Function

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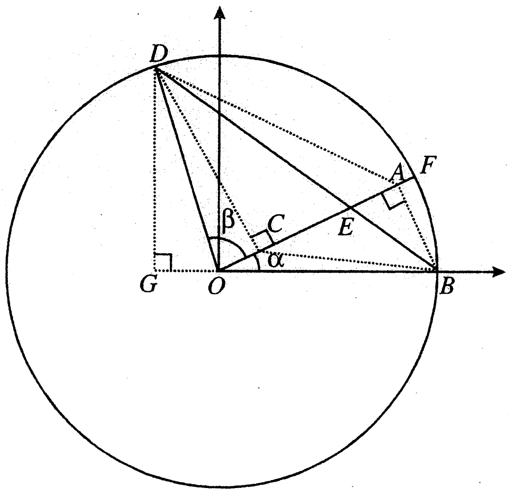

Department of Mathematics & Statistics – Monash University

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Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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* \$13 for *bona fide* secondary or tertiary students.

EDITORIAL

Welcome to all readers to this new issue of *Function*!

In this issue, we include three feature articles. The first one deals with an old problem: given that k letters are randomly placed in k envelopes, what is the probably that all letters end up in the wrong envelopes? This problem has been already stated in many ways—some of them published in *Function*—but this time the author takes a different approach to suggest a solution.

The second feature article is also another approach on solving an old problem: the very useful sine sum formula. The author, Anthony Lun, shows an elegant proof based on a simple geometric result.

The third feature article is a contribution by our reader from Sarajevo, Šefket Arslanagič, who stresses the importance of taking different approaches in solving a problem; the author always gains from deeper understanding of the problem.

We received another puzzling letter from our mysterious correspondent Dr Dai Fwls ap Rhyll, who once again, sent us a few somewhat intriguing mathematical observations. We invite readers to send us their thoughts about these observations.

The debate about the start of the millennium has not come to an end yet. In the *History of Mathematics* column, Michael Deakin points us to further history facts and documents which conflict with the definition given by Peter Grossman in our last issue of *Function*. In the *Computers and Computing* section we include a list of interesting mathematical resources on the web. We invite you to contribute with your own favourite bookmarks, to keep building up a list of links that could be of interest to our readers.

Finally, we include the solutions to the problems published in the October 1999 issue, as well as new problems to keep your mind active until our next issue. We encourage our readers to send their solutions.

LETTERS AND ENVELOPES

Lachlan Harris

There are k letters and k envelopes. If the letters are randomly placed in the envelopes, what is the probability that all the letters are in wrong envelopes? Or, in other words, what is the probability that no letter is in its correct envelope?

For one letter and one envelope ($k = 1$), it is obvious that it is impossible to get the letter in the wrong envelope. Hence the number of ways that no letter is in its correct envelope is zero.

For two letters and two envelopes ($k = 2$), either both letters are in the wrong envelope or both are in correct envelopes. Hence the number of ways that no letter is in the correct envelope is 1.

For three letters and three envelopes ($k = 3$), there are only two ways that no letter is in its correct envelope, as shown below.

Letters:	b	c	a
Envelopes:	A	B	C

Letters:	c	a	b
Envelopes	A	B	C

For four letters and four envelopes ($k = 4$), we find the total number of ways in which 0, 1, 2, 3 of the letters can be put in wrong, and subtract this from the total number, $4! = 24$ ways of putting 4 letters in envelopes.

Here are the different cases:

No. of letters in wrong envelopes	**No. of ways**
0 (i.e. all correct)	1
1	0
2	6, because the two wrong envelopes can be chosen in ${}^4C_2$ ways, and the other two correct envelopes can only be chosen in one way.

3	8, because the three incorrect envelopes can be chosen in $^4C_3$ ways, but there are two ways in which they are arranged incorrectly. Hence $^4C_3 \times 2$.
4 (i.e. all wrong)	x

Therefore, $1 + 0 + 6 + 8 + x = 24$, so $x = 9$.

Let N_k be the number of ways in which no letter is in the correct envelope when k letters are placed in k envelopes. The results so far are shown below:

k	1	2	3	4
N_k	0	1	2	9

Repeating this method for the first ten values of k, I found N_k for the first ten values of k:

k	1	2	3	4	5	6	7	8	9	10
N_k	0	1	2	9	44	265	1854	14833	133496	1334961

From the above table, I found that the following formula appears to be true:

$$N_k = (k \times N_{k-1}) + (-1)^k$$

For example:

$N_2 = (2\times0) + 1 = 1$

$N_3 = (3\times1) - 1 = 2$

$N_4 = (4\times2) + 1 = 9$

$N_5 = (5\times9) - 1 = 44$

$N_6 = (6\times44) + 1 = 265$

$N_7 = (7\times265) - 1 = 1854$

$N_8 = (8\times1854) + 1 = 14833$

$N_9 = (9\times14833) - 1 = 133496$

$N_{10} = (10\times133496) + 1 = 1334961$

$N_{11} = (11\times1334961) - 1 = 14684570$

Therefore the probability that no letter is in its correct envelope when k letters are placed in k envelopes is

$$\frac{N_k}{k!}$$

where

$$N_k = 0 \text{ for } k = 1$$

$$\text{and } N_k = (k \times N_{k-1}) + (-1)^k \text{ for } k > 1$$

[*Ed*: This problem is similar to the one described in *Function Vol* 27, *Part* 1 about a magician getting wrong the 12 different star signs of 12 people.]

* * * * *

Proof requires a person who can give and a person who can receive ... .

A blind man said, As to the Sun,
I'll take my Bible oath there's none;
For if there had been one to show
They would have shown it long ago.
How came he such a goose to be?
Did he not know he couldn't see?

Not he.

A De Morgan
in *Budget of Paradoxes*, London , 1872, p.262.

* * * * *

Man is full of desires: he loves only those who can satisfy them all. "This man is a good mathematician," someone will say. But I have no concern for mathematics; he would take me for a proposition. "That one is a good soldier." He would take me for a besieged town. I need, that is to say, a decent man who can accommodate himself to all my desires in a general sort of way.

—Blaise Pascal

THE SINE SUM IDENTITY

Anthony Lun, Monash University

The aim of this article is to give an elementary proof of the sine sum identity

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \tag{1}$$

The proof uses the following simple geometric result:

> Let $ABCD$ be a trapezium such that the pair of sides AB and CD are parallel. The diagonals AC and BD intersect at point E (Figure 1). Then
>
> $$\text{area of } \Delta AED = \text{area of } \Delta BCE$$

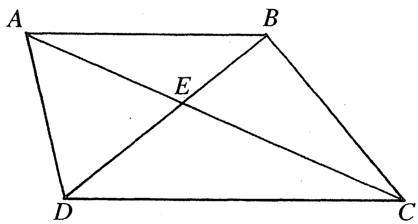

Figure 1

Let us see why this is the case.

Consider ΔACD and ΔBCD. These triangles share a common case CD, and they are of the same height, which equals the distance between the pair of parallel sides AB and CD. Therefore

$$\text{area of } \Delta ACD = \text{area of } \Delta BCD.$$

Subtracting the area of the common part ΔECD from the areas of ΔACD and ΔBCD, gives the result

$$\text{area of } \Delta AED = \text{area of } \Delta BCE.$$

Now on to the sine sum identity. Consider 2 cases: first for $0 < \alpha, \ \beta \leq \pi$, and then for either α or β greater than π, but smaller than or equal to 2π.

Case 1: Both α and β less than π

The sine sum identity (1) holds for all values of α and β. For simplicity, we first give details of a proof for the more restrictive case:

Case 1a: $0 < \alpha \leq \dfrac{\pi}{2}, \ 0 < \beta \leq \dfrac{\pi}{2}$

These imply $0 < \alpha + \beta \leq \pi$.

Let OB, OF and OD be radii of a circle with centre O such that $\angle BOF = \alpha$, $\angle FOD = \beta$, and $\angle BOD = \alpha + \beta$. From B draw a perpendicular line to OF intersecting it at A. Similarly, from D draw a perpendicular line to OF intersecting it at C, and another perpendicular line to OB intersecting it at G. Hence the lines AB and CD are parallel, and $ABCD$ is a trapezium. The diagonals AC and BD of the trapezium intersect at point E. By the simple geometric result stated at the beginning of this article,

$$\text{area of } \Delta AED = \text{area of } \Delta BCE \tag{2}$$

From Figure 2, we have

$$\text{area of } \Delta ODB = \text{area of } \Delta ODE + \text{area of } \Delta OEB, \tag{3.a}$$
$$\text{area of } \Delta ODA = \text{area of } \Delta ODE + \text{area of } \Delta AED, \tag{3.b}$$
$$\text{area of } \Delta OEB = \text{area of } \Delta OCB + \text{area of } \Delta BCE, \tag{3.c}$$

$$\begin{aligned}\therefore \text{ area of } \Delta ODB &= (\text{area of } \Delta ODA - \text{area of } \Delta AED) + \\ &\quad (\text{area of } \Delta OCB + \text{area of } \Delta BCE) \\ &= (\text{area of } \Delta \mathrm{ODA} + \text{area of } \Delta OCB) + \\ &\quad (\text{area of } \Delta BCE - \text{area of } \Delta AED)\end{aligned} \tag{4}$$

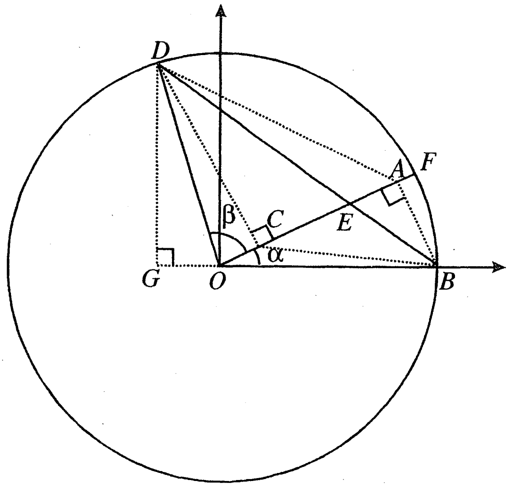

Figure 2

Hence, using (2), we obtain

$$\text{area of } \Delta ODB = \text{area of } \Delta ODA + \text{area } \Delta \text{OCB} \tag{5}$$

Finally, we will show that (5) is equivalent to the sine sum identity (1). From the diagram, we have

$$\sin\alpha = \frac{AB}{OB} \quad \text{and} \quad \cos\alpha = \frac{OA}{OB}, \tag{6.a}$$

$$\sin\beta = \frac{CD}{OD} \quad \text{and} \quad \cos\beta = \frac{OC}{OD}, \tag{6.b}$$

$$\sin(\alpha+\beta) = \frac{GD}{OD}. \tag{6.c}$$

(Note that when $\frac{\pi}{2} < \alpha + \beta \leq \pi$, $\cos(\alpha+\beta) = -\frac{OG}{OD}$, which is negative.)

Using (6), we can deduce that

$$\text{area of } \Delta ODB = \frac{1}{2} \times OB \times GD = \frac{(OB \times OD)}{2} \sin(\alpha+\beta). \tag{7}$$

$$\text{area of } \Delta ODA = \frac{1}{2} \times OA \times CD = \frac{(OB \times OD)}{2} \cos\alpha \sin\beta. \tag{8}$$

$$\text{area of } \Delta OCB = \frac{1}{2} \times OC \times AB = \frac{(OD \times OB)}{2} \cos\beta \sin\alpha. \tag{9}$$

Therefore (5) is equivalent to (1).

The remaining cases, where either one of α or β is greater than $\frac{\pi}{2}$, and α and β are greater than $\frac{\pi}{2}$ and less than π, can be proved following the same procedures as given above but with sign changes to appropriate terms in equations (4) to (9).

Case 1b: $0 < \alpha \le \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta \le \pi$ (see Figure 3)

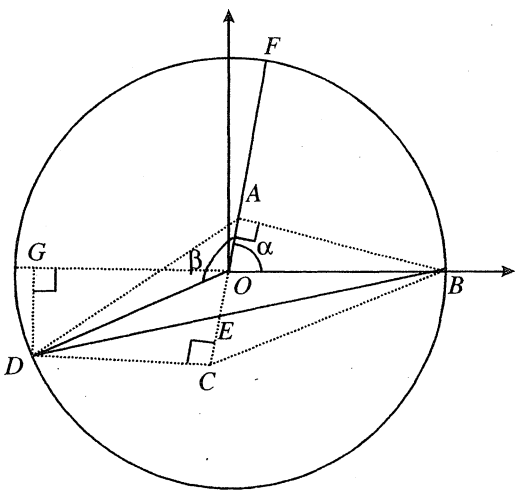

Figure 3: The sine sum identity with $0 < \alpha \le \frac{\pi}{2}, \frac{\pi}{2} < \beta \le \pi$

Case 1c: $\dfrac{\pi}{2} < \alpha \le \pi$ and $0 < \beta \le \dfrac{\pi}{2}$ (see Figure 4)

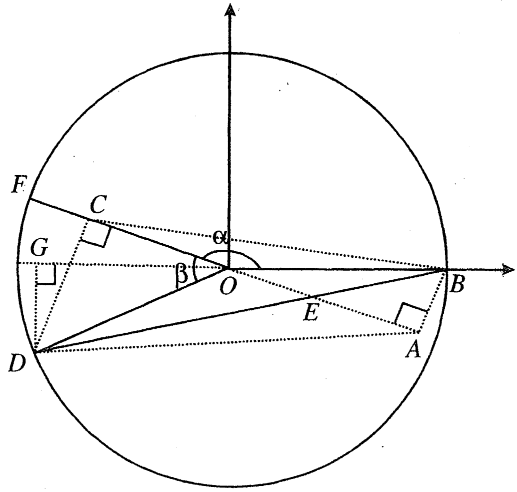

Figure 4: The sine sum identity with $\dfrac{\pi}{2} < \alpha \le \pi$, $0 < \beta \le \dfrac{\pi}{2}$

Case 1d: $\dfrac{\pi}{2} < \alpha,\ \beta \le \pi$ (see Figure 5).

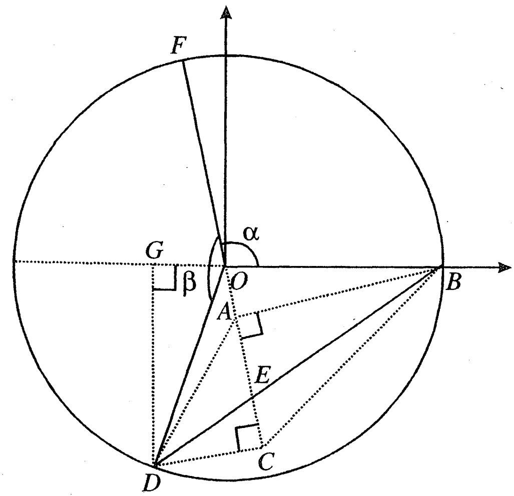

Figure 5: The sine sum identity with $\dfrac{\pi}{2} < \alpha,\ \beta \le \pi$

We leave details of their proofs as exercises.

Case 2: One of two angles greater than π.

When one of α or β is greater than π, there are subtle changes to the basic construction given in case 1. We now present an outline proof for the case where

(Case 2a:). $$0 < \alpha \le \frac{\pi}{2} \text{ and } \pi < \beta \le 2\pi ,$$

and leave the remaining case

(Case 2b:) $$\pi < \alpha \le 2\pi \text{ and } 0 < \beta \le \frac{\pi}{2},$$

as exercise.

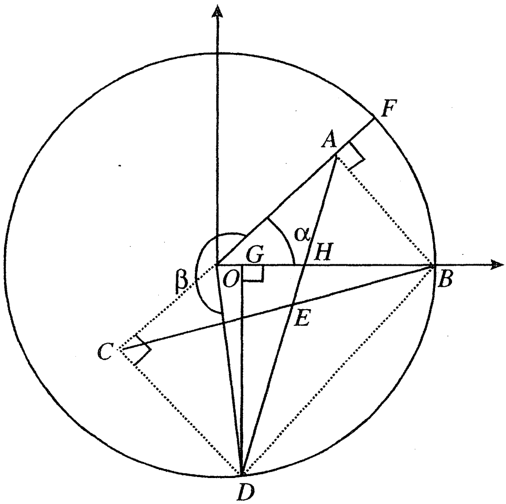

Figure 6: The sine sum identity with $0 < \alpha \le \frac{\pi}{2}$, $\pi < \beta \le 2\pi$.

Figure 6, $ABDC$ is a trapezium[1] with AB parallel to CD. The diagonals AD and BC of the trapezium intersect at point E. Furthermore AD cuts OB at point H. By the simple geometric result stated at the beginning,

[1] Notice the difference between the trapezium $ABCD$ in Figure 6 and those trapezia $ABCD$ in Figure 2 to Figure 5.

$$\text{area of } \Delta AEC = \text{area of } \Delta BDE. \tag{10}$$

From Figure 6, we have

$$\text{area of } \Delta OBD = \text{area of } \Delta OHD + \text{area of } \Delta HBD, \tag{11.a}$$
$$\text{area of } \Delta OAD = \text{area of } \Delta OHD + \text{area of } \Delta OAH, \tag{11.b}$$
$$\text{area of } \Delta HBD = \text{area of } \Delta HBE + \text{area of } \Delta BDE, \tag{11.c}$$

$$\begin{aligned} \therefore \text{ area of } \Delta OBD &= (\text{area of } \Delta OAD - \text{area of } \Delta OAH) + \\ &\quad (\text{area of } \Delta HBE + \text{area of } \Delta BDE) \\ &= (\text{area of } \Delta OAD + \text{area of } \Delta HBE) + \\ &\quad (\text{area of } \Delta BDE - \text{area of } \Delta OAH) \end{aligned} \tag{12}$$

(12) implies

$$\begin{aligned} \text{area of } \Delta OBD &= (\text{area of } \Delta OAD + \text{area of } \Delta HBE) \\ &\quad + (\text{area of } \Delta BDE - \text{area of } \Delta OAH). \end{aligned}$$

Using (10), we obtain

$$\begin{aligned} \text{area of } \Delta OBD &= (\text{area of } \Delta OAD + \text{area of } \Delta HBE) \\ &\quad + (\text{area of } \Delta AEC - \text{area of } \Delta OAH) \\ &= \text{area of } \Delta OAD + (\text{area of } \Delta HBE \\ &\quad + \text{area of } \Delta AEC - \text{area of } \Delta OAH) \\ &= \text{area of } \Delta OAD + \text{Area of } \Delta OBC \end{aligned} \tag{13}$$

Note the similarity between (5) and (13).

We now show that (13) is equivalent to the sine sum identity (1). From the diagram, we have

$$\sin\alpha = \frac{AB}{OB} \text{ and } \cos\alpha = \frac{OA}{OB}, \tag{14.a}$$

$$\sin\beta = \frac{CD}{OD} \text{ and } \cos\beta = \frac{OC}{OD}, \tag{14.b}$$

$$\sin(\alpha+\beta) = \frac{GD}{OD}. \tag{14.c}$$

Using (14), we can deduce that

$$\text{area of } \Delta ODB = \frac{1}{2}\times OB\times GD = \frac{(OB\times OD)}{2}\sin(\alpha+\beta). \tag{15}$$

$$\text{area of } \Delta ODA = \frac{1}{2}\times OA\times CD = \frac{(OB\times OD)}{2}\cos\alpha\sin\beta. \tag{16}$$

$$\text{area of } \Delta OCB = \frac{1}{2}\times OC\times AB = \frac{(OD\times OB)}{2}\cos\beta\sin\alpha. \tag{17}$$

From here (1) follows.

Conclusion

One can verify that the construction for other values of the angles α and β, can be reduced to one of those given in either Case 1 or Case 2. The crucial idea behind the construction lies in the two basic orientations of the trapezia $ABCD$ (see Figure 2 to Figure 5) and $ABDC$ (see Figure 6).

* * * * *

...Euclid alone
Has looked on Beauty bare.
He turned away at once;
Far too polite to stare.

—Riskin, Adrian
(after Edna St. Vincent Millay)
The Mathematical Intelligencer, Vol. 16, no. 4 (Fall 1994), p. 20.

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AN ALGEBRAIC INEQUALITY AND THREE DIFFERENT SOLUTIONS

Šefket Arslanagič, University of Sarajevo

Solving one problem
in two or more ways
is of greater value than
solving a hundred problems
always in one and the same way

G Polya

Solving independently a small number of hard problems is, no doubt, more beneficial to the reader than solving a large number of simple problems. If readers have access to someone else's solution, it is therefore highly recommended that they look at it only after having found their own. If they have not succeeded in doing so, they ought at least try to obtain a thorough insight into the problem. Quite often, one's own solution will deviate from that worked out by the author of the problem. This is highly desirable, for it is usually in such a situation that one obtains deeper understanding of the essence and the mathematical substance of the problem.

We are now going to describe an algebraic inequality and provide and discuss those different proofs. Here is the inequality:

If $x+y+z = 1$ (x, y, z real numbers), prove that $xy+yz+zx \le \frac{1}{3}$.

First proof

As $\quad (x-y)^2+(y-z)^2+(z-x)^2 \ge 0 \quad$ (squares of real numbers are non- negative)

$$(x^2-2xy)+y^2+y^2-2yz+z^2+z^2-2xz+x^2 \ge 0$$

$$\therefore \quad 2(x^2+y^2+z^2)-2(xy+yz+xz) \ge 0$$

we have $\quad xy+yz+zx \le x^2+y^2+z^2 \qquad (1)$

Hence
$$\begin{aligned} xy + yz + zx &= \tfrac{1}{3}(xy + yz + zx + 2xy + 2yz + 2zx) \\ &\le \tfrac{1}{3}(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) \text{ from (1)} \\ &= \tfrac{1}{3}(x + y + z)^2 \\ &= \tfrac{1}{3} \text{ (since } x + y + z = 1.) \end{aligned}$$

Also, equality holds if and only if $x = y = z = \frac{1}{3}$. Can you see why?

Second proof

As $1 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$, we have

$$x^2 + y^2 + z^2 = 1 - 2(xy + yz + zx) \qquad (2)$$

Further, by the power-means inequality

$$\frac{x + y + z}{3} \le \sqrt{\frac{x^2 + y^2 + z^2}{3}}$$

Now $x + y + z = 1$, so that $\frac{1}{3} \le \sqrt{\frac{x^2 + y^2 + z^2}{3}}$. Hence

$$\frac{1}{9} \le \frac{x^2 + y^2 + z^2}{3}$$

So
$$x^2 + y^2 + z^2 \ge \frac{1}{3}. \qquad (3)$$

Thus, by (2) and (3),

$$1 - 2(xy + yz + zx) \ge \frac{1}{3}$$

Hence
$$xy + yz + zx \le \frac{1}{3}$$

Furthermore, as the power-means inequality becomes equality if and only if $x = y = z$, so that equality holds if and only if $x = y = z = \frac{1}{3}$.

Third proof

We use the following substitution:

$$x = a+\tfrac{1}{3},\ y = b+\tfrac{1}{3},\ z = \tfrac{1}{3}-a-b\ \ (a,\ b\ ,c \text{ real numbers}).$$

Then

$$\begin{aligned} x^2+y^2+z^2 &= (a+\tfrac{1}{3})^2 + (b+\tfrac{1}{3})^2 + (\tfrac{1}{3}-a-b)^2 \\ &= a^2+\tfrac{1}{9}+\tfrac{2}{3}a+b^2+\tfrac{1}{9}+\tfrac{2}{3}b+\tfrac{1}{9}-\tfrac{2}{3}(a+b)+(a+b)^2 \\ &= \tfrac{1}{3}+a^2+b^2+(a+b)^2 \\ &\geq \tfrac{1}{3}. \end{aligned}$$

as squares of real numbers are non-negative.

Since

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$$

Hence

$$\begin{aligned} xy+yz+zx &= \frac{1-(x^2+y^2+z^2)}{2} \\ &\leq \frac{1-\frac{1}{3}}{2} \\ &= \frac{1}{3}, \end{aligned}$$

Note that equality holds if and only if $a = b = 0$, i.e. $x = y = z = \frac{1}{3}$.

* * * * *

An idea which can be used only once is a trick. If you can use it more than once it becomes a method.

—George Polya and Gábor Szegö
in *Problems and Theorems in Analysis*, Springer Verlag, 1972

LETTER TO THE EDITOR

This can indeed be called a red-letter year. Not only is it the first of a new millennium, but I have also heard for two years in a row from my wayward correspondent, Dr Dai Fwls ap Rhyll. Dr Fwls, as experienced readers of *Function* will know, makes a habit of challenging established mathematical wisdom. He has a tendency to raise challenges, and not to explain his background or his motivation; for this reason he has had a hard time of it having his work taken seriously by the mainstream media and the mathematical establishment.

He has also proved to be a most erratic correspondent. Sometimes I hear from him; all too often not.

However, he has once more written and I would like to share his latest conundrum. His letter comprised a short challenge and a sealed envelope.

The letter read:

"Solve

$x^6 - 21.015x^5 + 175x^4 - 735x^3 + 1624x^2 - 1764x + 720 = 0$."

That was all. No mention of how I was supposed to carry out this calculation. So, feeling rather lazy, I reached for the package *Scientific Notebook*, which gave me the answers: 0.99988, 2.0216, 2.7908, 6.3236 and two complex numbers, $4.3396 \pm 0.85603i$. I did check these and, within the quoted accuracy, they worked.

This left me wondering why he had bothered to send me such an apparently pointless problem. Then I thought to open the sealed envelope, in which was another letter that pointed out that the equation was the same in five of its six coefficients as the simpler equation

$$x^6 - 21x^5 + 175x^4 - 735x^3 + 1624x^2 - 1764x + 720 = 0.$$

Even the sixth of the coefficients differed by only 0.7% from one another. Clearly the very similar equations should have very similar solution sets. But now came the crunch!

The left-hand side of the second (simpler) of the two equations factorises as:

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$$

and so the roots of *this* equation are obviously 1, 2, 3, 4, 5 and 6.

I checked the factorisation twice by hand (using two different methods) and twice by machine (again using two different methods) and there is no question that it is correct.

But ...

Of the six roots of the first equation, the first two seem to lie, as we should expect, close to the corresponding values arising from the second equation, but after that, things go wildly wrong. 2.7908 is only very crudely approximated by 3, and 6.5236 is even less accurately approximated by 6. (At this level of approximation, 7 would be better!) And then there are those complex numbers! One ought to be a real number near 4 and the other a second real number near 5. As it is, both lie well off the real axis and well away from either of the relevant roots of the second equation.

As is usual with him, Dr Fwls has again successfully put a cat among the mathematical pigeons.

Kim Dean
Erewhon-upon-Yarra.

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Erratum: In the last issue of *Function*, we omitted the author of the Letter to the Editor which suggested an alternative construction for $\sqrt{x}$. The letter was sent by our regular contributor Julius Guest. We apologise for the omission.

* * * * *

I've read that things inanimate have moved,
And, as with living souls, have been inform'd,
By magic numbers and persuasive sound.

—Richard Congreve in *The Morning Bride, Act 1, Scene 1*

HISTORY OF MATHEMATICS

The Millennium Question

Michael A B Deakin, Monash University

I am going to use this column to take issue with my friend and colleague, Peter Grossman and his article in *Function, Vol* 23, *Part* 5 (Oct '99). Peter argued the case for saying that the 20th century and the second millennium will both actually end at midnight on December 31st, 2000, and did not end at the end of calendar year 1999, as most public celebrations had it.

Essentially his argument ran like this: that if (say) a baby turns 1, then this means that its birth took place one year ago from the day when the birthday is celebrated. We don't celebrate our 18th birthday until we have truly been around for a full 18 years; it would be premature to hold *that* birthday party at the start of our 18th year of independent life!

Now the usual calendar began by celebrating the birthdays of a historical figure, Jesus of Nazareth, seen by many as the founder of the Christian religion and so also called Jesus Christ. Thus, we see the year 2000 as 2000 AD, where AD stands for *Anno Domini* (meaning "Year of Our Lord", and thus signifying that *at the end of it* exactly 2000 years will have elapsed since Jesus' birth).

[In an increasingly secular (and culture-sensitive) age, we often today replace the letters AD by CE, which stands for "common era", and implies that by mutual consent, we number our years in this way, but ascribe no particular significance to the actual number thus generated.]

The calendar so generated was however the product of a Christian culture reacting to the previous custom of using the Roman method of dating. That reaction was quite a long time coming about. In 463 AD, Pope Hilarius appointed Victorinus of Aquitaine to sort out the date of Easter (see my column in *Function, Vol* 17, *Part* 4). The date of Easter depends on both the motions of the sun and the moon, each following different cycles. The combined motions occupy (as a good approximation) 532 years. In the 6th century AD (i.e. the one with numbers 5xx), Dionysius Exiguus (Dionysius, or Dennis, the Small) proposed a simplification and gave the year we now call 532 AD exactly that name. The first year of the Great Paschal (Easter) Cycle was thus termed 1 AD and the last year of the great Paschal Cycle was thus 532 AD.

If this was intended to reflect accurately the year of Jesus' birth, then it encountered two problems. The first was the Little Dionysius failed to include a year 0. His calendar (and in consequence ours) went straight from 1 BC ("before Christ") to 1 AD. The second problem was that he had the date wrong in any case. Matthew's Gospel has Jesus being born during the reign of King Herod the Great, but it is now known that Herod who died in the year we now call 4 BC. So Jesus was born in either that year or quite possibly even earlier.

[The other Gospel that recounts Jesus's birth is Luke's. Luke dates it to a census undertaken by the Roman Emperor Augustus. This census was however carried out in the year we now call 6 AD. Luke has it embracing "the whole world", which is wrong. It was a local affair confined to the province of Judea, and did not even extend to Galilee, where Jesus is said to have been born. For this reason, Matthew's is seen as the more accurate account, and a date of no later than 4 BC is preferred.]

Dionysius' calendar received popular acceptance with its use by the theologian and church historian, Bede of Jarrow (St Bede, or the Venerable Bede). Bede was born in either 672 or 673 and died in 735 (all AD of course!). He wrote a number of books and is one of the most significant writers of Anglo-Saxon England. He propounded the use of Dionysius' system in two books: *de Temporibus* ("On Time", 703) and an expanded edition of this same work, *de Temporibus Ratione* ("On the Reckoning of Time", 725). More influentially, he used this new system in his histories of Britain and its conversion to Christianity. Bede was actually aware that the base-date (the "origin") of the system he used was wrong, but he did nothing about this, and nor have those who followed him, even including ourselves.

The other question, but a less publicised one, was the question of when each calendar year began. January 1st is actually the Roman convention. Janus (after whom it is named) was a figure of Roman mythology, who faced both forward and back, as perhaps we tend to do as a new year begins. The Christians however adopted a different convention. The Gospels tell us that Jesus was circumcised on the 8th day of his life, and so his life (in the sense of his acceptance into the community of Jews among whom he was to live) began then. The date of his circumcision was made to coincide with the start of the Roman year and so, counting backwards, they celebrated Jesus' birth on December 25th. Clearly therefore there is no historical warrant for this date, but nonetheless, counting back further through a presumed 9-month pregnancy, gives us March 25th as the date on which he was conceived.

[Jesus's conception is held by most Christians to have been miraculous, so that his mother Mary was not inseminated in the normal way, but owed her pregnancy to the direct intervention of God. The relevant doctrine is that of the Virgin Birth – *not* that of the Immaculate Conception, with which it is sometimes understandably confused. The happy event was supposedly announced to Mary by the Angel Gabriel, and so is called "the Annunciation".]

Devout Christians therefore took the Feast of the Annunciation (or "Lady Day") as the start of the calendar year, although this made for confusion with the year starting mid-month, so to speak. As late as the 1660s, the English diarist Pepys used both January 1st and March 25th as the dates on which each new year began, though he used the different conventions for different purposes. (We in Australia begin each new financial year on July 1st, so perhaps this should cause no real surprise.)

When we come to smaller divisions of time, the same arbitrariness prevails. We in Victoria celebrated the New Millennium when we did because we embrace "daylight saving", and so we differ from Queensland, which does not. This meant that Brisbane celebrated the New Millennium an hour after we did. (But had the New Year fallen in Winter, as it does in the Northern Hemisphere, then this discrepancy would not have appeared.) On an even smaller scale, although we do have standard times, individual TV stations and other such public timekeepers do not always follow them. Thus Coober Pedy beat the rest of the nation to the New Millennium by 10 seconds, because they went by a clock fast by this amount. More spectacularly, Brisbane went by a clock 30 seconds slow! (Was I the only person to notice this?)

So here in summary is the situation. There were an unknown number of years (but about a thousand of them) to which we now assign numbers with fewer than 4 digits. There were 1000 identifiable years with dates of the form 1xxx. And there will be a further 1000 years with dates of the form 2xxx (all going well!).

Now focus on the first of these three eras. Of the two problems we have seen, let's look again at the first: the lack of a zero. In fact we have lots of other instances of this same method of numbering. While we Australians, like good little Vegemites, by and large still number the floors of our buildings as "Ground", 1, 2, etc, Americans don't. They use ordinal arithmetic and have the first floor (the one we enter), the second floor (the one above this), and so on. We Melburnians do something similar with our tramstops.

So, we have just entered the 2000th year, and (forgetting for a while the mistake in identifying correctly the year of Jesus' birth) this is cause for celebration. In consequence, the new dates "look different" from the old ones, even if there were only (on a literal and likely incorrect reading of events) 1999 of these, not 2000.

And now look at the question from another angle. We had 100 years with dates of the form 19xx, and if we count years in integers, then this number is exact. But did this century actually last precisely 100 years? Almost certainly not. Individual differences in time-keeping could put the duration out by an hour or so quite easily, and likely did.

Something similar happened (but on a somewhat more drastic scale) with the last millennium. The old Julian calendar was slightly incorrect. (See *Function*, *Vol* 17, *Part* 4, p 104.) At various times in various countries, some ten or more days were simply deleted from the calendar. Thus the last millennium was shortened because the previous counts (incorporating errors made in both the 1st millennium and the 2nd) were out of line.

Then is all this so very different from having a first millennium that may have been a year or so short of the full 1000? I don't see why it should be. And if, on one view, that first millennium was only 999 years long, on others it could have been 1004 years. And then, after all, there *are* other numbers in between. The claim of there being a year's discrepancy in the count pales beside the other known sources of possible error.

Let's now turn to the last time there was a New Millennium. Was this celebrated at the start of the year 1000 or was it 1001?

Well, both actually, and other dates beside.

The most quoted figure from this time is a Burgundian monk called Rodulphus Glaber. Around the year 1000 (but both before and after it!), Glaber made a noise preaching (essentially) "Repent, for the end of the world is nigh". Almost certainly he was not alone in this, but it is his words that have been most thoroughly documented.

Now most readers will be aware that Glaber was hardly the last person to preach such a message. More recently there have been several movements of like mind; the best known of these is the Seventh Day Adventist Church. In one early manifestation among several, this cult held that the world would come to an end on October 22nd, 1844. This clearly has not taken place as initially understood, and so nowadays this date is accorded a more mystical and spiritual significance.

All such movements take their inspiration from the Book of Revelation (also called "The Apocalypse") – the final book of the New Testament. In Chapter 20 of that work it is prophesied that souls of the just will live and reign with Jesus Christ for a thousand years. The words have been variously interpreted, but a common gloss has it that after this thousand years, a time of struggle will ensue ("Armageddon") and that following the successful outcome of this battle between right and wrong, good will triumph.

Clearly, the words, even if one tries to take them literally, lend themselves to various interpretations. How will we know when the thousand years begin and end? How long may we expect Armageddon to last? Has it yet begun? Etc.

It is now widely accepted that the early Christians, and even Jesus himself, thought the end of the world was approaching very fast. This is most clearly evident in the Epistles of the Apostle Paul, but the most widely quoted passages come from Matthew's Gospel. Chapter 24 of that work details the events to occur just before the end of the world. However they were very quickly applied instead to the sack of Jerusalem by the Romans in the year 70 AD. (Which almost certainly occurred *before* Matthew's Gospel attained its final form.) Face-saving has a long history!

So there is plenty of room for people to begin and end the thousand year period at all sorts of dates! Those movements that make such beliefs central to their religions are termed *millenarian* and this term embraces a wide variety of possible positions.

[And here note another piece of trivia: while the words *millennium* and *millennial* each have a double *n* in their spelling, *millenarian* does not. The reason for this apparent anomaly is explained in a footnote to Gould's book (detailed in the "Further Reading" section of this article). Just occasionally Gould's editors let him down and get this matter wrong, and they are hardly alone in this!]

Anyhow, Glaber seems to have had several tries at getting the matter right. One was to defer the end of the world till 1033AD because it was commonly thought that Jesus died at the age of 33 and so the count ought to begin at 33AD. All in all, Glaber had quite a time of it. It helped of course that back then the "average man in the street" (or in the fields) had little need for precise counting of the years, and quite likely had no idea of the exact date.

Now come to a more recent time: the end of the 19th Century. Was this held on December 31st, 1899 or on December 31st, 1900? Well in this case the latter date was a clear winner. It is a matter particularly pertinent to Australia, because our federation was timed to begin on January 1st, 1901, "the first day of the twentieth century". Back then, the consensus was that espoused by Peter Grossman in his analysis of the present question.

However, even so, there were dissenting voices. Kaiser Wilhelm II of Germany was one, but not even his official statement on the matter convinced his own nation. Sigmund Freud was another, and so was the physicist Lord Kelvin. Influential as these voices were, they failed then to carry the day. But *this* time round, the vast weight of public opinion is otherwise. Despite the voices of reason raised by Peter and others of like mind, the ordinary folk "just went ahead anyway". Gould puts this down to the rise of this century's respect for "popular culture", as opposed to the prevailing elitism of 100 years ago. He may well be right.

However, he's certainly right on another point: the way to start a good ding-dong row is to have people take sides on a question that is ultimately undecidable, or even meaningless!

Further Reading

Two general books are readily accessible and I have used both. Stephen Jay Gould's *Questioning the Millennium* is an excellent read and is in the shops as I write. (Gould incidentally is one of the few "educated" authors to agree with the position I adopt in this article.) Also around is *The Year 1000* by Robert Lacey and Danny Danzinger, recently reissued in paperback for the occasion. A more scholarly account is that by Henri Focillon, also called (in its English translation) *The Year 1000*. Glaber's work has been translated into English and edited and is available as *Rodvlfi Glabri Opera* (published by the Clarendon Press, Oxford in 1989). For questions on the dating of events, see the article *Calendar* in the *Macropedia* section of the *Encyclopedia Britannica*.

* * * * *

The theory of numbers is particularly liable to the accusation that some of its problems are the wrong sort of questions to ask. I do not myself think the danger is serious; either a reasonable amount of concentration leads to new ideas or methods of obvious interest, or else one just leaves the problem alone. "Perfect numbers" certainly never did any good, but then they never did any particular harm.

—J E Littlewood
in *A Mathematician's Miscellany*, Methuen Co. Ltd., 1953.

COMPUTERS AND COMPUTING

A wealth of mathematical resources

Cristina Varsavsky

For many people, browsing the net is increasingly becoming the starting point in the search for information of any kind—academic and mundane alike. On the Web you can find the latest joke about the President of the United States, as well as the latest scientific discovery. And there is also a wealth of mathematical resources. I include here some of my favourites.

- American Mathematical Society–Mathematical reviews on the Web:
  http://www.ams.org/mathscinet

- *Virtual Reality Polyedra* by George Hart:
  http://www.georgehart.com/virtual-polyhedra/vp.html

- *The Mathematical Atlas–A Gateway to Modern Mathematics*: This is a collection of short articles designed to provide an introduction to the areas of modern mathematics and pointers to further information, as well as answers to some common (or not!) questions. The present Mathematical Atlas evolved from a much more informal collection of mathematical pointers collected by Prof. Dave Rusin of Northern Illinois University.
  http://www.math-atlas.org/

- Australian Mathematics Trust, with the latest news about the Australian Mathematical Olympiads, as well as pointers to the latest publications.
  http://www.amt.canberra.edu.au/AboutAMT.html

- *New Zealand Maths* which includes many problem solving, statistics and probability, and measurement activities for children.
  http://www.nzmaths.co.nz/

- Erik Weisstein's *World of Mathematics*
  http://mathworld.wolfram.com/

- *MathGym*, and introduction to mathematical problem solving.
  http://www.mathgym.com.au/

- *Math Goodies*, interactive mathematics lessons with a problem solving approach: http://www.mathgoodies.com/

- World Mathematical Year 2000, where you will find agendas and project descriptions of this international series of events supported by the UNESCO. http://www.math.jussieu.fr/~jarraud/wmy2000/ma2000.html

- *ExploreMath*: a suite of highly interactive mathematics activities for students and educators. http://exploremath.com/

- *Mathematics World Wide Web library*: a collection of resources maintained by the Department of Mathematics of Florida State University. http://euclid.math.fsu.edu/Science/math.html

- The Gallery *maths online*, for senior secondary students and teachers. http://www.univie.ac.at/future.media/moe/galerie.html

- A catalogue of Isohedral Tilings by Symmetric Polygonal Tiles. http://forum.swarthmore.edu/dynamic/one-corona/

- *Maths Net*, with puzzles, resources and many interesting links. Created by Brian Dye. http://www.anglia.co.uk/education/mathsnet/

- *Mathsearch*, a tool for searching mathematical content on the Web. http://www.maths.usyd.edu.au:8000/MathSearch.html

- Fibonacci numbers and the golden section. http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html

- *Cut the knot!* An interactive column using Java applets by Alex Bogomolny http://www.maa.org/editorial/knot/knot-index.html

- Suzanne's mathematical lessons, with interactive materials on crystals, fractals, factoring, magic squares, tessellations, circle designs, etc. http://forum.swarthmore.edu/alejandre/

I invite readers to send their favourite mathematical bookmarks. We promise to include an updated list every now and then, acknowledging your contribution. Our e-mail address is function@maths.monash.edu.au.

PROBLEM CORNER

PROBLEM 23.5.1 (from Mathematical Spectrum)

Let the complex numbers $a, b,$ and c correspond to points $A, B,$ and C in the Argand plane. Find an equation for the bisector of angle BAC.

SOLUTION (Keith Anker)

The bisector of any angle of a triangle divides the opposite side in the ratio of the two adjacent sides. Hence a parametric equation for the bisector of angle BAC is $z = [\lambda(b-a) + \beta(c-a)]t + a$ where $\lambda = |c-a|$, $\beta = |b-a|$ and t is a real valued parameter.

Solutions were also received from Carlos Victor and Julius Guest.

PROBLEM 23.5.2 (from Crux Mathematicorum with Mathematical Mayhem)

The fraction $\frac{1}{6}$ can be represented as a difference in the following ways:

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3}; \quad \frac{1}{6} = \frac{1}{3} - \frac{1}{6}; \quad \frac{1}{6} = \frac{1}{4} - \frac{1}{12}; \quad \frac{1}{6} = \frac{1}{5} - \frac{1}{30}.$$

In how many ways can the fraction $\frac{1}{2175}$ be expressed in the form

$$\frac{1}{2175} = \frac{1}{x} - \frac{1}{y}$$

where x and y are positive integers?

SOLUTION (Carlos Victor)

The equation $\dfrac{1}{2175} = \dfrac{1}{x} - \dfrac{1}{y}$ can be re-arranged as $(2175 - x)(2175 + y) = 2175^2$. Let $a = 2175 - x$, $b = 2175 + y$; since x, y are positive integers we see that a and b are divisors of 2175^2, and $0 < a < 2175$. By considering the factors of 2175 we find that 2175^2 has 45 positive divisors, one of which is its square root 2175. Also the other divisors of 2175 come in pairs , one less than 2175 the other greater. Exactly half of the remaining 44 factors are less than 2175 so this gives 22 solution pairs (x, y).

Solutions were also received from Keith Anker and Julius Guest.

PROBLEM 23.5.3 (from Crux Mathematicorum with Mathematical Mayhem)

Let $a = \sqrt[1992]{1992}$. Which number is greater,

$$a^{a^{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}}\ \text{1992 times} \qquad \text{or} \qquad 1992?$$

SOLUTION (Carlos Victor)

Let n be any positive integer greater than 1 and let $a = \sqrt[n]{n}$, then $a < n$ and also $a^a < a^n$, so that $a^a < n$. But then $a^{a^a} < a^n$, so $a^{a^a} < n$.

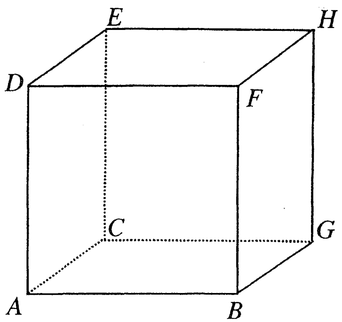
Continuing in this manner we arrive at $a^{a^{a^{\cdot^{\cdot^{a}}}}} < n$, for a tower of exponentials of any height . Taking the height as n with $n = 1992$ gives the desired result that 1992 is the greater number.

A solution was also received from Keith Anker and a numerical solution was received from Julius Guest.

PROBLEM 23.5.4 (from Mathematics and Informatics Quarterly)

A spider crawls randomly around the edges of a cube. At any vertex it moves to an adjacent vertex with probability $\frac{1}{3}$. The spider starts at one vertex. What is the expected value for the number of moves it will take the spider to reach the opposite vertex?

SOLUTION



Label the vertices of the cube as in the figure. Suppose the spider starts from A and on reaching H stops and goes to sleep. In one step from A he arrives at B or C or D each with probability $\frac{1}{3}$. On his second step, he can go from B to F or G or return to A, and similarly from C or D. So after 2 steps he is at E or F or G each with probability $\frac{2}{9}$ or at A with probability $\frac{3}{9}$.

In his third step he can go from E to H or return to E or D and similarly from F or G, and from A he can go to B or C or D as in the first step. So after 3 steps he is at B or C or D with probability $2 \cdot \frac{2}{9} \cdot \frac{1}{3} + \frac{3}{9} \cdot \frac{1}{3} = \frac{7}{27}$ each or at H with probability $3 \cdot \frac{2}{9} \cdot \frac{1}{3} = \frac{6}{27}$. H now drops out of this calculation, hence he starts his fourth step from B to C or D the same as at the beginning of the second step but with probability $\frac{7}{27}$ instead of $\frac{1}{3}$. So the situation after 5 steps is the same as after 3 steps but with probabilities multiplied by $\frac{7}{9}$, that is at B or C or D with probability $\frac{49}{3^5}$ each or at H with probability $\frac{42}{3^5}$. Similarly the probability of arriving at H in 7 steps is $6 \cdot \frac{7^2}{3^7}$ and in general in $2r + 1$ steps.

$$6 \cdot \frac{7^{r-1}}{3^{2r+1}} = \frac{2}{9} \cdot \left(\frac{7}{9}\right)^{r-1}$$

The expected number of steps $(\sum np_n)$ is therefore

$$E = \frac{2}{9} \cdot \left[3+5 \cdot \frac{7}{9}+7 \cdot \left(\frac{7}{9}\right)^2 + \ldots + (2r+1)\left(\frac{7}{9}\right)^{r-1} + \ldots \right].$$

Let $\quad S = 3+5x+7x^2+9x^3+\ldots+(2r+3)x^r+\ldots,$

Then $\quad xS = 3x+5x^2+7x^3+\ldots+(2r+1)x^r+\ldots,$

and $\quad x^2S = 3x^2+5x^3+\ldots+(2r-1)x^r+\ldots.$

Now $\quad (2r+3)-2(2r+1)+(2r-1) = 0,$

Hence $\quad S-2xS+x^2S = 3-x,$

$$S = \frac{3-x}{(1-x)^2}.$$

Finally, noting that $E = \frac{2}{9}S'$, with $x = \frac{7}{9}$.,

$$E = \frac{2}{9} \cdot \frac{3-\frac{7}{9}}{(1-\frac{7}{9})^2} = 10.$$

A similar solution was received from Keith Anker.

PROBLEM 23.5.5 (Julius Guest, East Bentleigh)

Determine

$$\int \frac{dx}{(1+\frac{1}{4}\cos x)^2}$$

SOLUTION (J A Deakin)

The substitution $t = \tan(\frac{x}{2})$ leads us to the integral

$$\begin{aligned} K(x) &= 32\int \frac{1+t^2}{(5+3t^2)^2}dt \\ &= \frac{32}{3}\int \frac{dt}{5+3t^2} - \frac{64}{3}\int \frac{dt}{(5+3t^2)^2} \end{aligned}$$

The first integral on the right is routine and we obtain

$$K(x) = \frac{32}{15}\sqrt{\frac{5}{3}} \quad \operatorname{artan}\left(\sqrt{\frac{3}{5}}\tan(\frac{x}{2})\right) - \frac{64}{3}I$$

where $I = \int \frac{dt}{(5+3t^2)^2}$. To evaluate I let $\sqrt{3}t = \sqrt{5}\tan\theta$, and we find that

$$I = \frac{1}{10\sqrt{15}}\left(\theta + \frac{1}{2}\sin 2\theta\right) = \frac{1}{10\sqrt{15}}\operatorname{artan}\left(\sqrt{\frac{3}{5}}t\right) + \frac{1}{10}\left(\frac{t}{5+3t^2}\right).$$

Collecting up terms finally results in

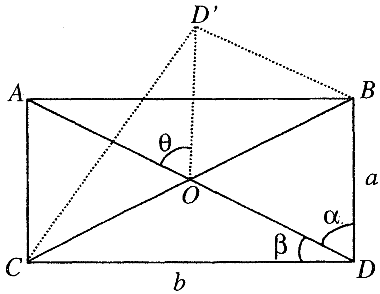
$$K(x) = \frac{128}{15\sqrt{15}} \quad \operatorname{artan}\left(\sqrt{\frac{3}{5}}\tan(\frac{x}{2}) - \frac{32}{15}\,\frac{\tan\left(\frac{x}{2}\right)}{5+3\tan^2\left(\frac{x}{2}\right)}\right) + C$$

Solutions were also received from J.C.Barton, Keith Anker, Carlos Victor and the proposer.

PROBLEMS

PROBLEM 24.2.1 (H C Bolton, Melbourne)

The figure below shows a rectangle $ABCD$ and its corner D folded about the diagonal BC into the position D'. The rectangle has sides of length b, a with $b > a$. Express $\tan\theta$ in terms of a, b.



PROBLEM 24.2.2 (from Crux Mathematicorum with Mathematical Mayhem)

In a quadrilateral $P_1P_2P_3P_4$ suppose that the diagonals intersect at the point $M \neq P_i$ and points P_2, P_4 lie on opposite sides of the line P_1P_3. Let the angles $MP_1P_4, MP_3P_4, MP_1P_2$ and MP_3P_2 be $\alpha_1, \alpha_2, \beta_1$ and β_2 respectively.

Prove that

$$\frac{|P_1M|}{|MP_3|} = \frac{\cot\alpha_1 \pm \cot\beta_1}{\cot\alpha_2 \pm \cot\beta_2}$$

where the $+(-)$ sign holds if the line segment P_1P_3 is located inside (outside) the quadrilateral.

PROBLEM 24.2.3 (from Crux Mathematicorum with Mathematical Mayhem)

Find all solutions to the inequality

$$n^2+n-5 < \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n+2}{3} \right\rfloor < n^2+2n-2,$$

where n is a natural number and $\lfloor x \rfloor$ denotes the least integer not exceeding the real number x.

PROBLEM 24.2.4 (from Mathematical Spectrum)

Evaluate the improper integral

$$\int_a^b \left(\frac{x-a}{b-x} \right)^{\frac{1}{2}} dx$$

PROBLEM 24.2.5 (from Mathematics and Informatics Quarterly)

In decimal notation , suppose that $\overline{cabadb} = (\overline{ab})(\overline{acbba})$ where a, b, c, d are all different natural numbers. Find a, b, c, d. [Note: $\overline{abc}$ denotes the number made up with the digits a, b, and c.]

* * * * *

> The mathematics, then, is an art. As such it has its styles and style periods. It is not, as the layman and the philosopher (who is in this matter a layman too) imagine, substantially unalterable, but subject like every art to unnoticed changes form epoch to epoch. The development of the great arts ought never to be treated without an (assuredly not unprofitable) side-glance at contemporary mathematics.
>
> —Oswald Sprengler, in *The Decline of the West.*

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