Function

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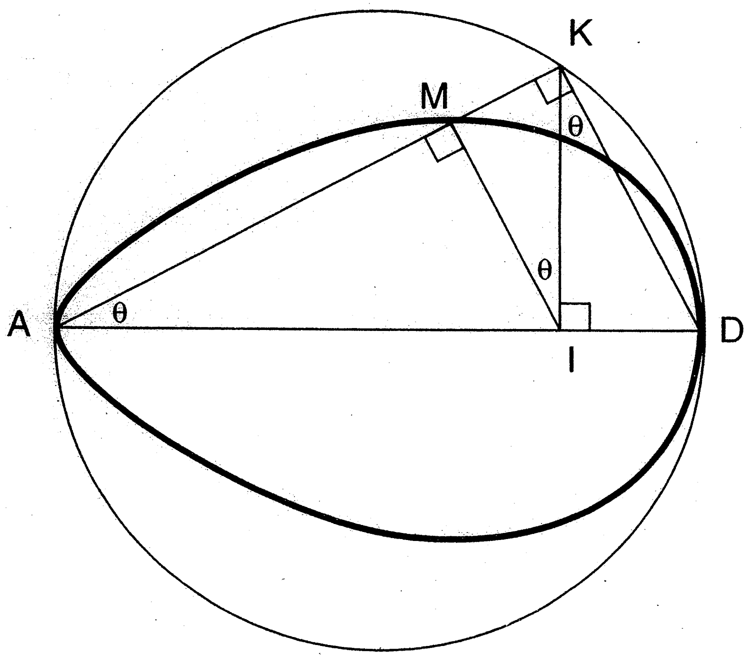

Mathematics Department - Monash University

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Function is a mathematics magazine produced by the Department of Mathematics at Monash University. The magazine was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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EDITORIAL

We welcome new and old readers alike with our twentieth volume of *Function.* We hope you find in it many interesting and enjoyable items.

The leaf-shaped curve depicted on the front cover, called the *unifolium*, arises in a construction by geometric means of the cube roots of numbers. The *Front Cover* article explains how this is done.

In the feature article, H C Bolton looks at the mathematics related to the shapes most pebbles take after abrasion, and the shapes of bars of soap as they are worn down.

In the *History of Mathematics* column you will find a description of the armillary sphere, the mathematics behind it, and also a practical example of how it can be used for deciding where to plant the trees in your back yard, by estimating the shade they will make at different times of the year.

In the *Computers and Computing* section there is a short introduction to automated reasoning, more particularly to programming languages which can be used to program the computer to make deductions and to answer questions.

The number of articles written about π seems to be endless, but there is always something interesting to say about this popular number. In this issue of *Function* we present a less well known way of approximating π.We also include another news item on sundials, and as usual, solutions to problems we have published, and a few more new ones to work on.

Finally, we thank all those readers who sent us letters, comments, solutions, and articles. We always welcome your contributions.

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THE FRONT COVER

A Curve that Gives us Cube Roots

Michael A B Deakin

One of the classic problems of ancient geometry was the so-called "duplication of the cube"[1] – the construction by geometric means of the number $\sqrt[3]{2}$. One solution to this problem has been reconstructed by Wilbur Knorr, a historian of ancient mathematics.

Here is how it proceeds. Let AD be the diameter of a circle and let K be a point on the circumference of that circle. For future reference, let $\angle KAD = \theta$ and choose $AD = 1$.

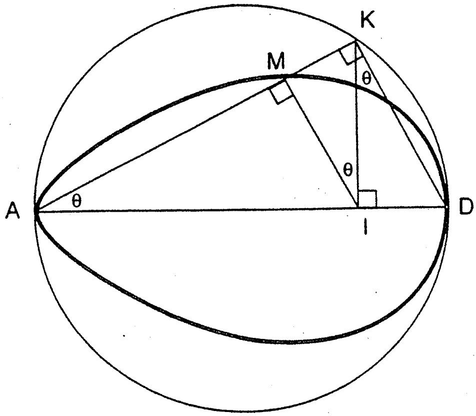

Figure 1

From K draw a line $\overline{KI}$ perpendicular to $\overline{AD}$ and meeting $\overline{AD}$ in I. Then from I draw a line $\overline{IM}$ perpendicular to $\overline{AK}$ and meeting $\overline{AK}$ in M.

Now $\angle AKD$ is an angle inscribed in a semicircle and thus it is a right angle. In consequence, $AK = \cos\theta$. Since $\triangle KAD$ and $\triangle MAI$ are similar, we have

$$\frac{AM}{AK} = \frac{MI}{KD}$$

[1]There are generally reckoned to be three such problems. (See for example V Katz, *A History of Mathematics*, pp 47-48.) The others are the "squaring of the circle" (i.e. a geometric construction giving a straight line of length π times a given unit length) and the trisection of an arbitrary angle. All three are now known to be impossible with ruler and compass constructions alone. For more on angle trisection, see *Function*, *Vol* 3, *Part* 3.

and then

$$AM = \cos\theta \frac{MI}{KD} = \cos\theta \frac{KI\cos\theta}{KI/\cos\theta} = \cos^3\theta.$$

We now have $AK \leq 1$ and

$$AM = AK^3. \tag{1}$$

Now for every position of K on the circumference of the circle a corresponding M may be constructed, and in each case equation (1) will be satisfied. So we may imagine K moving around the circle and M tracing out a curve as shown on the cover and reproduced as Figure 1.

This curve is known as the *unifolium* (meaning "single leaf") and it *may* have been used in ancient times as a means of constructing cube roots – some authorities think it was, others are not so sure.[2]

Here is how it can be made to work.

The value of AM can be made to take any value from 0 to 1 (0 when M coincides with A, 1 when it coincides with D). So let us start with numbers in the range $(0, 1)$. Suppose for example we choose $AM = \frac{1}{2}$. We move around the unifolium to the required point (it occurs where the unifolium intersects a circle centred at A and with radius $\frac{1}{2}$) and we then join this M to the point A. If we now extend the line AM so produced to meet the circle in K, then

$$AK = AM^{1/3} \tag{2}$$

in consequence of equation (1). If $AM = \frac{1}{2}$, then $AK = \sqrt[3]{\frac{1}{2}}$.

We may in this way construct the cube root of any number between 0 and 1. To generate cube roots of numbers larger than 1 ($\sqrt[3]{2}$ for example) a minor modification will do the trick. Look at Figure 2. On the line $\overline{AK}$ (extended) mark off a point L such that $AD = AL$ $(= 1)$. Through L draw a line $\overline{LN}$ parallel to $\overline{KD}$ and meeting $\overline{AD}$ (also extended) in N. Then if $AM = \frac{1}{2}$, it follows that $AN = \sqrt[3]{2}$. (I leave the proof of this as a simple exercise to the reader.)

It should be noted that, although, given any K, we may construct the corresponding M using only the classical tools of ruler and compasses, we cannot construct the entire unifolium in this way, as there are infinitely

[2] Some ascribe this use of the curve to the much later figure Juan Bautista Villalpando. Villalpando was a 16th-century Spanish Jesuit, a mathematician and architectural theorist.

many points on it. Thus the existence of this construction does not contradict the theorem that cube roots cannot, in general, be constructed using only classical means.

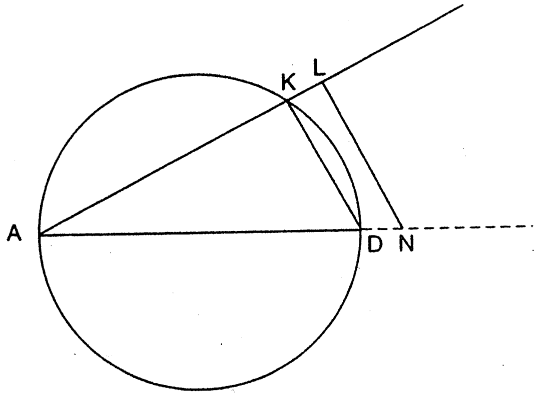

Figure 2

I learned of this material via an Internet newsgroup on the history of mathematics. The fullest fairly accessible discussion is to be found in Wilbur Knorr's book *The Ancient Tradition of Geometric Problems* (especially pp 50-52), but one of my principal sources for this article is a posting by Danny Otero of Xavier University, Cincinnati. He in turn quotes from several other books, but these alas are rare and were not available to me.

* * * * *

The words or language, as they are written or spoken, do not seem to play any role in my mechanisms of thought. The physical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be voluntarily reproduced and combined.

– Albert Einstein

* * * * *

THE SHAPES OF PEBBLES AND BARS OF SOAP

H C Bolton, University of Melbourne

The relationship between mathematics and physics is very profound. Here is a physical problem where a little is known of some observations of natural objects and where mathematics can help to organise these observations.

Most persons like pebbles, especially those that they find in rivers or on the beach, or in digging in a garden. The smoothness and roundness of the pebbles are attractive. I have two pebbles from a favourite beach near my home town, where I enjoyed swimming, climbing the cliffs and watching the sea birds; I carry those pebbles about with me. When we think how the pebbles were formed, we immediately find a physical problem. They all start from pieces of rock broken from cliffs or boulders. These pieces of rock gradually move downhill, falling into a stream or into the sea. In both cases, the rocks get tumbled over and over in all directions and the once sharp edges become rounded by abrasion with other rocks or sand. Also, rocks can disintegrate by the growth of minute organisms on their surfaces, such as lichens and mosses. The abrasion arising in the tumbling motion can be made artificially. It is possible to buy tumbled pebbles from a gemmological shop; small collections of these pebbles, often of differently coloured rocks, are very attractive as ornaments. These pebbles are made by putting rough stones into a closed tin and rotating them for several hours on a lathe.

We would expect that the sharpest corners and edges of a rough piece of rock would abrade most quickly and this suggests that the ultimate shape of a pebble would be a sphere which had its roundness, however defined, the same at all points. There are indeed some spherical pebbles but not many and it is found that many pebbles have a long axis which predominates even for small pebbles. There are many scientific parts to the story that have to be just mentioned; thus the crystals out of which some rocks are made are often inhomogeneous and this probably controls the rates of abrasion. Leaving this aside, we ought to ask if there is a collection of photographs of pebbles so that their shapes can be studied. One of the best books was written in Melbourne by E J Dunn in 1911, when he was Director of the Victorian Geological Survey. The book is called "Pebbles" and has a large collection of photographs of pebbles of all kinds of rock and from all over the world. The author makes an interesting point that the shape of a pebble contains information from which its history can be unfolded. Our mathematical discussion here is only part of the story of pebbles. The book by Dunn is very rare. The State Library of Victoria in Swanston Street, Melbourne, has a copy.

Dunn says that there are three common types of shapes. The first is the sphere, the second is an ovoid which has one of the long ends more pointed than the opposite end, and thirdly there is the ellipsoid, which has three axes at right angles and the cross-sections of the pebble are ellipses. The discussion on the shapes of pebbles was presented mathematically by Lord Rayleigh and his articles were written from England during the 1939-1945 war. He was interested in collecting more pebbles but, at that time, many beaches, especially in the south of England, were forbidden to the general public because of the fear of invasion. The articles are in the *Proceedings of the Royal Society of London* (1942-43) *Vol* 181, pp 107-118 and (1943-44) *Vol* 182, pp 321-335.

The ellipse has the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

and it is conventional to have $a > b$. Figure 1 shows an ellipse (the full curve) for $a = 3, b = 2$. The ellipse is close to the shape of a cross-section of many pebbles and is also similar to the shape of many pieces of soap when most of the original corners of the bar have been worn away in use. We will return to the soap shapes later. Notice that the graph of equation (1) has symmetry about the x and y axes; the equation remains unchanged when x is replaced by $-x$ and when y is replaced by $-y$.

The concept of roundedness is expressed by the *curvature* at any point on the curve and this is defined as the reciprocal of the *radius of curvature* at the point. We consider a circle going along the curve near the point in question. Figure 1 shows the circle centred at the point $(C_A, 0)$ which fits the ellipse near the point $A = (a, 0)$. The radius of curvature at this point is called R_A and

$$R_A = a - C_A. \tag{2}$$

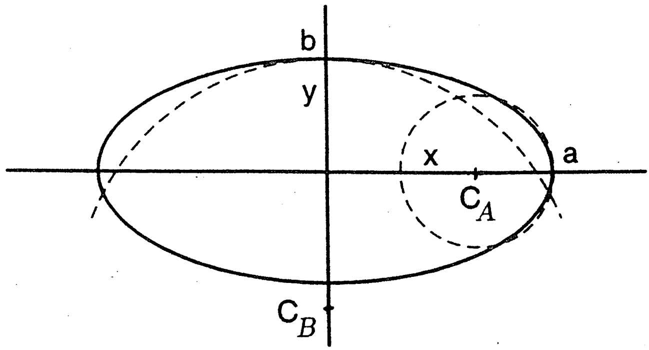

Figure 1. Sketch of the ellipse equation (1) with $a = 3, b = 2$. The centres of curvature at the ends of the axes $(a, 0)$ and $(0, b)$ are given by C_A and C_B and the circles fitting the ends of the axes are shown dotted.

There is a sophisticated way of getting R_A but we give a simple derivation. Consider the part of the ellipse near $(a, 0)$ where y small. Rewrite equation (1) as

$$x^2 = a^2 - \frac{a^2y^2}{b^2}$$

We add to the right hand side of this equation the term $\frac{a^2y^4}{4b^4}$ which is negligible for small y. This gives an approximation for x^2 which permits us to write it as a square:

$$x^2 \approx a^2 - \frac{a^2y^2}{b^2} + \frac{a^2y^4}{4b^4} = \left(a - \frac{ay^2}{2b^2}\right)^2 \tag{3}$$

Equation (3) then leads to the following approximation for x:

$$x \approx a - \frac{ay^2}{2b^2} \tag{4}$$

The points in question also lie very nearly on the circle of radius R_A and centre C_A, which has the equation

$$(x - C_A)^2 + y^2 = R_A^2 \tag{5}$$

Much in the same way as we obtained the approximation (4) for x we can obtain another approximation for x but now using equation (5):

$$x \approx (C_A + R_A) - \frac{y^2}{2R_A} \tag{6}$$

If we now compare equations (4) and (6), making use of equation (2), we find

$$R_A = \frac{b^2}{a} \tag{7}$$

For $a = 3, b = 2$ we get $R_A = 1\frac{1}{3}$ and this circle is shown dotted in Figure 1.

The expression (7) holds for the ellipse if x and y and simultaneously a and b are interchanged. The radius of curvature at the point $B = (0, b)$ is

$$R_B = \frac{a^2}{b} = 4\frac{1}{2}$$

Part of the corresponding circle is also shown dotted in Figure 1.

The pebble is in three dimensions and its common shape is an ellipsoid with an equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{8}$$

with conventionally, $a \geq b \geq c$.

Special cases are:

$a > b = c$; a *prolate spheroid*
$a = b > c$; an *oblate spheroid*

The shape of the earth is very close to an oblate spheroid with the polar axis less than those on the equator. The reason for this is that in earlier geological times, the material of the earth was more fluid than it is now and it moved outwards along the equator, whilst rotating, because of the centrifugal effect.

The radius of curvature of an ellipsoid depends on the plane section being considered. From equation (8), if z is put equal to zero, an ellipse is given with radius of curvature at the point $(a, 0, 0)$ being b^2/a. If y is put equal to zero, an ellipse is given with radius of curvature c^2/a. In three dimensions the *specific curvature* is defined as the product of the two two-dimensional curvatures. For the end of the x axis of the ellipsoid, the specific curvature at the point $(a, 0, 0)$ is a^2/b^2c^2. Rayleigh performed experiments on pebbles, tumbling them as the gemmologists make their pebbles, and showed that a prolate spheroid could maintain its shape under abrasion, that is, the ratio of the lengths of the axes $a, b, c(= b)$ were unaltered. From this he deduced that the rate of abrasion was proportional to the fourth root of the specific curvature. Since the fourth roots of numbers greater than one are considerably smaller than the numbers themselves, the regions of a pebble with large curvatures are only slowly abraded and the spheroidal shape is preserved down to small sizes.

Some of Rayleigh's experiments were on soapstone, a soft mineral with a "soapy" feel, and it is almost certain that he saw the way a bar of soap abrades when used for washing. A bar of soap initially is basically a cuboid with sides of length $2a, 2b, 2c$. Because abrasion takes place more rapidly at corners where the curvature is greatest, we might expect that the bar will change into an ellipsoid. But we must be careful: the pebbles of stone found in nature and in Rayleigh's experiments are produced by tumbling in all directions. But a bar of soap is not used in this way; when we wash our hands with a new bar of soap we rotate it about its longest axis, which we have called the x-axis. Also, when the soap is thin, we may only rub it between our palms. So the physics of abrasion of soap may not be the same as for a pebble.

Looking at an old piece of soap on a soap dish, we see the x and y axes horizontal and the z axis vertical. It looks like the ellipse in Figure 1 but it is wise to check this. In Figure 2 the outline of a piece of soap is reproduced and the ellipse is drawn with the same values of a, b. We see that the ellipse lies inside the soap shape. This was confirmed with another piece of soap.

The ellipse is one member of a family of curves of curves given by[1]

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1, \quad x \geq 0 \text{ and } y \geq 0 \tag{9}$$

where m is any number, integral or non-integral greater than or equal to 2, and the rest of the curve can be given by reflection in the axes. At this point in the argument a computer program could be written to draw the curves in (9) for various values of m. As m tends to infinity the curves will

[1]See *Function Vol* 14, *Part* 2.

be found to approach more and more closely to the vertical line through $(a, 0)$ and the horizontal line through $(0, b)$, which is the original shape of the cross-section through the bar of soap. After some trials, it was found that the value of m in (9) that gave a curve fitting the soap curve was $m = 3$. Indeed, the soap curve in Figure 2 has been produced using $m = 3$ and equation (9) for $x \geq 0$ and $y \geq 0$.

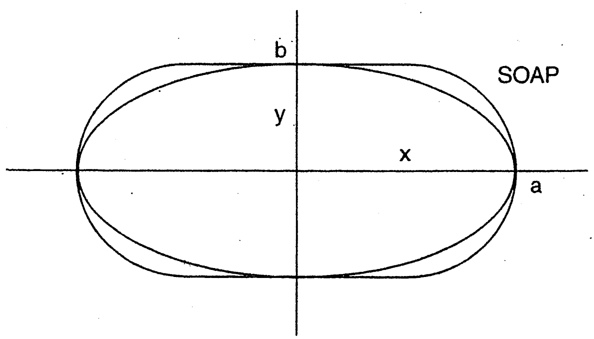

Figure 2. Sketch of the shape of a bar of soap after much use. The ellipse is shown and the curve representing the soap is the same as that given by equation (9) with $m = 3$.

One characteristic of the sets of curves obtained with (9) is that for m other than 2 the curvature is zero at the ends $x = \pm a, y = 0$ and $y = \pm b, x = 0$. We can repeat the argument used to obtain equation (5) to get

$$\left(a - C_a - \frac{ay^m}{mb^m}\right)^2 + y^2 = R_a^2.$$

By approximating as we did before we obtain

$$\frac{2ay^m R_a}{mb^m} = y^2 \quad \text{so} \quad \frac{2ay^{m-2} R_a}{mb^m} = 1$$

and for $m \neq 2$, R_a is no longer a constant. In fact, as y tends to zero, R_a tends to infinity which means a flat surface. This does not imply that the soap does not abrade; its surfaces dissolve in water.

As an exercise, plot some of the computed curves described by equation (9) and try to find the shape of your own pieces of soap. Is $m = 3$ a universal number?

* * * * *

HISTORY OF MATHEMATICS

The Armillary Sphere

Michael A B Deakin

Figure 1 shows an instrument known as an "armillary sphere". It is in essence a stylised model of the heavens, and it allows astronomical calculations to be made by means of computations relating to its geometry, or else to allow the results of such computations to be derived from analogue measurements made on the device itself.

It later gave place to a different instrument: the *astrolabe*, and this will be the subject of a later article in this series. But first we need to understand the theory of the armillary sphere.

If we look at the sky at night, we see a fixed pattern of stars which moves as a whole over the course of a 24-hour period and also (more slowly) over the course of a year. The pattern itself, however, never alters. The Southern Cross, for example, retains its shape always and it also retains the same relation to neighbouring stars like the Pointers, whatever the time of day or season of the year. Against this fixed pattern, the moon and the planets move, as also does the sun (as evidenced by the positions of its rising and setting).

The ancients, who did not have distractions like television, inconveniences like smog or tall buildings, or competing sources of illumination like streetlights, were much more familiar than we with the pattern of the night sky and were also much more able than we to use it to tell the time or for purposes such as navigation. The science of astronomy was perhaps the first science to reach a high level of development. Certainly by the time of Ptolemy (the second century AD), it had advanced to the extent that the movements of the planets against the fixed pattern of stars could be predicted, as could unusual events like eclipses. (Because Ptolemy's system of astronomy has since been superseded by Copernicus's, we tend nowadays to discount it. This however is unfair; the Ptolemaic system is actually very good.)

It was in the context of this background that first the armillary sphere and later the astrolabe were developed. The armillary sphere comprised a model of the heavens, conceived as a sphere representing the fixed pattern of stars; the sphere we see as we look up at the night sky. The earth's axis passes through this sphere (the celestial sphere) at points known as the north and south celestial poles. (The first of these is the approximate position of the star Polaris.) The circle connecting all the points equidistant between these poles is the celestial equator.

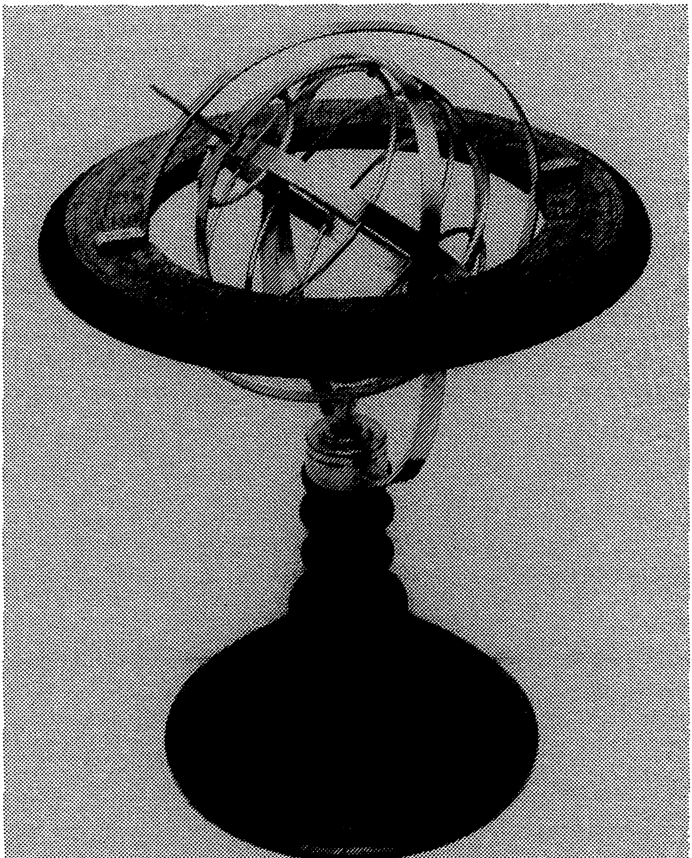

Figure 1. An armillary sphere. This example is in possession of the author and was designed by Brian Greig of Southern Skies Astronomy Pty Ltd of Melbourne. Photograph by Steve Morton (Physics Department, Monash University).

The armillary sphere was a metal replica of some aspects of the celestial sphere, comprising various circles representing the celestial equator, the ecliptic (the apparent path of the sun through the heavens), various lines of latitude and longitude, and possibly a plumb line to adjust the device to the local horizontal, a job done by the stand in Figure 1.

An armillary sphere could be set up with its "equator" parallel to the real equator and so with its axis parallel to that of the earth (as the gnomon of a sundial is so aligned), and then used to determine (e.g.) the day of the equinox, when the shadow cast by the upper half of the instrument's "equator" exactly covered the lower half. Other observations could also be made, sometimes with the aid of holes drilled in the metal circles, or by means of graduated scales carried upon them.

Armillary spheres came in a wide range of sizes, from large instruments that, in essence, constituted parts of ancient observatories, to small hand-held models that were in effect instructional toys. The large ones achieved accuracy and practicality at the expense of being unwieldy and not easily portable, but they could be adapted to a variety of uses as clocks, calendars and directional aids.

The larger ones allowed for careful measurement and so allowed practical problems to be solved without computation as such; they were rather like special purpose analogue computers. Smaller ones could not realistically be used in this manner as the errors involved in the measurement were too great.

Ptolemy wrote, it is thought, a treatise on the armillary sphere, and it is also thought that the later Theon of Alexandria wrote a commentary on that book. All this work is now lost, but there are grounds for the suggestion that it is partially preserved as the common source of several later works, some of which present aspects of it in translation (into Arabic and Syriac). These are among the first astronomers known to have worked on the subject, but it is generally believed that an earlier figure, Hipparchus, was also aware of much of the theory.

Ptolemy and Theon lived and worked in Alexandria, which is in Egypt, but in those days was a centre of Greek culture. Theon was the father of Hypatia, an early woman mathematician (see *Function, Vol* 16, *Part* 1). Much of what we know of Hypatia comes from her best-known pupil, whose name was Synesius. We will see in the sequel that Synesius learned much of this theory from his teacher, and is himself an important figure in the relevant history.

The celestial sphere, the apparent shape of the heavens with its fixed pattern of stars, may be analysed exactly as a replica of the terrestrial globe. Because of the rotation of the earth the celestial sphere will appear, to observers on the surface of the earth and thinking of themselves as fixed in space, to rotate once over the course of each 24 hours. This apparent rotation will take place about an axis which passes from the north to the south celestial pole. Each star, therefore, will trace out a circle in the observer's sky. This circle will be a circle of latitude on the celestial sphere. In particular, one such circle of latitude is the celestial equator.

North and south of the celestial equator are the tropics of Cancer and of Capricorn respectively. On December 22nd, the path of the sun through the heavens follows the tropic of Capricorn, and this is our longest day and the northern hemisphere's shortest; on June 22nd, the path of the sun follows the tropic of Cancer, and this is our shortest day and the northern hemisphere's longest. Between these dates, it follows lines of latitude on the celestial sphere and Figure 2 shows how these vary in the course of the year. We have $\pm 23.5°$ as the latitudes of the two tropics. If d is the number of days since December 22nd, and if α is the latitude of the sun's path, then we have[1]:

$$\sin\alpha = \sin 23.5° \cos(0.9856d)° = 0.3987\cos(0.9856d)°.$$

(The number 0.9856 is the number of degrees in a circle (360) divided by the number of days in a year (365.25).)

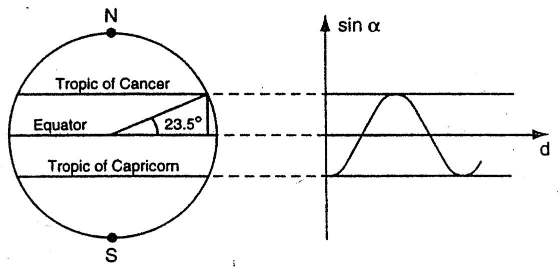

Figure 2. A cross-section of the celestial sphere and the sun's apparent path through the heavens.

This is to take a geocentric (or earth-centred) view of the matter, and there is nothing wrong with this; it served us well for some 2000 years! Let us continue to think in this way[2]. Look again at Figure 1. The armillary

[1]For another, more detailed, derivation of this law, see Aidan Sudbury's article in *Function*, *Vol* 19, *Part* 3.

[2]Aidan Sudbury's article uses the heliocentric (sun-centred) approach, but the results are the same in both cases.

sphere has been aligned in such a way that the axis is parallel to the axis of the earth as seen in Melbourne. Melbourne lies at latitude 37.75° south and the photograph shows the axis of the armillary sphere set at an angle of about this amount to the horizontal circle of the mount.

This horizontal circle of the stand represents the horizon. So now, in order to find the length of the day for any day of the year, we need to determine the length of the arc of a latitude circle of the relevant latitude α, that lies above the horizon. This is a relatively simple exercise in geometry, and indeed with a large enough and sufficiently detailed armillary sphere, measurement, rather than calculation, would do the job for us. With a smaller instrument such as that shown here, this is insufficiently accurate and so we need to calculate and we merely use the device to help us visualise the situation.

The relevant mathematics forms the subject matter of *spherical trigonometry*[3]. The answer is that the day is $(12 + 2\psi/15)$ hours long where ψ (in degrees) satisfies the equation

$$\sin\psi = \tan\alpha\tan 37.75°,$$

α being the angle found before and 37.75° being the latitude of Melbourne[4].

More detailed calculations, involving the longitude as well as the latitude, could tell us the times of sunrise and sunset.

I had occasion to use my armillary sphere the other day when two of my friends, who have just built a new house, wanted to get some advice on where to plant some trees in their garden. Their new back yard faces west and they wanted to know the approximate position of the sun in summer and winter. The extremes occur on the shortest day (June 22nd) when the sun travels along the tropic of Cancer in the celestial sphere. Figure 3 shows a vertical cross-section of Figure 1, and P represents the position of the sun at midday[5]. Then the sun will appear to be due north of their back wall. It will be at an elevation of 28.75° when it first shines over all their back yard. Similarly on the longest day, their back yard will be in full sun

[3]See *Function*, *Vol* 6 *Parts* 4 and 5, and for related articles, *Vol* 4 *Part* 1, *Vol* 5 *Part* 5 and *Vol* 15 *Part* 2.

[4]Again, see Aidan Sudbury's article for an alternative (and more fully detailed) derivation.

[5]This is the *true*, or *local*, midday; because of the way in which, for social and legal reasons, we actually measure time, it will not correspond *exactly* with 12 noon.

after midday and will see the sun at an angle of 75.75°.

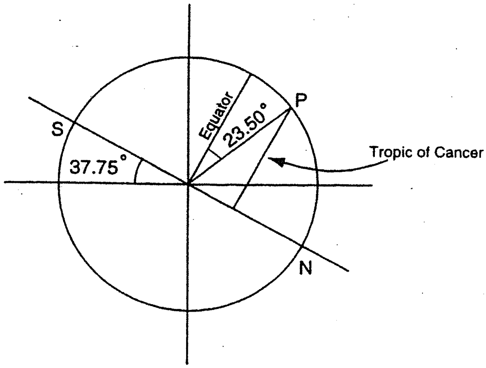

Figure 3. A vertical cross-section of an armillary sphere aligned for the latitude of Melbourne

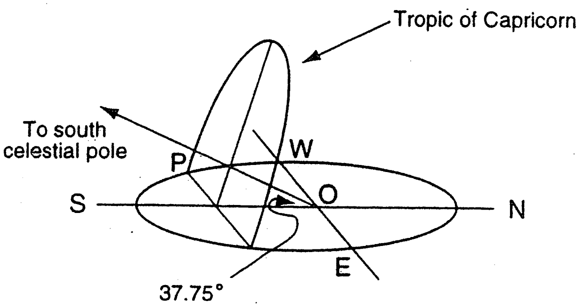

Figure 4. A schematic representation of a similarly aligned armillary sphere

Figure 4 now shows a schematic picture of the armillary sphere. The horizontal circle NWSE represents the horizon, and the circle representing the tropic of Capricorn is indicated. The point P represents the position of the setting sun on the longest day. A little 3-dimensional geometry shows that the angle POW (not drawn on the diagram) is very nearly 30°. Thus on the longest day, the sun sets some 30° south of west. Similarly on the shortest day it sets some 30° north of west.

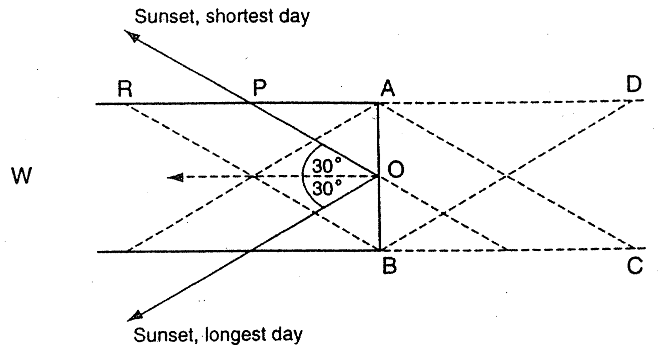

Figure 5. Positions of sunset for a west-facing house in Melbourne

Figure 5 now shows my friends' back yard. At true noon, the sun appears above the point marked A. On a fine day, the entire back yard will be in sunlight from this time until sunset and as the sun moves westward, its rays will shine into the back window. In the depths of winter, the sun's rays will shine at sunset along the parallel lines RB, PO and AC, so that only the triangular area ABC will receive direct sunlight. As it is desirable to preserve this, the area ABR should be kept clear (at least of evergreen shrubs). Once the weather warms up, we will have sunlight across the entire back wall and so entering the back of the house. Even at sunset, the area ABD will be illuminated directly. Particularly during the hotter parts of the day, we will want to shade the wall AB. We could plant evergreens west of the line RB to this end. Deciduous trees and shrubs planted in the triangle ABR would shade the back of the house in summer, but not so greatly impede the light in winter.

Further Reading

Much material is given on the armillary sphere in a good and reasonably accessible account of the astrolabe by J D North and published in *Scientific American* (January 1974). I will have more to say on this article in my subsequent account of the astrolabe. For some related material, see the articles on the Monash Sundial in earlier issues of *Function* or, better still, C F Moppert's booklet *The Monash Sundial.*

* * * * *

COMPUTERS AND COMPUTING

Answers from Facts and Rules

Cristina Varsavsky

Automated reasoning is a young but rapidly progressing field of computer science. Researchers are producing expert systems that play an important role in assisting professionals in medical diagnosis, geological exploration, solving engineering problems, answering legal questions, etc.

The purpose of automated reasoning is to write programs that answer questions which require reasoning. You are already familiar with some programming languages; in this section we usually use QuickBasic, but you have probably written, or at least seen, programs in Pascal, C, or perhaps Fortran. The programs written in these programming languages are all very focused: they clearly and unambiguously define the steps to be followed in order to perform a task. This time we will use a different programming language, a language that has the in-built facility to draw conclusions from facts.

This is best explained with an example: suppose we want to write a program that could answer questions about family blood relations, questions like "Who is Peter's grandfather?" or "Does he have a child?" or "Is Gabi married?" But our program does not have any information yet about this family. So for any reasoning to take place, first we have to supply the computer with the base knowledge: a set of facts and reasoning rules that adequately describe the situation. Let us start with a few facts. We enter the first line

(1) child(Sofi, Andrew)

meaning that Sofi is Andrew's child. We enter a few more facts:

(2)	child(Carol,Gustav)	(3)	child(Sofi,Gabi)
(4)	child(Ivy,Maria)	(5)	child(Peter,Gabi)
(6)	child(Sonia,Ivy)	(7)	child(Catalina,Ludvig)
(8)	child(Frank,Mary)	(9)	child(Gabi,Rosa)
(10)	child(Gustav,Rosa)	(11)	child(Valeria,Sonia)
(12)	child(Andrew,Ivy)	(13)	wife(Sonia,Jeremy)
(14)	wife(Gabi,Andrew)	(15)	wife(Ivy,Frank)
(16)	wife(Maria,Ludvig)		

We could go on with the list and enter all relations such as grandparent, husband, uncle, etc. But since these are related to the ones we have already defined, it is preferable to enter rules for reasoning. For example wife(X,Y) also means that Y is the husband of X – this we express as

(17) husband(Y,X):- wife(X,Y)

and a few more:

(18) female(X):- wife(X,Y)
(19) parent(X,Y):- child(Y,X)

or a slightly more complicated one:

(20) grandchild(X,Y):- child(X,Z), child(Z,Y)

and finally

(21) grandparent(X,Y):- grandchild(Y,X)

While the entries (1) to (16) are simple facts, the entries (17) to (21) are the rules for automated reasoning. Single capital letters such as X, Y, Z represent variables; they do not have values – or rather names – assigned to them. When we enter (17) the program understands "if for a pair X,Y the statement wife(X,Y) is true, then the statement husband(Y,X) is also true". The entry (20) is interpreted as a procedure. We translate this into English as follows:

> To see whether X is a grandchild of Y, first find out if there is a Z such that X is a child of Z. If so, the next step is to determine if Z is a child of Y.

Now that we have supplied the computer with facts about this family and procedures for finding out more facts, we can start making queries. We ask questions using the same syntax; the question "Does Ivy have a child?" is entered into the computer as

Question- child(X,Ivy)

When the program reads this question, it searches through the list of facts to see if there is one of the kind of child(X,Ivy) for some X. As it finds the statement (6) it returns

Sonia

It also finds (12) and returns

Andrew

Now try another query: "Who is a grandparent of Sofi?" – this we ask as

Question- grandparent(P,Sofi)

The only entry related to grandparent is (21). With this, the program converts the query to grandchild(Sofi,P). Next it finds the definition for grandchild(X,Y) in the entry (20). This procedure tells the program how to go about finding out grandparents for Sofi. There are two steps involved; each of them is called a *subgoal* and they have to be performed in the order they appear from left to right: the first subgoal is to find entries of the form child(Sofi,Z), and the second subgoal is to match child(Z,P) for each Z found in the first subgoal. The search corresponding to the first subgoal gives (1) child(Sofi, Andrew) and (3) child(Sofi, Gabi). For each of them the program proceeds to the second subgoal. First it searches a matcn for child(Andrew, P). It finds (12) and returns

Ivy

Now the program goes back to the second match in the first subgoal and searches for child(Gabi,P); it finds (9), returns

Rosa

and stops. Although every child has four grandparents, the program working on the list of facts and procedures can find only two. If we add the line

(22) child(X,Z):- child(X,Y), wife(Y,Z)

then another grandparent is found for Sofi, namely Frank, because (12) and (15) give child(Andrew, Ivy). Note that (22) may not be true for every family.

The more information we feed the computer with, the more sophisticated are the questions we can ask. If we want the program to answer questions like "Who are the cousins of Peter?" we need to define the procedure for cousin, for which we also need the procedure to find siblings:

```
(23) cousin(X,Y):- child(X,Z), sibling(Z,W), child(Y,W)
(24) sibling(X,Y):- child(X,Z),child(Y,Z)
```

So for the query

```
Question- cousin(Peter,C)
```

the search goes as follows. The procedure cousin is defined in (23) through three subgoals. The first subgoal is to match child(Peter,Z). The program finds (5) child(Peter,Gabi), and through (22) and (14) it finds out that child(Peter,Andrew) is true. The program leaves the second match in memory, and goes to the second subgoal to find if Gabi has siblings, i.e, to start the search for sibling(Gabi,W), which according to (24) has to proceed through two further subgoals. It finds (9) child(Gabi,Rosa) and then (10) child(Gustav, Rosa). So Gabi has a sibling, Gustav. Now to the third subgoal of (23), to search for child(Y,Gustav); (2) child(Carol, Gustav) is the only one found and the program outputs

Carol

The next step is to go back to the second match that was left in memory, child(Peter,Andrew), and to proceed to the second subgoal: to match sibling(Andrew,W). For (24) it finds (12) child(Andrew,Ivy) and also, using (22) and (15), child(Andrew,Frank). Now with each of them it goes to the second subgoal of (24) and finds that sibling(Andrew, Sonia) is true.

Finally, the program moves to the third subgoal of (23) to search for child(Y, Sonia). It finds only Valeria; giving the last output

Valeria

The interesting feature of this programming language is that we can keep adding lines, facts and rules, to expand its knowledge base. I leave to you as an exercise to define the procedures uncle(X,Y), aunt(X,Y), brother(X,Y), sister(X,Y), and niece(X,Y), to pose related questions and trace your procedures. Note that not every question can be answered completely because the program doesn't have enough information. For example, if we leave the program with the 24 lines we entered, the task of finding all female members of this family will lead to a list of all married females, disregarding those that are not listed as wife(X,Y). Although it might well be common knowledge that Catalina is a female name, this is in *our* knowledge base, which has been built up through our living experience; but the program knows only the information we provide. Also, as with any other programming language, if the rules are not defined in a precise manner, we may get strange answers. Note that when we searched for Gabi's siblings, we found only Gustav, but the program would have also returned Gabi, because we did not specify that X and Y in (24) cannot be the same.

As another exercise, you could also try to imagine what a program that performed the same family relations searching task, would look like in a traditional programming language such as Pascal or Basic. How many lines will you need to do the job (23) does? How much processing would be involved to find all Peter's cousins? How would you ask the question in the first place?

With the above example we have given a very brief introduction to the area known as logic programming; this is one of the various ways of tackling automated reasoning. The syntax we used is very similar to that of PROLOG, one of the best-known logic programming languages. Using any such language requires a significant amount of practice to have the programs running effectively and efficiently: procedures must be defined precisely and the order of the lines of information must be carefully chosen to economise processing time.

Traditional programming languages are good at performing mechanical and focused tasks, while logic programming languages, because of their in-built logical power, are ideal for rather unfocused tasks like answering questions about facts entered into the computer.

If you would like to learn more about logic programming in particular and automated reasoning in general, a good book you can start with is *Automated Reasoning, Introduction and Applications* by Larry Wos et al, published by Prentice Hall in 1984.

APPROXIMATING PI[1]

We start with the triangle OAB (Figure 1) in which

$$OA = 1 + t^2, OB = 1 - t^2 \text{ and } AB = 2t, \quad 0 < t < 1$$

It is very easy to show that

$$OA^2 = OB^2 + AB^2$$

which implies immediately that $\angle B$ is a right angle. (See Figure 1.)

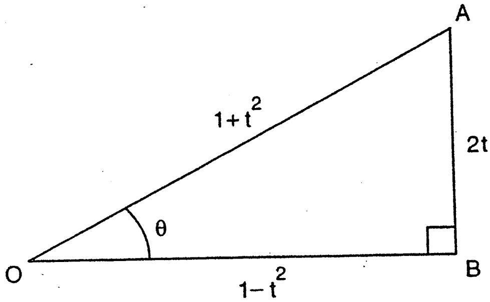

Figure 1

If we set $\angle AOB = \theta$, then we have

$$\tan\theta = \frac{2t}{1-t^2}.$$

Figure 2 is an elaboration of Figure 1. Here $OC = OA$ and thus we may readily prove that $\angle BAC = \frac{\theta}{2}$. We also find $BC = 2t^2$, and thus from the triangle ABC we have $\tan\frac{\theta}{2} = t$.

But now $\tan\theta = \frac{2t}{1-t^2}$ and if we put $T = \tan\theta$, by using the formula for the solutions to a quadratic equation and discarding the negative solution, we find

$$t = \frac{-1 + \sqrt{1+T^2}}{T}. \tag{1}$$

Now consider the special case $\theta = \frac{\pi}{4}, T = \tan\theta = 1$ and apply equation (1) to find

$$\tan\frac{\pi}{8} = -1 + \sqrt{2} \simeq 0.4142\ldots\ .$$

[1]This approximation of π is based on an article written by our reader A Woodman (Cranbourne, Victoria).

If we apply equation (1) yet again, we will find

$$\tan\frac{\pi}{16} = \frac{-1+\sqrt{4+2\sqrt{2}}}{\sqrt{2}-1} \simeq 0.1989\ldots\ .$$

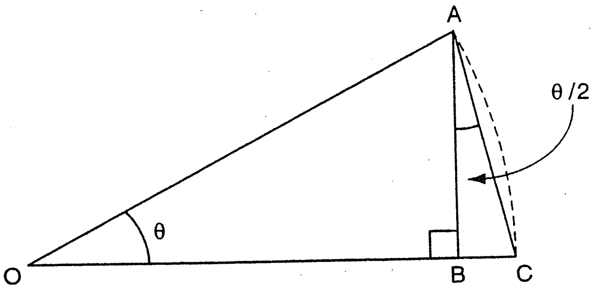

Figure 2

But we may continue in this way to find $\tan\frac{\pi}{32}$, $\tan\frac{\pi}{64}$, etc. These different numbers, $t(n)$ say, will give the values of $\tan(\pi/2^n)$ and as n gets larger, the angle $\pi/2^n$ gets smaller. Thus, because we are working in radians, we have[2]

$$t(n) = \tan(\pi/2^n) \simeq \pi/2^n,$$

and the approximation gets better the smaller the angle becomes, that is to say, the larger the value of n. It thus follows that

$$\pi \simeq 2^n t(n).$$

Table 1 shows the successive values of approximations to π calculated using a spreadsheet.[3] It will be seen that the successive values of $2^n t(n)$ approach π until we reach the limits of the available software.

[2]The graph of the function $\tan(x)$ is very well approximated by the straight line $y = x$ when x is very close to 0. You can check this by plotting their graphs with a computer program.

[3]See *Function, Vol* 17, *Part* 3, pp 76-82.

n	t(n)	2^n * t(n)
	1	
3	0.4142	3.313708499
4	0.1989	3.182597878
5	0.0985	3.151724907
6	0.0491	3.144118385
7	0.0245	3.14222363
8	0.0123	3.141750369
9	0.0061	3.141632081
10	0.0031	3.14160251
11	0.0015	3.141595118
12	0.0008	3.14159327
13	0.0004	3.141592808
14	0.0002	3.141592691
15	1E-04	3.141592676

Table 1

* * * * *

Once upon a time a professor accidentally (and temporarily) landed in another world right at the feet of Socrates. The following conversation ensued.

PROFESSOR: But, Socrates, I had so many questions to ask you. Instead, *you* have asked all the questions.

SOCRATES: I am sorry. It's what I do. I have always maintained my ignorance, but when I ask questions, I learn and my students learn. The moment I start giving answers, their sense of inquiry goes to sleep, and the lesson is over.

–*Kennesaw State College (GA) Newsletter*, Spring 1991

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Letter to the Editor

In the interesting article *Sand or Water: Telling them Apart* in *Function, Vol* 19, *Part* 5, a statistic z was calculated, on page 149, as having the value 0.536, because "the value of $\sqrt{n}$ cancels out". Unfortunately, z should be 1.268; there are really two different $\sqrt{n}$'s which do not cancel out. In the expression for σ, namely $\sqrt{npq}$, the appropriate n is the number of guesses made by each person, namely 10, not the "number of persons in the group". On the other hand, in the expression

$$z = \frac{\overline{x} - \mu}{\sigma/\sqrt{n}}$$

the n here is the "number of persons in the group" over which the average $\overline{x}$ is calculated, namely 56.

The revised value of z does not change the conclusion that the Skeptics could have been just guessing, without having any real skill at distinguishing between the boxes which contained sand and those which contained water.

There are two other matters of importance in this experiment:

- It is because $\sqrt{n}$ does not really cancel out, that the experiment becomes better, the more people attempt it. If the same results had been achieved by a larger group of people, then that may have been convincing evidence in favour of water-divining powers.
- The formula for σ assumes that each person makes their 10 guesses as statistically independent trials, so that the number of correct guesses has the binomial probability distribution. If, however, some people assume that there will be five boxes of sand and five of water (I note that they were not told this) and that their task is simply to decide which are which, then a different (smaller) value applies for σ because the number of correct guesses would then have a hypergeometric probability distribution.

Geoff Watterson

Ed Dr Watterson is correct, and his z-value should replace that quoted in the article. As a result of this amendment some of the other numerical values given in the subsequent discussion also require revision. However, as Dr Watterson remarks, the conclusions remain intact. A corrected version of the article can be obtained by writing to Dr M Deakin.

NEWS: ANOTHER SUNDIAL

We have had several articles on the unusual Monash sundial. (See *Function*, *Vol* 5, *Part* 5; *Vol* 14, *Part* 4; *Vol* 15, *Part* 3.) This tells us both the time and the date by the position of a shadow on the north wall of the Union building. The scale on the wall comprises a number of "analemmas" (figure-8 shapes) which identify the hours and a further set of curves representing dates.

A similar principle underlies a new (and even more elegant) sundial to be seen in the central mall of the main (Hobart) campus of the University of Tasmania. The accompanying photograph shows the sundial and its designer: Dr Tony Sprent of the Department of Surveying in Hobart.

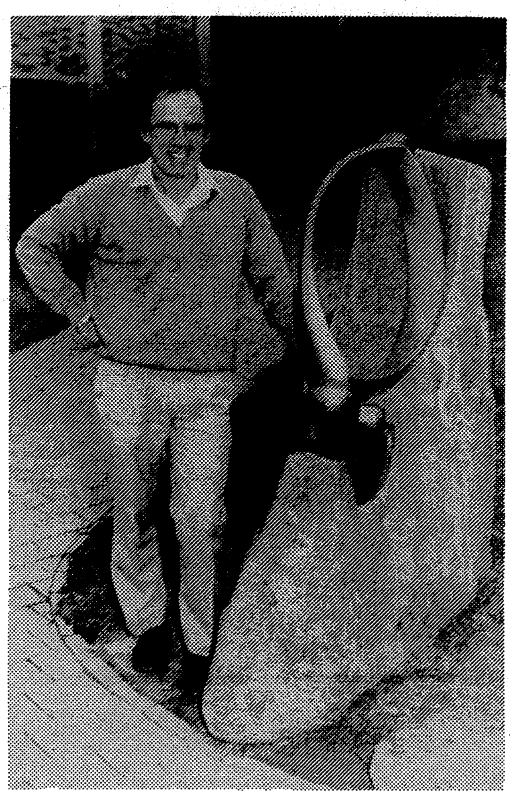
The main component of the instrument is a bronze loop housed in two pivots, one at the top and the other at the bottom. The loop may thus turn about an axis running between these and this rotation may be brought about by means of the knob at lower right of the movement.

The axis of rotation is so aligned as to be perpendicular to the earth's axis and also perpendicular to the line from east to west. Through its upper loop, a pinhole (not visible in the photograph) has been drilled and the geometry is so set up that this may be lined up precisely with the rays of the sun merely by turning the knob.

The point of light then falls onto a carefully engraved analemma (again not visible in the photograph) on the lower loop. The point of the analemma on which the light shines gives the date on a scale engraved in the brass itself. The hours and minutes come from scales at the base of the instrument. Hours are on the lower pivot and minutes on the turning knob.

Dr Sprent sees his instrument as combining Science, the Arts and Technology in a unity fitting for the university, which (as its name implies) seeks a synthesis of knowledge. The detailed calculations he himself performed supply the Science. The sculptor (Marti Wolfhaggen) who made the beautiful instrument supplied the Art, and Kirwan Engineering and Retlas Bronze, who between them completed the actual fabrication, represent Technology.

The sundial is accurate to within about a minute and Dr Sprent recalls seeing a student use it to see if he was running late for a lecture. At least, this is how things should be and usually are. But like our Monash sundial, Tasmania's has had its ups and downs. Recently a water main burst and the subsequent flood caused the stone base to settle at the wrong angle and so to be about 10 minutes (or, if you like, a week) out. It had to be realigned to allow the sundial once more to demonstrate its full potential.



Dr Tony Sprent next to the new sundial in the central mall, Hobart campus (Photo taken from the *Alumni Review* UTESAA, April 1994)

* * * * *

Once, when Isaac Newton was asked how he made all of his discoveries, he replied "If I have seen further than others, it is by standing on the shoulders of giants."

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 19.4.1 (K R S Sastry, Dodballapur, India)

In the convex quadrilateral $ABCD$, the diagonal AC is the bisector of angle DAB and a trisector of angle BCD, and the diagonal BD is a trisector of angle CDA and a quadrisector of angle ABC, as indicated in Figure 1. Also, $\alpha, \beta, \gamma, \delta$ are natural numbers of degree measures of the angles indicated. Find the possible degree measures of the angles of $ABCD$.

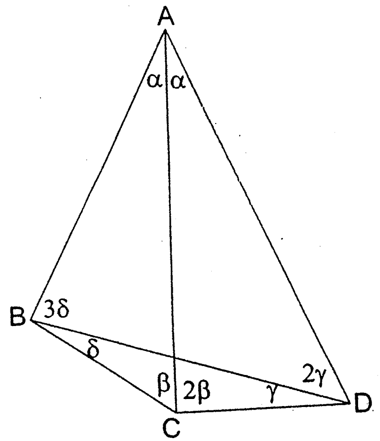


Figure 1

SOLUTION by K R S Sastry

From the figure it follows that

$$\begin{aligned} \alpha + 2\gamma &= \beta + \delta & (1)\\ \alpha + 3\delta &= 2\beta + \gamma & (2)\\ \alpha + \beta + 4\delta &= 180 & (3) \end{aligned}$$

From (1) and (2) we have

$$\alpha = -5\gamma + 5\delta \qquad (4)$$

$$\beta = -3\gamma + 4\delta \qquad (5)$$

Then from (3), $-8\gamma + 13\delta = 180$, and $\delta > \gamma$ from (4), i.e.

$$5\delta + 8(\delta - \gamma) = 180. \qquad (6)$$

Since 5 and 8 do not have common divisors and 5 does not divide 180, it must be that 5 divides $(\delta - \gamma)$.

Suppose $\delta - \gamma = 5$. Then from (6), $\delta = 28$; from (4), $\alpha = 25$; $\gamma = 23$; from (5), $\beta = 43$. Hence $A = (2\alpha)^\circ = 50^\circ$, $B = (4\delta)^\circ = 112^\circ$, $C = (3\beta)^\circ = 129^\circ$, $D = (3\gamma)^\circ = 69^\circ$.

Suppose $\delta - \gamma = 10$. This yields $\alpha = \beta = 50$, $\gamma = 10$, $\delta = 20$, and so $A = 100^\circ$, $B = 80^\circ, C = 150^\circ$, $D = 30^\circ$.

$\delta - \gamma > 10$ is impossible.

Solutions were also received from John Barton (Carlton North, Vic) and Keith Anker (Monash University).

PROBLEM 19.4.2 (Juan-Bosco Romero Márquez, Departamento de Algebra, Geometría y Topología, Universidad de Valladolid, Valladolid, Spain)

Find all solutions in positive integers of the Diophantine equation $4xy = 9(x + y)$.

SOLUTION by Keith Anker (Monash University)

Multiply the equation by 4 and rearrange to obtain $4x \cdot 4y - 9(4x + 4y) = 0$. By adding 81 to both sides and factorising, the equation can be written $(4x - 9)(4y - 9) = 81$. Therefore $4x - 9$ is a factor (positive or negative) of 81, so $4x - 9 \in \{\pm 1, \pm 3, \pm 9, \pm 27, \pm 81\}$. Checking each value in turn, we obtain $(x, y) = (3, 9)$ and $(x, y) = (9, 3)$ as the only solutions.

Also solved by John Barton and the proposer.

PROBLEM 19.4.3 (South African 1993 Old Mutual Mathematics Olympiad)

For which values of n is the number $S_n = 1! + 2! + \ldots + n!$ the square of an integer?

SOLUTION by John Barton

Every factorial beyond 4! is congruent to 0 modulo 10. The sums up to 4! are 1, 3, 9, 33, of which the first and third are square. All sums beyond this point are congruent to 3 modulo 10, and hence are not square. The required values of n are 1 and 3.

Also solved by Derek Garson (Lane Cove, NSW) and Keith Anker.

PROBLEM 19.4.4 (South African 1993 Old Mutual Mathematics Olympiad)

For every pair of natural numbers p and q a circle is drawn above the x-axis; it has diameter $\frac{1}{q^2}$ and it touches the x-axis at the point $\left(\frac{p}{q}, 0\right)$. Prove that no two circles intersect, and determine the condition for tangency.

SOLUTION by John Barton

Consider two of the circles, with contact points $\left(\frac{p}{q}, 0\right)$ and $\left(\frac{p'}{q'}, 0\right)$ and radii $\frac{1}{2q^2}$ and $\frac{1}{2q'^2}$ respectively (Figure 2). The separation s of their centres is given by

$$s^2 = \left(\frac{p'}{q'} - \frac{p}{q}\right)^2 + \left(\frac{1}{2q'^2} - \frac{1}{2q^2}\right)^2$$

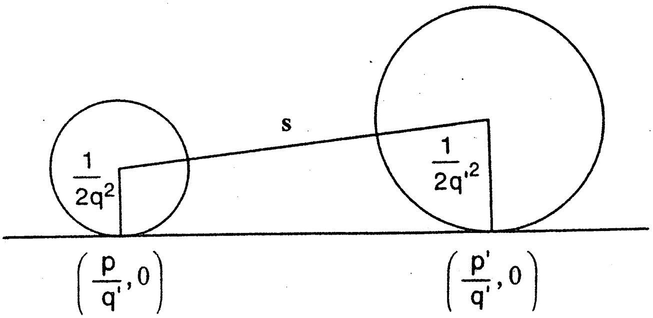


Figure 2

The sum of the radii is $\frac{1}{2q'^2} + \frac{1}{2q^2} = \sigma$, say, and we have

$$\begin{aligned} s^2 - \sigma^2 &= \left(\frac{p'}{q'} - \frac{p}{q}\right)^2 + \frac{1}{4}\left(\frac{1}{q'^2} - \frac{1}{q^2}\right)^2 - \frac{1}{4}\left(\frac{1}{q'^2} + \frac{1}{q^2}\right)^2 \\ &= \frac{(p'q - pq')^2 - 1}{q'^2 q^2} \\ &\geq 0, \text{ because } p'q - pq' \text{ is an integer.} \end{aligned}$$

Equality occurs only when $|p'q - pq'| = 1$ and this will give a sufficient condition for tangency. If $|p'q - pq'| > 1$ then $s^2 - \sigma^2 > 0$, so $s > \sigma$ and the circles do not intersect.

We also note that if $p'q - pq' = 0$, that is, $\frac{p'}{q'} = \frac{p}{q}$, then the circles are tangent at $\left(\frac{p}{q}, 0\right)$. If p and q are coprime, all the circles corresponding to the pair of natural numbers αp and αq ($\alpha = 2, 3, 4, \ldots$) touch the circle corresponding to p and q, and lie inside it.

We can summarise the condition for tangency as $p'q - pq' \in \{0, \pm 1\}$.

PROBLEM 19.4.5 (from Trigg C W, *Mathematical Quickies*, 1967, McGraw-Hill)

During a period of days, it was observed that when it rained in the afternoon, it had been clear in the morning, and when it rained in the morning, it was clear in the afternoon. It rained on 9 days, and was clear on 6 afternoons and 7 mornings. How long was this period?

SOLUTION by Benito Hernández-Bermejo, Madrid

From the conditions of the problem we observe that the only possible combinations are clear/rain, rain/clear and clear/clear in the morning/afternoon, respectively. That is, the only forbidden possibility is rain/rain. We shall make the following definitions:

$$\begin{aligned}
x &= \text{number of days in which we have clear/rain} \\
y &= \text{number of days in which we have rain/clear} \\
z &= \text{number of days in which we have clear/clear}
\end{aligned}$$

Then, from the data of the problem:

$$\begin{aligned}
x + y &= \text{number of days with rain} = 9 \\
x + z &= \text{number of clear mornings} = 7 \\
y + z &= \text{number of clear afternoons} = 6
\end{aligned}$$

And, of course, $x + y + z$ equals the number of days of the period, which is the unknown. Adding the three equations, we are led to:

$$2(x + y + z) = 22, \text{ so } x + y + z = 11,$$

which is the solution. Notice that we do not need to solve the system in full detail to find the answer. However, you can check that $x = 5$, $y = 4$ and $z = 2$.

A misprint in the problem when it was originally stated led to an awkward "solution" involving fractions of a day. Two readers, John Barton and Keith Anker, solved the problem in this form.

PROBLEM 19.4.6 (from Trigg C W, *Mathematical Quickies*, 1967, McGraw-Hill)

Albert and Bertha Jones have five children: Christine, Daniel, Elizabeth, Frederick and Grace. The father decided that he would like to determine a cycle of seating arrangements at their circular dinner table so that each person would sit by every other person exactly once during the cycle of meals. How did he do it?

SOLUTION by John Barton and Keith Anker

There is no loss of generality if we assume that the family is seated in alphabetical order for the first meal: $ABCDEFG$. The neatest solution is obtained by taking every second name in cyclic order for the second meal, $ACEGBDF$, and every third name for the third meal, $ADGCFBE$.

The extent to which this approach can be generalised to other family sizes is left to our readers to explore.

Solution to an earlier problem

The problem below appeared in the April 1992 issue of *Function*. A solution has not been published previously in *Function*.

PROBLEM 16.2.4 (from Juan-Bosco Romero Márquez, Valladolid, Spain)

Let T and T' be two right-angled triangles. Let R, r and R', r' be the circumradius and inradius of T and T' respectively. Prove that if $R/R' = r/r'$, then T and T' are similar. Does this theorem hold for a larger class of triangles?

SOLUTION

Let the triangle T have vertices at A, B and C, with the right angle at C. Let a and b denote the lengths of the sides opposite A and B respectively. The centre of the circumcircle of a right-angled triangle is the midpoint of the hypotenuse, so the length of the hypotenuse must be $2R$, i.e. twice the circumradius. Let D be the incentre of the triangle, and let E, F and G be the bases of the perpendiculars to the sides $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ respectively, passing through D. (See Figure 3.)

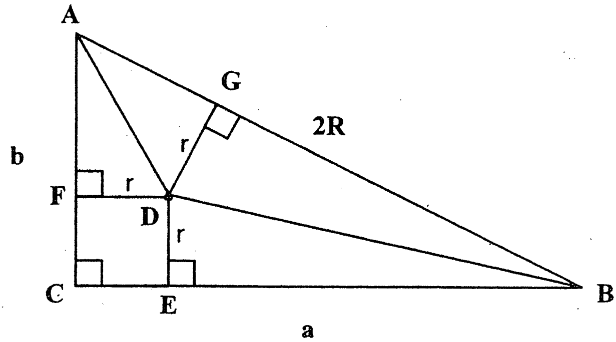


Figure 3

From the figure, triangles BDE and BDG are congruent, so $BE = BG$ and thus $BG = a - r$. Similarly, $AG = b - r$. Therefore $a + b - 2r = AB = 2R$, so

$$a + b = 2(R + r) \tag{1}$$

From the original triangle T, we also have the equation

$$a^2 + b^2 = 4R^2 \tag{2}$$

From equations (1) and (2) we obtain

$$\frac{(a+b)^2}{a^2+b^2} = \frac{(R+r)^2}{R^2}$$

This equation can be written in the form

$$\frac{\left(\frac{a}{b}+1\right)^2}{\left(\frac{a}{b}\right)^2+1} = \left(1+\frac{r}{R}\right)^2 \tag{3}$$

Upon replacing a/b and r/R by u and v respectively, equation (3) becomes a quadratic equation in u. You can solve it if you wish, but there is no need; all that matters is that equation (3) has at most two solutions for u in terms of v. In fact, if $u = u_0$ is one solution then $u = 1/u_0$ is the other, since swapping a and b does not change the geometry of the problem, but corresponds merely to relabelling the two sides of the triangle.

Now, if we have two triangles, T and T', and if $R/R' = r/r'$, then equation (3) must apply to T and T' with the same value of v. Therefore either $a/b = a'/b'$ or $a/b = b'/a'$, so T and T' are similar.

If T is not a right-angled triangle, we can calculate the ratio r/R for T and try to solve equation (3) using this value. If a real solution a/b exists, form a right-angled triangle T' whose sides adjacent to the right angle are in the ratio $a : b$. Then $R/R' = r/r'$ but the triangles are not similar, so the theorem could not be extended to a larger class of triangles containing T. However, there are classes of triangles for which the ratio r/R is such that equation (3) has no real solutions – equilateral triangles (for which $r/R = \frac{1}{2}$) are an example, as you may check. Thus the theorem can be extended, albeit rather artificially, to larger classes of triangles – for example, the class comprising all right-angled triangles and all equilateral triangles.

PROBLEMS

Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received in sufficient time will be acknowledged in the next issue but one, and the best solutions will be published.

Just for a warm-up, the first problem for 1996 is an amusing exercise in proportions from Keith Anker of Monash University.

PROBLEM 20.1.1

If a hen and a half lay an egg and a half in a day and a half, how many hens will lay two eggs in three days? (Complaints to the editors about the unreality of this problem are not encouraged!)

And now for some "real" problems:

PROBLEM 20.1.2 (Claudio Arconcher, São Paolo, Brazil)

Find solutions in positive integers for

(a) $\quad x^x + y^y = 2xy$ $\qquad$ (b) $\quad x^x + y^y + z^z = xy + xz + yz.$

PROBLEM 20.1.3

Prove that for any positive integer n, $\lfloor n + \sqrt{n} + \frac{1}{2} \rfloor$ is not a perfect square. (The notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

PROBLEM 20.1.4 (1994 Old Mutual Mathematics Olympiad, South Africa; reprinted from *Mathematical Digest*, January 1995, University of Cape Town)

A, B, C, D and E are distinct points in three-dimensional space, such that A, B and C lie on the surface of a sphere. Prove that at most one of the four-sided figures $XYDE$, where X and Y are two of the points A, B and C, can be a parallelogram.

PROBLEM 20.1.5 (from *Mathematical Mayhem*, Vol 7, Issue 5, University of Toronto)

Let $G(n)$ be the number of strictly increasing or decreasing sequences formed using the values $1, 2, \ldots, n$, e.g. for $n = 2$ there are 4 sequences (1), (2), (1,2), (2,1). Find an explicit formula for $G(n)$ in terms of n.

PROBLEM 20.1.6 (from Sweedish Mathematical Olympiad, 1979 Qualifying Round; reprinted from *Mathematical Mayhem, Vol* 8, *Issue* 1, University of Toronto)

For which real values of a, $a \geq 1$, is $\sqrt{a + 2\sqrt{a-1}} + \sqrt{a - 2\sqrt{a-1}} = 2$?

PROBLEM 20.1.7 (based on a problem seen on the Internet)

1. Let two circles C_1 and C_2 be given, with C_1 inside C_2. A third circle, C_3, moves around the region between C_1 and C_2, in such a way that it is always tangent to both circles. Prove that the locus of the centre of C_3 is an ellipse.
2. Now suppose the problem in (1) is modified so that C_1 is outside rather than inside C_2. What type of figure is described by the locus of the centre of C_3 in this situation?
3. What happens if C_1 and C_2 overlap?

PROBLEM 20.1.8 (from the Internet, author unknown)

From each corner of a unit square draw a quarter of an inscribed unit circle. Find the area of the central diamond shape where the four quarter circles overlap.

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