calculation] pretends to judge the chances of the sun's rising tomorrow from the *assumed* risings in the past. But the assumed rising of the sun on February 5, 3123 B.C., is by no means more certain than that the sun will rise tomorrow. We believe in both for the same reasons."

And finally some excellent advice from Feller:

"The beginner is advised always to [calculate conditional probabilities directly from the definition] and not to memorise [the formula for Bayes' theorem]."

* * * * *

PROBLEM CORNER

PROBLEM 24.1.1 (from the AMATYC Review)

Let p be a prime number. Prove that p is of the form $8n \pm 1$ if and only if there exists an integer k such that $p = \sqrt{48k+1}$.

SOLUTION (Keith Anker, Glen Waverley)

If $p = 8n \pm 1$ then $p^2 = 16(4n^2 \pm n)+1$ so that p^2-1 is divisible by 16. Since p is prime and greater than 3 we also have $p = 3m \pm 1$ and hence $p^2 - 1$ is also divisible by 3. So we now see that $p^2 - 1$ must be divisible by 48, hence $p = \sqrt{48k+1}$ for some integer k. For the converse, let $p = \sqrt{48k+1}$ where p is prime. The either $p = 8n \pm 1$ or $p = 8n \pm 3$. If $p = 8n \pm 3$ we have $p^2 = 16(4n^2 \pm 3n)+9$ and hence p^2 canot be written as $16m + 1$, let alone as $48k + 1$.

Solutions were also received from John Barton, Carlos Victor and J.A. Deakin.

PROBLEM 24.1.2. (from Mathematics and Informatics quarterly)

Let $n \geq 2$ be a positive integer and let $P(n)$ denote the product of the positive divisors (including 1 and n) of n. Find the smallest n for which $P(n) = n^{10}$.

SOLUTION

First observe that if n is a positive integer, than for every positive divisor m of n, the number n/m is also a divisor of n, and that their product, $m(n/m)$, is n. Hence $P(n)$ is always of the form $n^{d(n)/2}$, where $d(n)$ is the number of divisors of n. Since for each $m < \sqrt{(n)}$, the number n/m is greater than m, we find that $d(n)$ is even, unless n is a perfect square; in that case, $\sqrt{(n)}$ is an additional divisor of n, and hence $d(n)$ is odd. In the present problem, $d(n)/2 = 10$, hence $d(n) = 20$.

Next observe that if $n = \Pi\, p_i^{a_i}$ is the prime factorization of n, then $d(n) = \Pi\,(a_i + 1)$. Thus the exponents of the prime divisors of n must be one less than divisors of 20. Since $20 = 10 \cdot 2 = 5 \cdot 4 = 5 \cdot 2 \cdot 2$, these exponents must be (19), $(9, 1)$, $(4, 3)$ or $(4, 1, 1)$. Since we wish to find the smallest n for which $P(n) = n^{10}$, we match the largest exponent with the smallest prime 2, the second largest exponent with the next smallest prime 3, and the third with 5, to obtain the following candidates for the smallest such n:

$$2^{19} = 524288,\ 2^9 \cdot 3^1 = 1536,\ 2^4 \cdot 3^3 = 432,\ 2^4 \cdot 3^1 \cdot 5^1 = 240.$$

The smallest of these is clearly 240; that's the answer to our problem.

A similar solution was received from Keith Anker and a similar argument was received from Carlos Victor.

PROBLEM 24.1.3 (from Mathematics and Informatics Quarterly)

Using my pocket calculator I divide one positive integer by another giving answer 0.5876578. Both integers were less than 1000. What were the two integers?

SOLUTION

The received solutions used continued fractions. Here is a solution using the properties of Farey fractions. If two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are such that the determinant $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 1$, then $\frac{a+c}{b+d}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$, furthermore no fraction with a denominator less than $b + d$ has this property. If we let $x = 0.5876578$ then x lies between $\frac{1}{2}$ and $\frac{2}{3}$ and $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$. Adding numerators and denominators we see $x < \frac{3}{5}$ so that x lies between $\frac{1}{2}$ and $\frac{3}{5}$. Continuing in this manner we find $\frac{4}{7} < x < \frac{3}{5}$, $\frac{7}{12} < x < \frac{3}{5}$, etc. Successive nested intervals containing x are obtained in this manner, each interval having rational endpoints. We continue until we obtain an interval $\left[\frac{x}{y}, \frac{w}{z}\right]$ with the property that $y + z > 1000$. The reader can verify that this leads to $\left[\frac{419}{713}, \frac{181}{308}\right]$.

Now, correct to seven places we have $\frac{419}{713} = 0.5876578$. Since no fraction with a denominator of less than 1021 lies between the endpoints of this interval we conclude that that two integers were 419 and 713.

Solutions were received from Keith Anker, John Barton, J.A. Deakin, David Shaw and Carlos Victor. Julius Guest solved the problem using a program.

PROBLEM 24.1.4 (from Mathematical Spectrum)

Determine the value of the definite integral

$$\int_2^3 \frac{dx}{\sqrt{5-x}+\sqrt{x-1}}$$

SOLUTION

The standard approach is to rationalise the denominator, however this leads to an improper integral. Our solution uses the identity $(5-x) + (x-1) \equiv 4$. This identity implies that for each x in $[1, 5]$ we can find θ such that $5-x = 4\cos^2\theta$, $x-1 = 4\sin^2\theta$.

The given integral reduces to

$$I = \int_{\pi/6}^{\pi/4} \frac{4\sin\theta\cos\theta}{\cos\theta+\sin\theta}\, d\theta$$

Now the identity $2\sin\theta\cos\theta = (\sin\theta+\cos\theta)^2 - 1$ enables us to write the above integral as

$$I = 2\int_{\pi/6}^{\pi/4} \left\{ (\sin\theta + \cos\theta) - \frac{1}{\sin\theta + \cos\theta} \right\} d\theta$$

Now $\int_{\pi/6}^{\pi/4} \sin\theta + \cos\theta \, d\theta = \frac{1}{2}(\sqrt{3} - 1)$, and, putting $\phi = \theta + \frac{\pi}{4}$

Gives

$$\int_{\pi/6}^{\pi/4} \frac{d\theta}{\sin\theta + \cos\theta} = \frac{1}{\sqrt{2}} \int_{5\pi/12}^{\pi/2} \frac{d\phi}{\sin\phi} = \frac{1}{\sqrt{2}} \left[\ln\left(\tan\frac{\phi}{2} \right) \right]_{5\pi/2}^{\pi/2} = \frac{-1}{\sqrt{2}} \ln\left(\tan\frac{5\pi}{24} \right)$$

Hence $\mathrm{I} = \sqrt{3} - 1 + \sqrt{2} \ln\left(\tan\frac{5\pi}{24} \right)$

Solutions were received from Keith Anker, J A Deakin, Julius Guest and Carlos Victor.

Correction to Problem 24.2.3

The statement of the problem defined $\lfloor x \rfloor$ as the least integer not exceeding the real number x; this should be corrected to the *greatest* integer not exceeding the real number x.

PROBLEMS

PROBLEM 24.3.1 (from Mathematics and Informatics quarterly)

Show that 1 is the sum of all numbers $\frac{1}{x \cdot y}$, where $1 \le x \le n$, $1 \le y \le n$, $x + y > n$ and x and y are relatively prime.

$\left(\text{For example, when } n = 4, \ \frac{1}{2 \cdot 4} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 1} + \frac{1}{4 \cdot 3} = 1 \right)$

PROBLEM 24.3.2 (proposed by Keith Anker, adapted from a Westpac junior mathematics competition.

Fifteen (equal sized) circular discs, each either red or green, are arranged in an equilateral triangular array with one disc in the top row, two in the second row, etc, down to five in the fifth row. Show that there are three disks of the same colour with centres at the vertices of an equilateral triangle.

PROBLEM 24.3.3 (proposed by Julius Guest, East Bentleigh)

Solve the equation $9x^4 + 12x^3 - 3x^2 - 4x + 1 = 0$.

PROBLEM 24.3.4 (from Mathematics and Informatics Quarterly)

Points M and N are drawn inside an equilateral triangle ABC. Given $\angle MAB = \angle MBA = 40^{\circ}$, $\angle NAB = 20^{\circ}$ and $\angle NBA = 30^{\circ}$, prove that MN is parallel to BC.

PROBLEM 24.3.5 (from Mathematics and Informatics Quarterly)

Let x and y be real numbers of the form $\dfrac{m+n}{\sqrt{m^2+n^2}}$ where m and n are positive integers. Show that if $x < y$ then there is a real number z of the same form such that $x < z < y$.

* * * * *

The main duty of the historian of mathematics, as well as his fondest privilege, is to explain the humanity of mathematics, to illustrate its greatness, beauty and dignity, and to describe how the incessant efforts and accumulated genius of many generations have built up that magnificent monument, the object of our most legitimate pride as men, and of our wonder, humility and thankfulness, as individuals. The study of the history of mathematics will not make better mathematicians but gentler ones, it will enrich their minds, mellow their hearts, and bring out their finer qualities.

—J Sarton

OLYMPIAD NEWS

The 2000 Australian Mathematical Olympiad

The contest was held in Australian schools on February 8 and 9. On either day, 125 students in years 8 to 12 sat a paper consisting of four problems, for which they were given four hours. These are the two papers.

The 2000 Australian Mathematical Olympiad
First Day

Tuesday, 8th February, 2000

Time allowed: 4 hours
NO calculators are to be used
Each question is worth seven points

1. Find all polynomials f with real coefficients such that

$$(x-27)f(3x) = 27(x-1)f(x)$$

for every real number x.

2. For each date of the current year (2000) we evaluate the expression

$$\text{day}^{\text{month}} - \text{year}$$

and then find the highest power of 5 dividing it.
(For example, the 293rd anniversary of Leonhard Euler's birth will be on 15 April, in which case we obtain:

$$\text{day}^{\text{month}} - \text{year} = 15^4 - 2000 = 5^3(405-16) = 5^3 \cdot 389,$$

a multiple 5^3 but not of 5^4.)

Find all dates for which the corresponding power of 5 is the greatest.

3. Let x_1, x_2, ..., x_n, y_1, y_2, ... be real numbers such that

(i) $0 < x_1 y_1 < x_2 y_2 < \ldots < x_n y_n$ and

(ii) $x_1 + x_2 + \ldots + x_i \geq y_1 + y_2 + \ldots + y_i$ for $1 \leq i \leq n$.

Prove that $\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} \leq \frac{1}{y_1} + \frac{1}{y_2} + \ldots + \frac{1}{y_n}$.

When does equality occur?

4. Let A, B, C, A', B', C' be points on a circle such that AA' is perpendicular to BC, BB' is perpendicular to CA, and CC' is perpendicular to AB. Further, let D be a point on that circle and let DA' intersect BC in A'', DB' intersect CA in B'', and DC' intersect AB in C'', all line segments being extended where required. Prove that A'', B'', C'' and the orthocentre of triangle ABC are collinear.

Second Day

Wednesday, 9th February, 2000

5. Let m and n be positive integers. Prove that $m^{n+2} + m^{n+1} + m^n$ is not a perfect square.

6. Let a, b and c be any real numbers not all of which are zero. Determine all functions f that assign real numbers to real numbers such that

$$af(xy + z^2) + bf(yz + x^2) + cf(zx + y^2) = 0$$

for all real numbers x, y, z.

Distinguish all possibilities for a, b and c.

7. Solve the following system of equations:

$$\begin{aligned} x + \lfloor y \rfloor + \{z\} &= 200.0 \\ \{x\} + y + \lfloor z \rfloor &= 190.1 \\ \lfloor x \rfloor + \{y\} + z &= 178.8 \end{aligned}$$

Note that if r is a real number, then $\lfloor r \rfloor$ denotes the largest integer not exceeding r, while $\{r\}$ stands for the fractional part of r, i.e. $r - \lfloor r \rfloor$.

8. Let A_1, A_2, A_3, A_4, A_5, A_6, A_7 be vertices of a heptagon lying in the plane Π, and let B and C be two different points in space, not lying in Π, such that no three of these nine points are collinear. Each of the 14 edges A_iB and A_iC $(i = 1, 2, \cdots, 7)$, the 14 diagonals of heptagon A_1, A_2, A_3, A_4, A_5, A_6, A_7, and the line segment BC are coloured either green or gold. Prove that there are three line segments among them, all of the same colour, that form a triangle.

As a result, 27 students were invited to represent Australia at the Twelfth Asian Pacific Mathematics Olympiad (APMO). This annual competition was started in 1989 by Australia, Canada, Hong Kong and Singapore. Since then the APMO has grown into a major international competition for students from about twenty countries on the Pacific Rim as well as from Argentina, South Africa and Trinidad & Tobago. It was held in the second week of March. Here is the contest paper.

The Twelfth Asian Pacific Mathematics Olympiad
March 2000

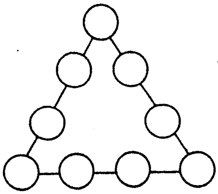
Time allowed: 4 hours
No calculators to be used
Each question is worth 7 points

1. Compute the sum

$$S = \sum_{i=0}^{101} \frac{x_i^3}{1 - 3x_i + 3x_i^2}$$

for $x_i = \dfrac{i}{101}$.

2. Given the following triangular arrangement of circles:



Each of the numbers 1, 2, …, 9 is to be written into one of these circles, so that each circle contains exactly one of these numbers and

(i) the sums of the four numbers on each side of the triangle are equal;
(ii) the sums of the squares of the four numbers on each side of the triangle are equal.

Find all ways in which this can be done.

3. Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA respectively, and O the point in which the perpendicular at P to BA meets AN produced.

Prove that QO *is perpendicular to* BC.

4. Let n, k be given positive integers with $n > k$. Prove that

$$\frac{1}{n+1} \cdot \frac{n^n}{k^k(n-k)^{n-k}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k(n-k)^{n-k}} .$$

5. Given a permutation $(a_0, a_1, \ldots, a_n)$ of the sequence $0, 1, \ldots, n$. A transposition of a_i with a_j is called *legal* if $i > 0$, $a_i = 0$ and $a_{i-1} + 1 = a_j$. The permutation $(a_0, a_1, \ldots, a_n)$ is called *regular* if after a number if legal transpositions it becomes $(1, 2, \ldots, n, 0)$. For which numbers n is the permutation $(1, n, n-1, \ldots, 3, 2, 0)$ regular?

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