Function

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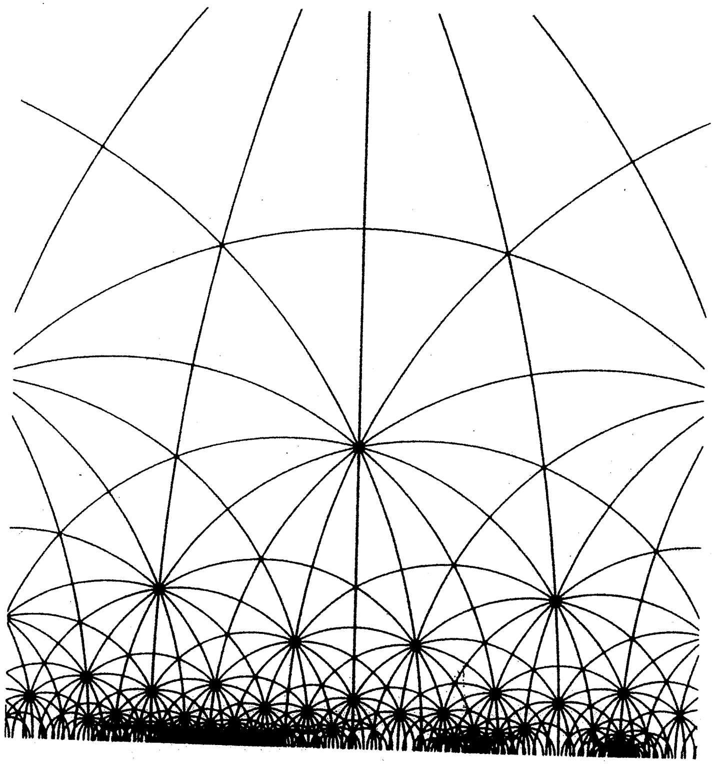

Department of Mathematics - Monash University

Function is a refereed mathematics journal produced by the Department of Mathematics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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*\$10 for *bona fide* secondary or tertiary students.

EDITORIAL

Welcome to this issue of *Function*! We hope it contains much that will interest you.

The figure on the front cover is related to the leading article written by our Honours student Selena Ng about non-euclidean geometries. Selena gives a very intuitive introduction to the geometry of the hyperbolic plane, a geometry in which parallels through a point are not unique, and in which the sum of the angles of a triangle is less than π radians. What you see on the front cover is a tessellation of the plane with hyperbolic triangles. There is much stimulating reading in this article.

The second feature article is a contribution from Rik King. The author poses the practical problem of finding the base distance between two leaning ladders. The problem is modelled with a polynomial equation of degree 8, and gives rise to the question of where the spurious solutions come from.

In the *History of Mathematics* column, our regular columnist Michael Deakin explains how in some cases two mathematicians think the same thought, but without knowing about each other. He illustrates this with a few classical problems and theorems which in each case were discovered independently by two mathematicians.

The Internet is a popular topic these days. In the *Computers and Computing* column of this issue you will find the serious limitations this medium currently has for the communication of mathematics, and the latest news about how the problem is being solved.

Many pages of this issue of *Function* are devoted to the *Problem Corner*, giving solutions to previous problems and presenting new ones. We are very pleased to see so many readers contributing to this section of our journal.

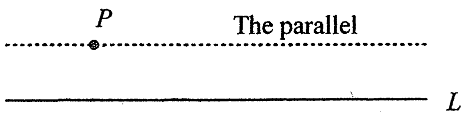
Happy reading!

* * * * *

A NON-EUCLIDEAN GEOMETRY

Selena Ng, Monash University

Many of you would already have met the Euclidean plane and its geometry (even if you didn't know it!). Triangles, circles and squares as we know them are all defined in the Euclidean plane, in which the angles of a triangle add up to π radians. Something else we also know, and use probably without thinking, is that there is a *unique parallel* to a given line through each point in the plane. That is, for every line L and point P not on L, there is *only one* line through P that never intersects L:



The Euclidean plane is named after the Greek mathematician Euclid (325 BC) who was one of the first to attempt to formalise the foundations of geometry in his work *Elements.* One of his postulates essentially stated that parallels are unique, and for 2000 years mathematicians attempted to prove this parallel postulate, and failed.

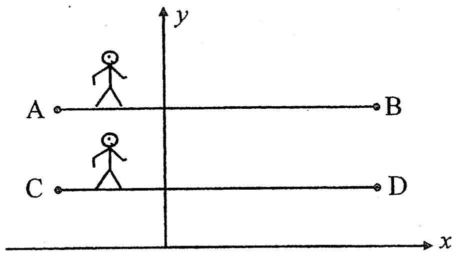
But in the 1820s, the German mathematician Gauss, the Hungarian Bolyai and the Russian Lobachevski began to conceive a new notion. Perhaps it was possible that a geometry other than the Euclidean geometry existed, a geometry where the parallel postulate does not hold, i.e. parallels are not unique (and are in fact infinite in number), and in which the sum of the angles of a triangle is less than π radians. And so was born the concept of a non-euclidean geometry, which has come to be known as hyperbolic geometry[1].

However, this non-euclidean geometry lacked intuitive appeal, and a few decades passed before mathematicians began to take notice and study these new ideas.

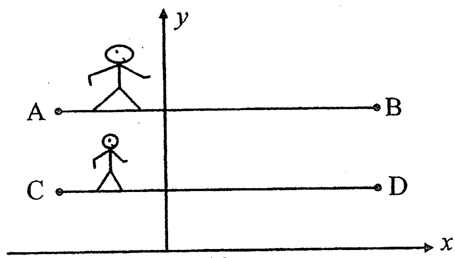
Let us try to gain an intuitive feel for the nature of hyperbolic geometry. Returning to the Euclidean plane for the moment, how do we define Euclidean distance? If we lay out our cartesian coordinates on a flat piece of ground and label points A, B, C, D such that the distance between A and B is equal to the distance between C and D, then it takes us the same time

[1] See also *Function Vol* 3 *Part* 2.

to walk (assuming we walk at the same speed both times!) from A to B as it does from C to D.



What happens in the hyperbolic plane? Now imagine a two-dimensional universe with cartesian coordinates but with an *infinitely cold* x-axis. Then, as the people in this strange land approach the x-axis, they contract so that it takes them *less* time to walk from A to B than it does from C to D! Even stranger, however, is that their rulers also contract as much as they do, so that it seems perfectly normal to them!



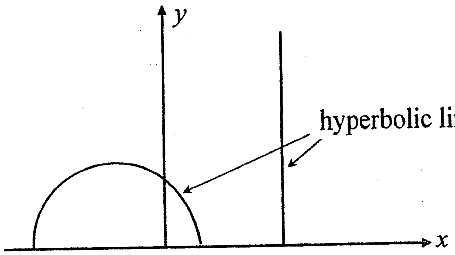
More formally, we can write that

$$\text{Hyperbolic distance} = \frac{\text{Euclidean distance}}{y},$$

where y is the Euclidean distance to the x-axis.

What we have essentially seen here is the French mathematician Poincaré's (1854-1912) model of the hyperbolic plane, which uses the Euclidean upper half-plane ($y > 0$) together with the hyperbolic distance defined above. This definition of distance tells us much about the geometry of the plane: in particular, the nature of the curves of shortest length between any two points in the plane.

For the Euclidean plane, the straight line connecting two points is clearly the curve of shortest length. Now if we take "lines" in the hyperbolic plane to be curves of shortest length, then these hyperbolic "lines" turn out to be (arcs of) semicircles sitting on the x-axis and (segments of) straight lines perpendicular to the x-axis (you can think of these as semicircles with infinite radius).



The Poincaré model allows us to see some of the geometrical features of the hyperbolic plane, and a beautiful way of demonstrating this is via *tessellations.*

A tessellation is a way of covering a surface with non-overlapping congruent polygons, called *tiles*. Let us consider in particular tessellations by triangles with angles $\pi/p, \pi/q, \pi/r$ radians, where p, q, r are integers; we will call these (p, q, r) tessellations. Now how do we generate a tessellation? And how do we determine what sorts of triangles are suitable for tiling the plane?

Let's start with the straightforward Euclidean case. Since the angles of a triangle sum to π radians, we must have

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1,$$

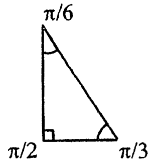
which leads to only three possible triangles

$$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{4}$$
$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}.$$

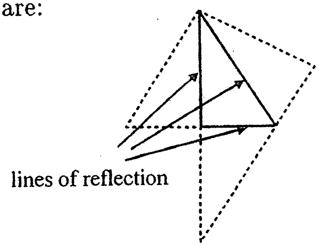
To generate a tessellation, we will assume that triangles can be obtained from each other by reflections in the sides of the triangles.

Take the (2,3,6) triangle as an example.

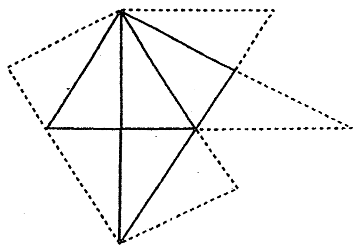
The starting triangle is:



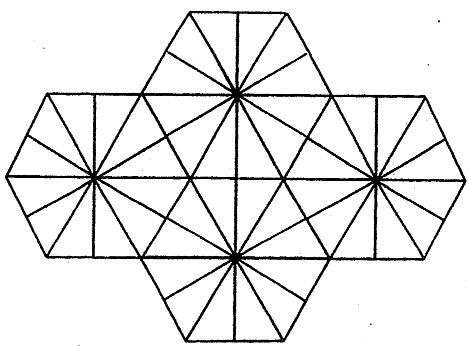
The first reflections are:



Further reflections are:



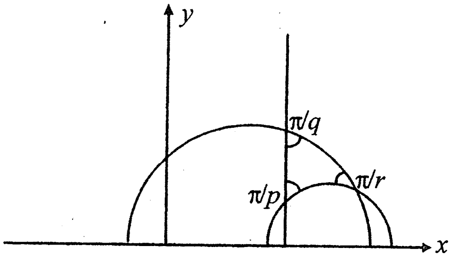
The (2,3,6) tessellation is:



This tessellates the Euclidean plane with hexagons as well as triangles.

Try the (3,3,3) and (2,4,4) by hand. You should find that they tessellate by equilateral triangles and squares respectively.

Now what does a hyperbolic triangle look like? The Euclidean triangle is the result of the intersection of three non-parallel Euclidean lines (excluding infinity as a possible value for p, q or r). So taking the intersection of three non-parallel hyperbolic 'lines', we have a (p, q, r) triangle.

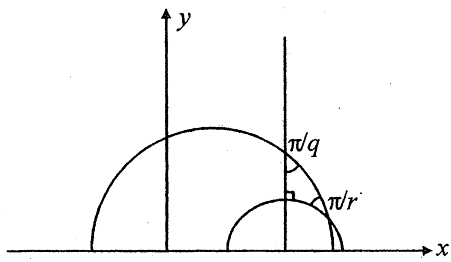


The angles of a hyperbolic triangle add up to less than π radians (measure it!), and so to tessellate the hyperbolic plane we require

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

You can quickly see that there is an infinite number of possible combinations of p, q, r which satisfy this inequality. So there is an infinite number of (p, q, r) tessellations of the hyperbolic plane.

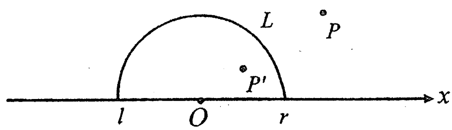
Let's choose as a starting triangle a $(2, q, r)$ triangle, where one of the angles is $\pi/2$ radians.



You can find the coordinates, angles.etc. of the starting triangle by ordinary geometry of circles, but this is quite hard.

Using the same idea as before, we now have to reflect in the lines along the sides of the triangle, but hyperbolic reflection is a little tricky as we have to take account of the hyperbolic distance. A formal definition:

> If L is a hyperbolic line with ends l, r and centre O, then the *reflection in* L of the point P is the point P' on the line $\overline{OP}$ where
>
> $$OP \cdot OP' = (Or)^2$$



Note that if we just have the Euclidean half-line with ends at $y = 0$ and $y = \infty$, then the equation above reduces to normal Euclidean reflection. (Keep r fixed and let l tend to $-\infty$ on the x-axis.)

So as for the Euclidean (2,3,6) tessellation, if we first reflect in the sides of the starting triangle, then keep reflecting in its reflections and so on, we can generate a $(2, q, r)$ tessellation of the hyperbolic plane.

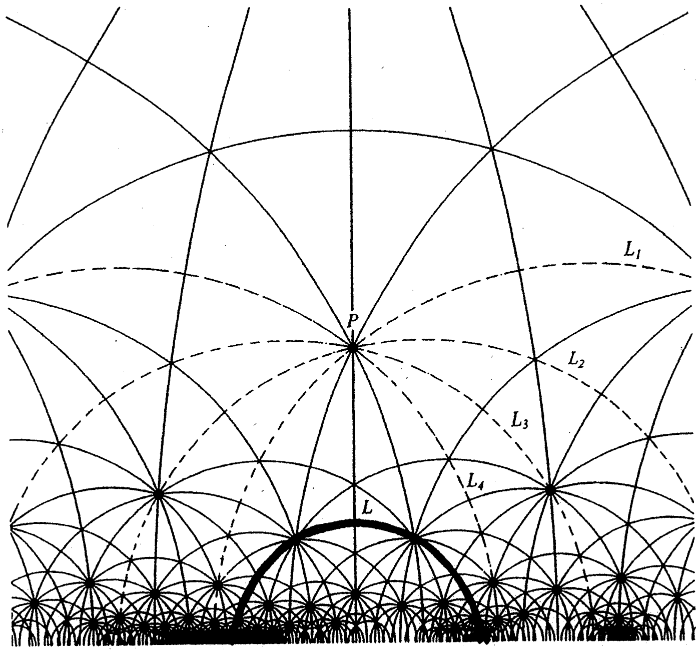
A word of advice: don't try *this* by hand! I wrote a program in C to generate the (2,3,7) tessellation shown on the front cover.

We gain a certain intuition for hyperbolic distance from the Poincaré model. All the (2,3,7) triangles in this tessellation are of equal hyperbolic size (since they have the same angles); however, they become infinitely small in this model as they approach the x-axis.

We can also see in the picture below some examples of non-unique parallels. The line L (thick line), for instance, has four parallels through the point P, namely l_1, L_2, L_3, and L_4 (dotted lines).

There are other models of the hyperbolic plane which you may wish to explore, in particular the Beltrami disc model which appears in many of the works of the Dutch artist M C Escher (see Circle Limit I, II, III, IV in

The Magic Mirror of M.C. Escher by Bruno Ernst). A related article also appeared in *Function Vol 3 Part 4*.



Acknowledgement

My sincerest thanks to Professor John Stillwell for all of his invaluable suggestions.

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Selena Ng is an honours student in the Department of Mathematics at Monash University. She began her university studies when she was 13 years of age. In 1995 she did a special project on non-euclidean computer graphics, which led to the present article. Her recent studies cover a wide range of topics in mathematics and mathematical physics, including an honours project on the question: can the total mass of the universe be negative?

* * * * *

LEANING LADDERS

Rik King, University of Western Sydney

Part of the fascination of mathematics is that, often, what seems to be a fairly simple looking problem gives rise to a complicated equation required for its solution. The famous "cattle problem", said to have been propounded by Archimedes (*circa* 500 BC), for the discomfiture of a rival, Eratosthenes, merely involved numbers of cattle of various kinds and different colours. It came down to solving an equation with two squares, yet the integer needed to satisfy that equation consisted of 206545 digits![1]

Hardly likely to achieve the same fame but, nevertheless, not without its own interest, is the problem following. Consider the situation shown in Figure 1 below:

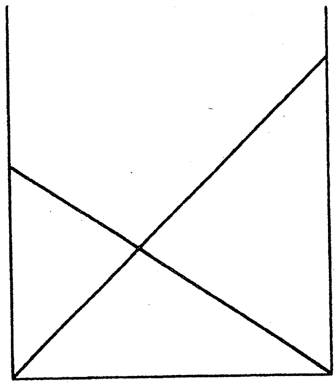

Figure 1

Two ladders of lengths 3m and 4m lean across a narrow path by making angles with vertical walls on each side. Their crossing point is at height 1m above the path – how wide is the path?

Suppose that the width of the path is d. It will be shown below that the final explicit expression from which d may be found turns out to be a polynomial which is of degree 8!

[1]The cattle problem is discussed in *Function Vol* 16 *Part* 3.

To establish the equation for d, we redraw as in Figure 2 below, with points labelled as shown:

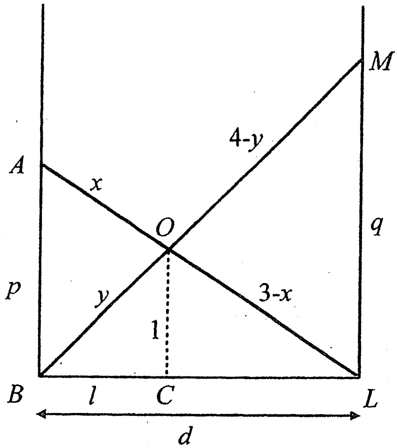

Figure 2

Let

$$\begin{aligned} AO &= x, \text{ so } OL = 3 - x \\ BO &= y, \text{ so } OM = 4 - y \end{aligned}$$

and finally

$$AB = p, \ ML = q, \text{ and } BC = l.$$

Now from this point, it is possible to work out many relationships. Some are shown below, although it is certainly not claimed that the following selection of equations is necessarily the most economical or elegant which might be devised.

From the triangles BCO, BLM the application of Pythagoras's theorem gives:

$$l^2 + 1 = y^2 \tag{1}$$

$$d^2 + q^2 = 16. \tag{2}$$

From the same triangles (which happen to be similar),

$$\frac{1}{y} = \frac{q}{4}. \tag{3}$$

Then, by substituting $y = \sqrt{l^2+1}$ (from (1)) and $q = \sqrt{16-d^2}$ (from (2)) into equation 3, we obtain

$$\frac{1}{\sqrt{l^2+1}} = \frac{\sqrt{16-d^2}}{4}. \tag{4}$$

In a fashion similar to the preceding, triangles LCO and LBA give the following relationships:

$$(d-l)^2 + 1 = (3-x)^2 \tag{5}$$

$$d^2 + p^2 = 9 \tag{6}$$

$$\frac{1}{3-x} = \frac{p}{3} \tag{7}$$

and, following the same procedure as before, using these three equations taken together, we get

$$\frac{1}{\sqrt{(d-l)^2+1}} = \frac{\sqrt{9-d^2}}{3} \tag{8}$$

Now equations (4) and (8) contain only l and d. Since the former is not required, an equation in d may be obtained by making l the subject of each of (4) and (8). Thus, after squaring and inverting each, we get:

$$\sqrt{\frac{16}{16-d^2} - 1} = l = d - \sqrt{\frac{d^2}{9-d^2}} \tag{9}$$

i.e.

$$\sqrt{\frac{d^2}{16-d^2}} = d - \sqrt{\frac{d^2}{9-d^2}}. \tag{10}$$

We notice, after rearranging, that d is a factor of (10), so that we have

$$d\left(\sqrt{\frac{1}{16-d^2}} + \sqrt{\frac{1}{9-d^2}} - 1\right) = 0, \tag{11}$$

whence $d = 0$ may be extracted as a solution of the equation (11), but not, of course, of the original problem!

From this point there are two paths which may be taken. First, if the reader is in the fortunate position of having access to computer algebra packages such as *Derive* or *Mathematica*, he or she will find that the commands soLve or NSolve (respectively) will give the solution of (11) as:

$$d = 2.60329,$$

which finishes the problem there. In fact, the only other solution technique is to use trial and error, beginning with some arbitrary choice for d, bearing in mind that for a starting point, $0 < d < 3$, which may be deduced from an examination of Figure 2.

Second, for the sake of completeness, can we progress to an explicit expression for d, that is, one free of radicals?

Rearranging (11) and squaring gives:

$$\frac{1}{16-d^2} = 1 - 2\sqrt{\frac{1}{9-d^2}} + \frac{1}{9-d^2}, \tag{12}$$

which, after squaring yet again, comes down to:

$$\left(\frac{-151+25d^2-d^4}{(16-d^2)(9-d^2)}\right)^2 = \frac{4}{9-d^2}. \tag{13}$$

Then, after multiplying out, the equation

$$d^8 - 46d^6 + 763d^4 - 5374d^2 + 13585 = 0 \tag{14}$$

appears. This is the degree 8 polynomial in d mentioned at the beginning of the article.

A plot of the polynomial of degree 8 in (14) is shown in Figure 3 below.

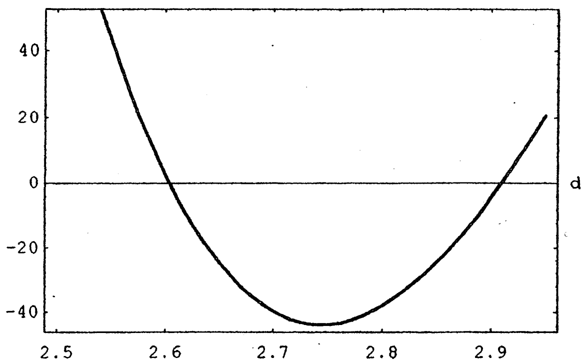

Figure 3

From this figure it may be seen that there are now *two* values of d which are roots of equation (14). We now make use of a computer algebra package to find the values of the roots of (14) more exactly.

Derive initially cannot cope with a polynomial of such high degree; but it may be encouraged to do so by making the substitution

$$d^2 = u,$$

after which the values below follow. *Mathematica*, however, (with the NSolve command) immediately gives all 8 roots, as expected:

$$\begin{aligned} d &= -3.92241 \pm 0.0721765i \text{ (two complex roots)} \\ d &= 3.92241 \pm 0.0721765i \text{ (two complex roots)} \\ d &= \pm 2.60329 \text{ (as previously found)} \\ d &= \pm 2.909072. \end{aligned}$$

Plainly, both the negative values of d are inadmissible for the original problem, but what about the new value of $d = +2.909072$?

Here is a point of interest. This value must satisfy equation (14); it does **not**, however, satisfy the original physical problem.

Can you explain this?

References

1. Ersoy Y, Moscardini A, 1993, *Mathematical Modelling Courses for Engineering Education*, Springer-Verlag, New York.

2. Allenby R B J T, 1989, *Introduction to Number Theory with Computing*, Edward Arnold, Melbourne.

* * * * *

The knowledge I had in mathematics, gave me great assistance in acquiring their phraseology, which depended much upon that science, and music; and in the latter I was not unskilled. Their ideas are perpetually conversant in lines and figures. If they would, for example, praise the beauty of a woman, or any other animal, they describe it by rhombs, circles, parallelograms, ellipses, and other geometrical terms, or by words of art drawn from music, needless here to repeat. I observed in the king's kitchen all sorts of mathematical and musical instruments, after the figures of which, they cut up the joints that were served to his majesty's table.

– Swift Jonathan in *Gulliver's Travels*

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HISTORY OF MATHEMATICS

Thinking Other People's Thoughts

Michael A B Deakin

In my column for this issue, I would like to revisit some of the topics I've discussed in the past, but from a single point of view: that of seeing how it can come about that someone can rediscover the thinking of someone else. In such cases, one mathematician and another mathematician think the same thought, but without either depending on or copying from the other.

Of course, in other areas, say literature, close correspondence of text indicates copying (or plagiarism as it is called), and it is rightly condemned. In all intellectual endeavours, mathematical or otherwise, it is good practice as well as being simple courtesy to acknowledge the contributions of others. Regrettably it is not always followed. However, my various cases to be discussed here are all matters of simple rediscovery. Indeed, for someone *rediscovering* a result it comes as a disappointment to learn that one has been preceded; one likes to be *first*.

I'll take five case studies, all discussed in *Function* previously, but drawing out this aspect of the matter. Four of them are geometric, the fifth is "algebraic".

1. Fasbender's Theorem

In my column for June of last year, I discussed the history of Fasbender's Theorem. In rough terms this says that if a floppy polygon is made from rods held together by hinges at their ends, then the polygon's area is maximised when its vertices (the hinges) are arranged to lie on a circle.

We saw that the case of four rods had been known to the ancient Greeks, and there the matter rested for almost two millenia. Then in 1843, a minor mathematician named Umpfenbach stated the general result and proved the case for five rods. He may have thought this was enough to establish a "pattern" for the general case or he may not have. If he did, then he was wrong. However, in very short order another minor mathematician, Eduard Fasbender, was able to show that the case of four rods did indeed generalise to that of any number of rods.

Fasbender's paper began by giving a very clumsy proof of the case of four rods. It is clearly his own, and equally clearly he was quite unaware that this much had been known from antiquity. The argument by which he extended the case of four rods to the general situation is, however, most elegant. We don't know who named the general results as "Fasbender's Theorem", but whoever it was did give credit where credit was due.

Another proof, also very elegant, is now known. I learned it from a book by Ivan Niven, a contemporary American mathematician. So far I haven't found out who first discovered it; I speculated that it was Niven himself, but I now know this is not the case.[1] Niven doesn't tell us where he got it, quite likely because he had forgotten where he learned of it, or else because he learned of it from someone not the actual originator.

But all this work *assumed that it was possible* to arrange the vertices so as to lie on a circle. Nobody seemed to realise that this was a matter requiring proof, and the proof that finally appeared, though simple to provide, is anything but simple to discover.

The gist of that proof was first given by D S McNab in 1981, but unfortunately McNab didn't consider all possible cases and so his proof required patching up. The first adequate proof was supplied in a book by Z A Melzak, published in 1983. Whether Melzak knew of McNab's work we can't say. He need not have drawn attention to it because, had he done so, he would also have been under the obligation of pointing out its unsatisfactory nature. Some authors decide to handle such matters one way; others the other.

Later Melzak's proof was rediscovered by Professor Chih-Han Sah (just last year). He sent it to an email group of which I am a member (and in direct response to a query of mine). Another member of the group, however, pointed out that Sah had rediscovered the exact same proof that already appeared in Melzak's book. There was no question but that Sah had had the idea, the same idea; he didn't try in any way to pass off Melzak's or McNab's work as his own. Doubtless he was disappointed to learn that Melzak had beaten him to it, but such is life!

2. The Four Circles Theorem

The Four Circles Theorem (also known as the Kiss Precise) was the subject of our cover story in August 1991. Imagine making four circles out of wire. Place three of them on a smooth table top and slide them about until each of the three touches the other two. Then try to place the fourth circle in such a way that it touches (i.e. *just* touches) each of these three. Generally

[1]See the update in my column for February this year.

this will not be possible, but in special circumstances it may be achieved. If the first circle has a radius of r_1, the second r_2 and so on, then the condition is

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}=\frac{1}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2. \tag{1}$$

This theorem was first discovered in 1643 by René Descartes, but it has been rediscovered several times since then.

The first of these rediscoverers was Jacob Steiner, a geometer who revived this branch of mathematics in the early part of last century. We shall have occasion to speak again of Steiner in the course of this article. He rediscovered the Four Circles Theorem in 1826. There is no question of his plagiarising Descartes. He investigated the same problem (and of course came up with the right answer) quite independently.

A few years later, an English amateur mathematician, Philip Beecroft, rediscovered the result, unaware that he had been twice anticipated.

Nor did this end the story, for a very eminent chemist, Frederick Soddy, also found the result in 1936. Soddy had been sickened by the military uses of chemistry in World War I and turned away from that discipline, in which he had advanced to the very highest levels. (He won a Nobel Prize!) One of the areas he turned to was mathematics. His announcement of the Four Circles Theorem took the form of a verse entitled "The Kiss Precise" and was one of two results he published in this unusual manner.

When I first wrote on this, I thought this catalogue was surely complete, but I later learned of even a further rediscovery. A Colonel R S Beard gave the result in 1955. This was certainly an independent discovery (as of course were the others) because Beard missed the particularly elegant form – Equation (1). You can retrieve Beard's form of the result by solving Equation (1) as a quadratic in one of the radii, say r_1.

3. Urquhart's Theorem

Suppose two lines intersect at A. On one line choose two points B and C (B being closer to A); on the other, choose two points D and E (D being closer to A). Join BE and CD, and let the intersection of these lines be F. Then Urquhart's Theorem states:

If $AB+BF=AD+DF$, *then* $AC+CF=AE+EF$.

The late Mac Urquhart was employed at the University of Melbourne and later at the University of Tasmania. He came to this theorem independently in 1964, and although he did not himself publish it (although he was very creative, he published nothing), others saw to it that proofs were published and that Urquhart received the credit. His original proof in fact appeared in *Function* in an article by John Barton, one of our regular contributors. (See *Volume* 2, *Part* 3.) A nicer version was soon produced, better than the original, and this is the one we now use.

For this and more, see my column for October 1990. This will also give the information that another proof may be given as a special case of a theorem discovered in 1860 by the French geometer Chasles. However, Chasles did not go on to make this further deduction.

So for many years it was believed that Urquhart had indeed found a new theorem in elementary geometry. In a sense this was true, but he wasn't after all the first to discover it. We now know that Augustus De Morgan, an English mathematician, published it semi-pseudonymously in 1841. Indeed, De Morgan had in essence the same simplified proof that is now standard.

Shortly after Urquhart's rediscovery (as we now must concede it to be), the Australian Mathematical Society held its meeting in Hobart. Urquhart had by then died but many of his colleagues were still around. One of the leading speakers at that conference was the American geometer Dan Pedoe, and I rather suspect that it was in this context that Pedoe learned of the theorem. Certainly he was attracted to it, wrote further on it and also delved into its history. Among other things Pedoe tells us is the information that a geometer called Yaglom, again independently of Urquhart, and only slightly later came up with a result from which the theorem easily could have been derived, but wasn't. Shades of Chasles!

Thus Pedoe was active in using Urquhart's result as a springboard for further investigation. Urquhart himself, however, rediscovered a result that De Morgan had found earlier (and which Chasles failed to notice). It may be just "the luck of the draw" that Pedoe and others took notice of Urquhart's discovery, whereas an earlier generation overlooked De Morgan's paper.

4. The Steiner Point

Take three points A, B, C. We seek a fourth point P so situated that the total distance $AP + BP + CP$ is mimimised. To fix matters, think of a triangle ABC with P as some point in its interior (or perhaps on its boundary).

This problem was discussed in *Function* in the October 1987 issue. First let me tell you the answer, because the main purpose of this present discussion is to look at a rather complicated history of discovery. The answer comes in two parts.

> If there is a vertex of the triangle for which the angle is 120° or more (of course there can be at most one such), then that vertex is the required point.
>
> Otherwise, there is a unique point P inside the triangle for which the angles APB, BPC, CPA all equal 120° and that is the point we want.

This problem was considered by Steiner in a pair of discussions (1835, 1837) and is often attributed to him in the belief that he was the first to consider the matter. In Steiner's honour the point P is sometimes referred to as the "Steiner point" of the triangle ABC.

A particularly elegant analysis of the Steiner point was provided by Hugo Steinhaus early this century. Steinhaus imagined a table with holes drilled in its top at points corresponding to A, B, C. Strings were supposed to be passed through these holes. Above the table they were knotted together; below they supported equal weights. This system adjusted itself so that it came to rest with the knot at the Steiner point.

One of the more remarkable discoveries that came to light after I had written on this matter was the unearthing of an obscure memoir by two French mathematicians, Lamé and Clapeyron. These two by at the latest 1829 had had both the Steiner and the Steinhaus ideas. But it seems that nobody noticed their work. For more on this matter, see my column for February 1990.

But recently I came on yet another discussion of the matter. You may be interested to hear how I found it. In *Function* recently (February 1997) I published an article jointly authored with Otto Steinmayer on the flight of arrows. Gordon Troup, one of our readers and occasional contributors, was taken by a footnote in that article and this led to some collaborative research by Gordon and me. In the course of that work we came across a series of articles by the nineteenth century physicist P G Tait analysing the flight of not arrows but golf balls.[2]

[2]Tait was, as well as being a physicist, a passionate golfer. His son Freddie was more than this, a champion. His career was cut short by his death in 1899 in the course of the Boer War.

While pursuing this work, we came across a brief paper by Tait on (you've guessed it) the Steiner problem. Tait in fact traces the origins of the problem back even further than the early 1800s. To Fermat, in fact; that is to say, some 200 years earlier. Tait then went on to solve the problem himself, although he dutifully documented the solutions he had already read.

But Tait wanted to use quaternions to do the work of solution. This would have been a quite natural thing for Tait to do, as Tait was the most enthusiastic follower of Hamilton's newly invented quaternions.[3] But next comes a surprise. "The quaternion investigation at once suggests the following kinematical solution of the problem." And there follows the Steinhaus solution, which we know to have already been thought of by Lamé and Clapeyron.

So this marvellous piece of lateral thinking was the independent product of not merely two but *three* separate investigations!

5. Thermodynamics and Inequalities

After all this geometry, some algebra. The result to be discussed is this.

> Let $a, b, c, \ldots$ be positive quantities, n in number. Then we may form the arithmetic mean A defined by $A = \dfrac{a+b+c+\ldots}{n}$ and the geometric mean G defined by $G = \sqrt[n]{abc\ldots}$. Then $A \geq G$.

A good proof of this result can be rather hard to find, but we reproduced one in *Function* (February 1984). It is the work of Ivan Niven, whom we met in Part 1 of this article. We also gave a demonstration (rather than a strict proof) in our issue for February 1981. This proceeds as follows.

> Consider n objects, identical in all respects except for their temperatures, which we will suppose to be $a, b, c, \ldots$. Put all these objects inside a thermally sealed calorimeter so that no heat can get in or out. Then the objects exchange heat among themselves and eventually reach a common temperature, which is A. This is in consequence of the *First Law of Thermodynamics*. However, there is also a *Second Law of Thermodynamics*. This tells us that a quantity called the "entropy" must increase as time goes by. The entropy of each of the objects is proportional to the logarithm of its

[3]Quaternions have been the subject of several articles in *Function*. The most recent discussion is in the issue for October 1995.

temperature. In fact, if we choose the right units, we can make the entropy *equal* to the logarithm of the temperature.

So when we start the entropy is $\log a + \log b + \log c + \ldots$. Because of the law of logarithms, we may write this as $\log(abc\ldots)$, which in its turn is $n \log G$. The entropy attained when the common temperature is achieved is $n \log A$, and this must be the larger of the two. That is to say, $n \log A > n \log G$, and it follows that $A > G$ (or we could have, in the very special case $a = b = c = \ldots$, that $A = G$). This completes the demonstration.

I first heard of this demonstration (it is *not* a proper mathematical proof) as a result of the work of P T Landsberg (University of Southampton) in the late 1970s. By 1979, however, Landsberg realised that he had been anticipated and that the physicist Arnold Sommerfeld had had the same idea and had put it into a book that first saw print (but after Sommerfeld's untimely death) in 1952.

Actually the idea goes back much further. While we were looking at Tait's work (see the previous section of this article), we found that Tait had also written on this question. Indeed, Tait used the technique to "prove" more complicated inequalities, as later did Landsberg.

However, Tait seems to have thought that these demonstrations were in fact proofs; Sommerfeld and Landsberg realised that they were not. All the same, it is interesting to see the same idea occurring, clearly independently, to three separate researchers.

6. Finale

It often happens that young mathematicians find for themselves some result, only to learn that it is already known. This disappoints them; they would prefer to have found something genuinely new. However, it is very rare for young people or amateur mathematicians these days to discover genuinely new results. But it is still an achievement to find for oneself things the great mathematicians of earlier years also found.

It should be a source of pride, rather than of disappointment, that one has thought for oneself the same thoughts as some great and famous mathematician.

* * * * *

COMPUTERS AND COMPUTING

Mathematics and the Internet

Cristina Varsavsky

The Internet has opened many new and exciting possibilities, and it is already shaping the way we communicate with others. Until recently, the Internet was available to a handful of people, mainly academics, who used it for the exchange of text based information. The standardisation of graphics formats which enriched the information that can be put on the World Wide Web – or the Web, as we tend to call it these days – was a major factor in the explosion of the use of the Internet around the world. Now not only academics, but also teachers, students, business people, children, and ordinary citizens use the Internet for communicating ideas, learning, searching, striking deals, buying goods, chatting, playing games, etc.

Unfortunately mathematics was not ready for this new type of communication. Web browsers use HTML (Hyper Text Markup Language) for the formatting of Web pages, the creation of links, and the embedding of images. But this language does not support the display of mathematical expressions, which makes the communication of mathematics on the Internet a very difficult task. However, mathematicians have not given up; despite the limitations they still manage to create Web pages with mathematics content, but they do so with much work and less than perfect results. The most common approach is embedding equations as graphic images. This has two major inconveniences for the page viewer. The embedding of graphics significantly slows down the downloading process, so reading a page with many equations may be a very frustrating experience. Also, the font size of the equations is fixed in their images, while the surrounding text varies with the user's setup, which usually makes the Web pages look a bit awkward. There are also other methods that rely on special HTML coding, which include viewers for commercial mathematics software, and specialist plug-ins for the conversion of LaTeX code – the universal typesetting language for mathematics – to be rendered by the browser. In any case, these attempts are made only for the static display of mathematics; they do not solve the problems of the input of mathematical expressions and the flexible manipulation of dynamic mathematical objects.

Just in the same way mathematicians have settled on a standard universal language for communicating mathematics on paper – a product of many centuries of refinement – now they have a strong need to settle on a standard universal language for communicating mathematics electronically, a language which does not depend on the computing environment of the users, nor on their preferred mathematics software. What is needed is a language to describe mathematical objects very precisely, using minimum space and time, so that a Web browser can render them locally and display them properly.

Fortunately there is some good news on this front. Several software companies and academics have joined forces to form the *World Wide Web Consortium HTML-Math group* (known as *W3C HTML-Math*, or simply *W3C*) to discuss the problem of communicating and preserving mathematics with the evolving technologies. They are focusing on the two important aspects of mathematics communication: the encoding of mathematical language and its implementation. This group has taken a long term view of the problem and they are directing more energy into the encoding aspect; they believe that if the encoding problem is solved well, documents authored today will still live on when better technology becomes available.

The *W3C* group has set as its goal to come up with a language – to be called *MathML* – that would be easy to use and would allow conversion to and from other mathematics formats used today, such as LaTeX, and the various syntaxes used by computer algebra systems. The output capabilities should include print media (including braille), graphics display, speech synthesizers, and any computer algebra system input. The language should also allow for future extensions.

At the same time, the *W3C* group also set out to implement a browser which could render the information contained in the markup language. The goals set by the group include the capability of rendering mathematical equations encoded in *MathML* in accordance with the viewer's preferences, and at the highest possible quality; proper printing; reaction of equations to mouse commands; and communication with other applications, particularly mathematics software, through the browser.

It seems to me that there are fascinating times ahead for on-line mathematics. A time when we will be able to browse nicely setup mathematics Web pages, see animations live, change parameters, cut mathematical objects and paste and edit them in our preferred mathematics software, lecture notes, assignments, or research work, and *vice versa*. I can hardly wait!

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 21.2.1

A 5×5 square is tiled with six L-shaped pieces, leaving one square not covered. Where can that square be?

SOLUTION

We will show that the uncovered square can be the centre square or one of the corners, and that it cannot be anywhere else.

Examples of the possible configurations are shown in Figure 1, in which an L-shaped piece can be inserted into the shaded region in two ways, leaving an uncovered square in either the centre or a corner.

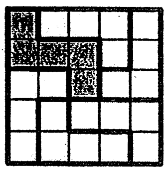

Figure 1

We now show that the uncovered square cannot be anywhere else. Colour the first, third and fifth rows of squares black and the others white. Each L-shape must cover either three black squares and one white or three white and one black. There are 6 L-shapes and 10 white squares. Suppose the uncovered square is white. Then 9 white squares are covered. If there are n L-shapes covering three white squares each, then there are $6 - n$ L-shapes covering one white square each. Therefore $3n + 6 - n = 9$, so $2n = 3$, which is impossible as n is an integer. Thus the uncovered square must be black, i.e. it must be in the first, third or fifth row. By symmetry, the uncovered square must also be in the first, third or fifth column. We are left to show that the squares at the centres of the sides must be covered. We will do this by showing that the uncovered square must be one of the black squares in Figure 2.

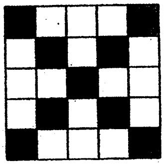

Figure 2

There are nine black squares, and it is easy to see that each L-shape must cover either one or two of them. Suppose all nine of these squares are covered. If there are n L-shapes covering one black square each, there must be $6 - n$ L-shapes covering two black squares each. Then $n + 2(6 - n) = 9$, so $n = 3$. Thus at most three of the corner squares are covered by L-shapes covering no other black square, so at least one corner square must be covered by an L-shape covering one other black square. Within symmetry, this can be done in only one way, shown by the shaded L-shape in Figure 3. Then two more L-shapes must be placed as shown, covering one black square each, and it is now impossible to place three L-shapes in the remaining space with two of them covering two black squares each. We conclude that not all of the black squares in Figure 2 can be covered, so the uncovered square must be one of these. This completes the proof.

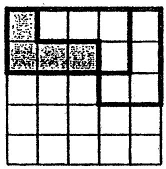

Figure 3

We received several solutions to this problem. Jennifer Palisse (Year 8) and Sarah Nguyen (Year 10) found arrangements with the uncovered square in the centre and in a corner, and described how one could be obtained from the other by moving just one L-shape. Natasha Chadha (Presentation

College, Windsor) and Paul Tescher (Bialik College) also found arrangements with the uncovered square in the centre and in a corner. In addition, they gave examples in which the uncovered square was placed in each of the other positions and the remaining space was filled with five L-shapes, with the other four squares left scattered. Finally, Chris Cheung found arrangements with the uncovered square in the centre and in a corner, and noted that these were the only possibilities. While these solutions fall short of a proof that no other arrangements can be found, they all show that the students had explored the problem and gained some insight into it.

PROBLEM 21.2.2 (28th Spanish Mathematical Olympiad – First Round, Question 8)

Let ABC be any triangle. Two squares $BAEP$ and $CADR$ are constructed, externally to ABC. Let M and N be the midpoints of $\overline{BC}$ and $\overline{ED}$, respectively. Show that $\overline{AM}$ and $\overline{ED}$ are perpendicular and $\overline{AN}$ and $\overline{BC}$ are perpendicular.

SOLUTION

The situation is depicted in Figure 4. (The points P and R, and the sides of the squares incident to them, are not needed in order to solve the problem, and are therefore not shown.) Rotate triangle AED through $90°$ so that $\overline{AE}$ coincides with $\overline{AB}$. Let D' be the image of D under the rotation. Then C, A and D' are collinear, so $CD'B$ is a triangle. Since $CD' = 2CA$ and $CB = 2CM$, triangles $CD'B$ and CAM are similar: Therefore $\overline{D'B}$ is parallel to $\overline{AM}$, so $\overline{AM}$ is perpendicular to $\overline{ED}$.

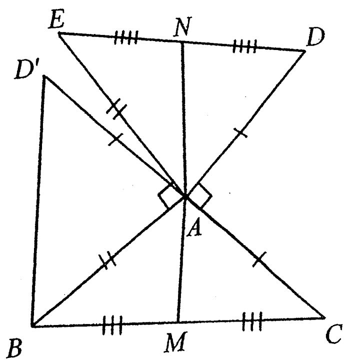

Figure 4

A similar argument, in which triangle ABC is rotated through $90°$ so that $\overline{AC}$ coincides with $\overline{AD}$, can be used to prove that $\overline{AN}$ and $\overline{BC}$ are perpendicular.

Claudio Arconcher (São Paulo, Brazil) supplied the following simple and elegant proof using vectors. Since $\vec{AM} = \frac{1}{2}(\vec{AB}+\vec{AC})$ and $\vec{ED} = \vec{AD}-\vec{AE}$, we have:

$$\begin{aligned}\vec{AM}.\vec{ED} &= \frac{1}{2}(\vec{AB}+\vec{AC}).(\vec{AD}-\vec{AE}) \\ &= \frac{1}{2}(\vec{AB}.\vec{AD}-\vec{AB}.\vec{AE}+\vec{AC}.\vec{AD}-\vec{AC}.\vec{AE}) \\ &= \frac{1}{2}(\vec{AB}.\vec{AD}-0+0-\vec{AB}.\vec{AD}) \text{ (since } \vec{AC}.\vec{AE}=\vec{AB}.\vec{AD}) \\ &= 0.\end{aligned}$$

Thus $\vec{AM}$ and $\vec{ED}$ are perpendicular. A similar argument shows that $\vec{AN}$ and $\vec{BC}$ are perpendicular.

PROBLEM 21.2.3

Three circles in the plane intersect to form seven bounded regions. In each region there is a token that is white on one side and black on the other. At any stage, you can either:

(a) flip all four tokens inside one of the circles,

or

(b) flip all tokens showing black inside one of the circles, making all the tokens in that circle white.

Starting with all tokens white, and using only (a) and (b) above, is it possible to get all the tokens white except for the one in the region common to all the circles?

SOLUTION

No, it is not possible. We can show this by starting at the final state and working backwards. Since each circle contains tokens of both colours, the last move cannot be of type (b). But if a move of type (a) results in each circle containing tokens of both colours, then each circle must contain tokens of both colours before the move is made. Therefore, as we step backwards from the final state, we never encounter a move of type (b), and each circle will always contain tokens of both colours. Hence we can never reach a state in which all tokens are white.

Keith Anker, formerly of Monash University, provided the following alternative argument. The number of black tokens in each circle is initially zero, which is an even number. Any circle with an even number of black tokens before a move of type (a) must still have an even number after the move. A move of type (b) always results in a circle with an even number of black tokens (namely zero). Therefore there will always be at least one circle with an even number of black tokens. But the final state does not satisfy this condition, so that state cannot be attained.

PROBLEM 21.2.4

A friend challenges you to the following game. You and your friend take turns to say any one of the numbers 1, 3 and 4, and a running total is kept. (For example: you begin by saying 3; your friend replies by saying 4, bringing the total to 7; you say 3 again, making the total 10; your friend says 1, making the total 11; and so on.) The player who says the number that brings the total to 100 is the winner. (The total is not permitted to exceed 100.) You are given the choice of going first or second. Which should you choose, and what is the winning strategy?

SOLUTION by Keith Anker

I will call second. I will contrive at each turn to make the total either a multiple of 7 or 2 more than a multiple of 7. (The starting total, 0, is of this form, as is the final total: $100 = 7 \times 14 + 2$.) I do so as follows:

If my opponent calls 3, I call 4, and *vice versa.*

If my opponent calls 1 when the total is a multiple of 7, so do I.

If my opponent calls 1 when the total is 2 more than a multiple of 7, I call 4.

Thus, I always make the total either a multiple of 7 or 2 more than a multiple of 7, and my opponent can never make the total either of those forms. (In chess terminology, I "keep the opposition".)

We can see that this solution works, but it is not obvious how it was obtained. We can gain some insight into this by starting at the final total and working backwards. If I bring the total to 99, then my opponent wins by adding 1. If I can bring the total to 98, however, my opponent cannot win, so 98 is "safe" for me. The totals 94, 95, 96 and 97 are unsafe, since my opponent can reach 98 from the first two and 100 from the others. The next number in descending order, 93, is safe, because from it my opponent can reach only unsafe numbers. Continuing in this way, we are eventually led to the solution described above.

For more information on how to solve such problems, look up the Sprague-Grundy theory for "subtraction" games in *Winning ways for your mathematical plays* by E R Berlekamp, J H Conway and R K Guy (Academic Press, 1982).

PROBLEM 21.2.5 (from *Mathematics and Informatics Quarterly*, 2/96)

(a) At least two of these statements, apart from this one, are true.

(b) At least two of these statements, apart from this one, are false.

(c) At least one of these statements is false.

(d) x of these statements are true.

Given that, if you knew the value of x, you could determine uniquely which statements are true and which are false, determine the value of x.

SOLUTION

Statement (c) must be true, since if we assume it is false then we are led immediately to a contradiction.

Suppose (b) is true. Then the assertion made in (a) is correct, so (a) is true. But then the assertion made in (b) is false, which is a contradiction. Therefore (b) is false.[1]

Since the assertion made in (b) is false, at most one of (a), (c) and (d) is false. We know that (c) is true, so at most one of (a) and (d) is false.

Suppose (a) is false. Then, by the reasoning above, (d) is true. But if the assertion made in (a) is false, then at most one of (b), (c) and (d) is true. This is a contradiction, so (a) must be true. From the assertion made in (a), we conclude that (d) is true. (Are you still with us?)

[1]This is an instance of *consequentia mirabilis*; see "The Wonderful Deduction" by M Deakin in *Function Vol* 17 *Part* 3, pp 83-88.

At this point, we have established that (a) is true, (b) is false, (c) is true and (d) is true, so three of the statements are true. Since (d) is known to be true, we must have $x = 3$.

PROBLEM 21.2.6

The minute and hour hands on a watch are interchanged. Prove that the resulting arrangement does not correspond to a valid time unless the positions of the two hands coincide.

SOLUTION

Keith Anker has pointed out to us that the assertion is incorrect! At $\frac{720}{143}$ minutes after midday (just after 12:05 pm), the hour hand has advanced $\frac{60}{143}$ minute marks since midday, and the minute hand has advanced $\frac{720}{143}$ minute marks. At $\frac{8640}{143}$ minutes after midday (just after 1:00 pm), the hour hand has advanced $\frac{720}{143}$ minute marks since midday, and the minute hand has advanced $\frac{8640}{143} = 60\frac{60}{143}$ minute marks, i.e. it is $\frac{60}{143}$ minute marks past the XII position. Thus the positions of the hands are interchanged, but they do not coincide.

We thank Keith Anker for pointing this out, and apologise to readers for the error.

Solution to an earlier problem

We continue providing solutions to problems which have appeared in *Function*, but for which we have not published solutions previously. In this issue, we present the solution to another of the Kürschák competition problems from the August 1990 issue.

PROBLEM 14.4.8

For any given positive integer n, denote by $S(n)$ the *sum of the digits* of n (in the decimal system). Determine all positive integers M for which $S(M) = S(kM)$ for all integers k for which $1 \leq k \leq M$.

SOLUTION

We claim that the required values of M are 1 and all numbers consisting entirely of the digit 9, i.e. 9, 99, 999, etc.

If $M = 1$ then k must equal 1, and the condition is trivially satisfied. If $2 \leq M \leq 8$ then the condition is not satisfied (just check each value of M

with $k = 2$). If $M = 9$ then it is easy to check that the sum of the digits of each multiple of 9 up to 81 equals 9.

Now assume that M has n digits, where $n \geq 2$. If M is a power of 10, then clearly the condition is not satisfied with $k = 2$. If M is not a power of 10, let k be the n-digit number 100...01, where there are $n-2$ zeros between the ones. Clearly $k \leq M$, so this is a valid value of k. Now consider the result of multiplying M by k. An example (we have chosen $M = 567$) will help to illustrate what happens:

$$\begin{array}{rrrrrr} & & & 5 & 6 & 7 \\ \times & & & 1 & 0 & 1 \\ \hline & & & 5 & 6 & 7 \\ & 5 & 6 & 7 & & \\ \hline & 5 & 7 & 2 & 6 & 7 \end{array}$$

Note that the last $n-1$ digits of M appear as the last $n-1$ digits in the answer. In this example, and for most values of M, the first digit of M appears as the first digit of the answer. For these values of M, the remaining digits of the answer cannot all be zero, so the sum of the digits of the answer must be greater than the sum of the digits of M. The only way that the first digit of the answer could fail to be the first digit of M is if, at the addition step, there is a "carry" at every column from the middle leftwards. This can happen only if the last digit of M is not 0 and the other digits of M are all 9s. We can eliminate the possibility that the last digit of M is a number in the range 1 to 8 by well-known "digital root" (or "casting out 9s") considerations. Therefore M must consist entirely of 9s.

It remains for us to show that if M is a string of 9s then M has the desired property. Suppose M is a string of n 9s, i.e. $M = 10^n - 1$. Let k be an integer for which $1 \leq k \leq M$. We need to show that $S(kM) = 9n$. If k ends in one or more zeros, say $k = b.10^a$ where $a \geq 1$ and b does not end in zero, then $S(kM) = S(bM)$, and the problem reduces to the same problem with a smaller value of k, not ending in zero. We may therefore assume that k does not end in zero. We have $kM = k(10^n - 1) = (k.10^n - 1) - k + 1$, where $k.10^n - 1$ comprises the digits of $k - 1$ followed by n 9s.

Perform the subtraction of k from $k.10^n - 1$; since k has at most n digits, the digits of k are all subtracted from 9s, so the result is the n digits of $k - 1$ (which are the same as the digits of k, except the last digit which is reduced by 1), followed by the complements modulo 9 of the n digits (including leading zeros, if necessary) of k. The sum of the digits of the result is therefore $9n - 1$. Finally, add 1 to obtain kM; this is done simply by adding 1 to the last digit, since that digit cannot be 9 (because k does not end in zero). Thus the sum of the digits of the final answer is $9n$, so $S(kM) = 9n$ as required.

PROBLEMS

Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received by 8 December 1997 will be acknowledged in the February 1998 issue, and the best solutions will be published.

PROBLEM 21.4.1

You have three calculating machines:

Machine A (an adder) accepts two numbers, a and b, as input, and calculates $a + b$.

Machine S (a subtractor) accepts two numbers, a and b, as input, and calculates $a - b$.

Machine Q (a "quarter-squarer") accepts one number, a, as input, and calculates $a^2/4$.

Explain how you could find the product, ab, of any two numbers a and b, using only these machines and no hand calculation.

PROBLEM 21.4.2

Find all three quadratic polynomials $p(x) = x^2 + ax + b$ such that a and b are roots of the equation $p(x) = 0$.

PROBLEM 21.4.3 (from *Mathematical Spectrum)*

A triangle has angles α, β and γ which are whole numbers of degrees, and $\alpha^2 + \beta^2 = \gamma^2$. Find all possibilities for α, β and γ.

PROBLEM 21.4.4 (Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain)

Find all possible sets of six two-digit numbers $M = xy$, $N = yz$, $P = zu$, $M' = yx$, $N' = zy$, $P' = uz$ (where x, y, z and u are decimal digits, and xy, etc. denote the decimal representations of the numbers), such that M, N, P

and M', N', P' are two geometric progressions with the same integer common ratio.

PROBLEM 21.4.5 (Claudio Arconcher, São Paulo, Brazil)

Let Γ be a circle of radius r, and let $\overline{BC}$ be a chord of Γ. A point A on Γ makes one revolution around Γ. Prove that the locus of the centroid of the triangle ABC is a circle with radius $r/3$, and that this circle divides the chord $\overline{BC}$ into three equal parts.

Garry O'Brien (Bunbury, WA) wrote to us asking about the following problem. It is in fact a well-known problem which has been around for a long time, and many readers will have seen it before. For those who haven't, we will just mention that the answer probably can't be expressed exactly in terms of known mathematical constants, but it can be approximated numerically.

PROBLEM 21.4.6

A farmer would like to graze his animal on his neighbour's circular paddock, but the neighbour stipulates that the farmer can only use half of the paddock and the animal must be tethered on the boundary line. What is the length of the tether as a function of the radius of the paddock?

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> Newton could not admit that there was any difference between him and other men, except in the possession of such habits as ... perseverance and vigilance. When he was asked how he made his discoveries, he answered, "by always thinking about them;" and at another time he declared that if he had done anything, it was due to nothing but industry and patient thought: "I keep the subject of my inquiry constantly before me, and wait till the first dawning opens gradually, by little and little, into a full and clear light."

– W Whewell, *History of the Inductive Sciences*

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