Function

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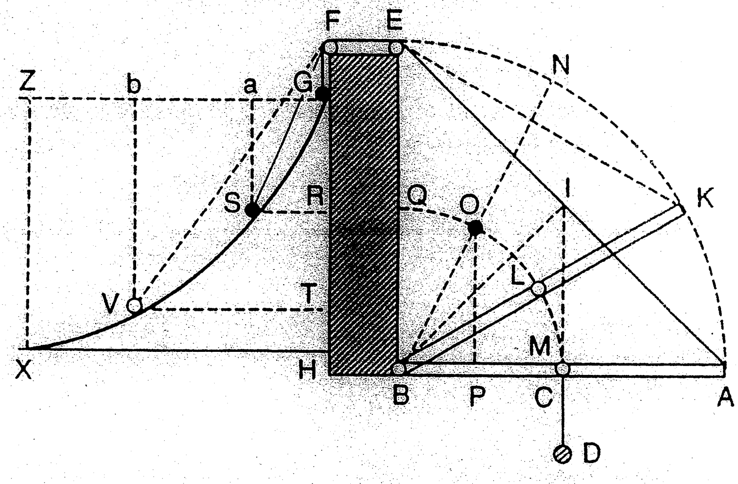

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Function is a mathematics magazine produced by the Department of Mathematics at Monash University. The magazine was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Alternatively, correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside back cover.

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EDITORIAL

We welcome readers to our first issue of *Function* for 1995.

In our first feature article, "Roland Percival Sprague and the Impartial Game", H Lausch explains a technique for recognising a "winner" position of a Nim game and finding a winning strategy. In the second feature article, "Multi-digit Numbers", A R Boyd explores some interesting properties of numbers that are multiples of the product of their digits.

The diagram on the front cover is a sketch of a drawbridge which is to be raised and lowered by means of a cable. M Deakin presents the underlying mathematics which involves the curve known as a cardioid.

In our regular *History of Mathematics* column, two earlier articles are updated: the first update explains how a curve acquired the name of "witch" due to poor translation; the second discusses the formula for the energy released by the first atomic explosion. In the *Computers and Computing* section you will find how pseudo-random numbers are generated by a computer.

As usual, in the *Problem Corner* there are problems for you to attempt and solutions to problems published in earlier issues. We also include the problems of the XXXV International Mathematical Olympiad. We are thankful to all those readers who sent solutions to problems. Some of them are acknowledged and their solutions published in this issue.

Our best wishes for 1995, especially to those readers who have just started their last year of secondary schooling.

Happy reading!

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THE FRONT COVER

The Mathematics of a Drawbridge

Michael A B Deakin

The front cover shows a diagram based on one in an 18th century book of military engineering: *La science des ingénieurs* by the mathematician and military theorist Bernard Forest de Bélidor (1697-1764). It depicts a drawbridge which is to be raised and lowered by means of a cable (the line $AEFG$ in the diagram when the movable span is down, the line $KEFS$ when the span is somewhat raised, etc.). In order to assist the lift, the cable is attached at its left-hand end to a counterweight (G, S or V in the diagram).

The counterweight is made to run down a curved track $GSVX$ and there is a reason for this. If the weight simply descended vertically behind the supporting tower $FHBE$, then it would exert a constant force on the cable. The *actual* force required to raise the movable span of the bridge (BA) is however greatest when the span is in its fully lowered position and progressively decreases as the lift gets under way. If the force were not somehow reduced as the span rose, then the span would crash into the tower and do a lot of damage. Moreover, once the span was raised, the full force of the counterweight would resist any attempts to lower the span once more.

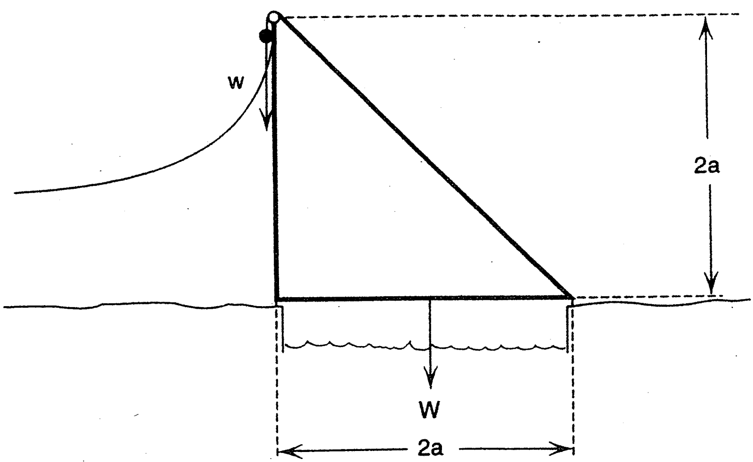

Figure 1

The shape of the track on which the counterweight runs is therefore so designed as to overcome these two problems. The idea is that all the moving parts of the bridge (span plus counterweight) be in equilibrium (i.e. perfect balance) at all stages of the lift.

Look at Figure 1, which depicts a somewhat idealised case. W is the weight of the movable span and w the weight of the counterweight. The length of the span is $2a$, and this is also the height of the tower. Here the bridge is shown in the "down" position. Figure 2 shows the same idealised bridge in its general position (part raised).

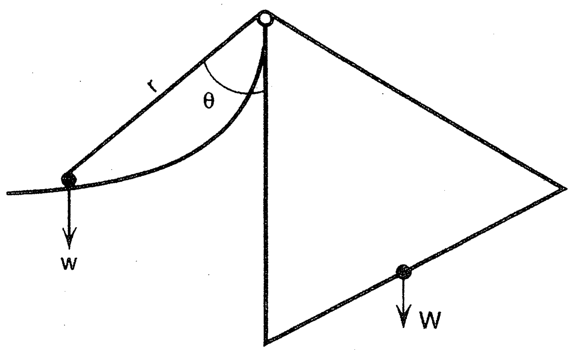

Figure 2

From this second diagram we can calculate the position of the counterweight. It is given in terms of polar co-ordinates (r, θ) where these are as shown in the figure.[1]

I will not give the details but the result is

$$r = 2l(1 - \cos\theta), \tag{1}$$

where l is the total length of the cable. (From Figure 1, we may see that, in this idealised case, $l = (2\sqrt{2})a$.)

This is quite an interesting (and also in some ways an ironic) result. The curve is one that is known as a cardioid, or more accurately it is an arc of a cardioid. The full cardioid is shown in Figure 3 and the arc AB is the part in question. The curve was named as a cardioid (or more

[1] For an article on polar co-ordinates, see *Function, Volume* 18, *Part* 4.

precisely by a closely related Latin word) by the Italian mathematician Johann de Castillon, writing in 1741. The name derives from the apparent resemblance of the full curve to a stylised picture of a heart. The first mathematician to study the curve, however, was one Phillipe de la Hire, in 1708. In fact, there were two men of this name, father (1640-1718) and son (1677-1719).

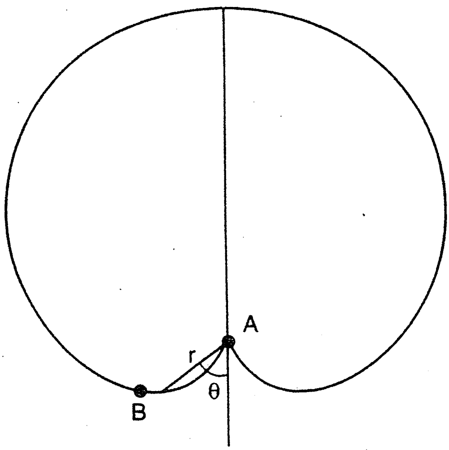

Figure 3

It was probably the older man who first studied the curve later termed the cardioid, and it may have been he also who was one of de Bélidor's teachers (although *this* person is more likely to have been the son). In any case, father and son collaborated very closely and so de Bélidor was perhaps in a better position than anybody to recognise the cardioidal shape of the track. However, he missed this opportunity. Indeed, the connection with the cardioid was not appreciated until very recently (1990).

De Bélidor in fact did not derive an equation for his curve. After his death, his book *La science des ingénieurs* was revised in two posthumous editions by the physicist and engineer Navier (1785-1836). (Navier's is one of two names now attached to the basic equations of fluid dynamics.)

Navier did derive an equation for the curve, but he did so in terms of co-ordinates that are not used nowadays and certainly would not be familiar to readers of *Function*. However, if we convert his equation into polar co-ordinates the result is not Equation (1) but the more general

$$r^2 + 2(A\cos\theta - l)r = B, \tag{2}$$

where A and B are constants. If $B = 0$ and $A = l$, then the simpler form (1) is regained.

The reason for the discrepancy is that Navier analysed a somewhat more general case. If we compare the cover diagram with Figure 1, we may notice that when the span is fully lowered, the counterweight is right at the top of the tower in Figure 1, but somewhat below it in the cover diagram. This simplification in Figure 1 makes for the simpler equation (1).

De Bélidor himself built no actual bridges along these lines. He may have been aware of a ninth century structure in Corsica that (vaguely) approximates his design, but in fact his principle was not used until 1895 when a bridge of this type was used to cross one of the branches of the Chicago river. Several more such were built in the US over the next dozen or so years.

It was in New South Wales, however, that probably an even greater number were built. The designer was Harvey Dare (1867-1949). Dare was a brilliant graduate of the University of Sydney, who made a life career in the NSW public service. In 1901 he designed the first of his Bélidor bridges – at Telegraph Point on the Pacific Highway. It was opened in 1902 and carried the highway for 72 years.

Dare designed a total of eight such bridges in the first quarter of this century. Three of them can still be seen intact (although they are no longer raised and lowered). These are to be found at Carrathool Crossing on the Murrumbidgee, at Maclean, and at Coraki in northern NSW. A railway bridge on Sydney's Botany Line was once also of this type, although this is no longer apparent as the towers and the counterweight track have been removed. At Darlington Point (also on the Murrumbidgee) the towers and track have been reassembled (on land, at the entrance to the town's caravan park) as a public monument. The other three, the Telegraph Point bridge, and others at Swansea and Kyalite, are now gone.

I have measured the Darlington Point structure to see how well the shape of the curved track approximates to Equations (1) and (2). From my measurements, I estimate $l = 33$ (metres), $A = 39$ (metres) and $B = 177$ (m^2), and this equation is an excellent fit. However, A is not particularly close to l, nor is B particularly small compared with l^2, so an attempt to fit Equation (1) will not give a particularly accurate result. I have similarly looked at the Telegraph Point bridge, using one of Dare's original design drawings – with a similar result.

Figure 4 shows the author standing on the bridge at Carrathool Crossing. The photograph is the work of Mrs R Jenkinson of Taylor's College.

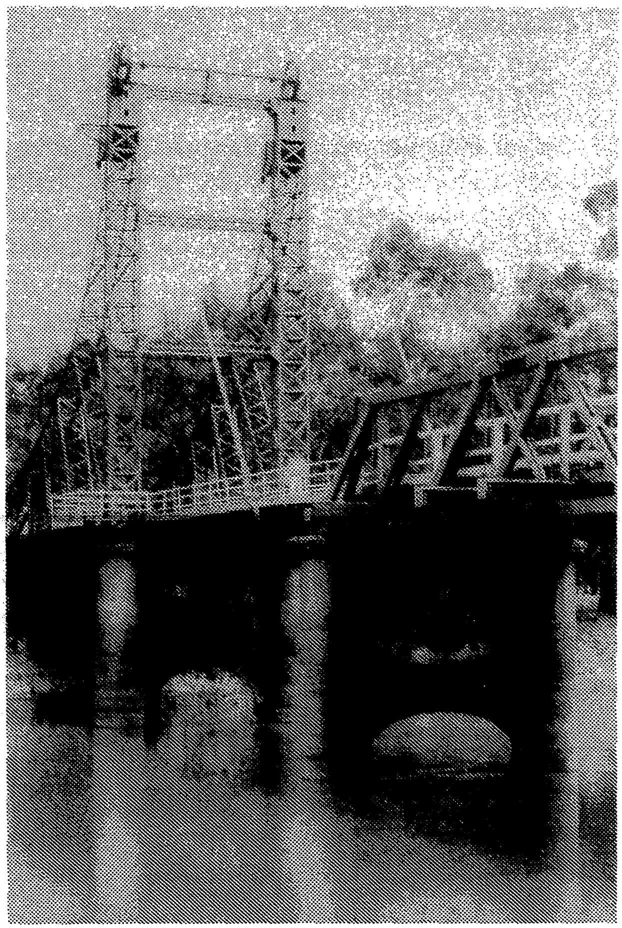

Figure 4

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ROLAND PERCIVAL SPRAGUE AND THE IMPARTIAL GAME

Hans Lausch, Monash University

Impartial games

"What is an impartial game and who was Roland Percival Sprague?" you may ask. To answer the first question, we consult the games manual written by Berlekamp, Conway and Guy (1982: 16-17).

A game of the type we are going to consider has:

1. two **players**, usually people,
2. **positions**,
3. clearly defined **rules** that specify how either player can **move**, i.e. into which positions a given position can be changed by either player.

Players and moves have to satisfy the following conditions:

4. players perform moves **alternately**, in the game as a whole;
5. the rules guarantee that play **will always come to an end**, because some player will be unable to move;
6. both players know what is going on, i.e. there is **complete information**;
7. there are no **chance moves** (such as rolling dice or shuffling cards).

A player unable to move is termed the **loser**, the other the **winner** – draws or ties are outlawed.

Among the activities that do *not* qualify as games in our restricted sense are:

- Noughts-and-crosses – ties can arise;
- Chess – stalemates and positions that are drawn by infinite play (e.g. by perpetual check) are possible;
- Backgammon – has chance moves, since it uses dice;

- Battleships – players do not have complete information about the disposition of their opponent's pieces;
- Bridge – has two "players", each being a couple of persons, but these "players" do not even have complete information about their own cards;
- Football – has two "players", each being a team, but the definition of appropriate "positions" and "moves" causes difficulties.

But don't worry, plenty of "real" games are still at hand, so many that we can afford to concentrate solely on **impartial** games. Besides all our conditions above, they meet a further one: from any position exactly the same moves are available to either player.

Let us first have a look at Nim, a game played with counters arranged in heaps. A Nim position is an array of such heaps. A Nim move is made by taking any positive number of counters from any *one* heap. Thus, the winner is the player who removes the last counter.

Jim and Kim play Nim. Initially four heaps are on the table top: heap 1 with 2 counters, heap 2 with 3 counters, heap 3 with 7 counters, and heap 4 with 5 counters. We denote the position briefly by (2, 3, 7, 5). First to move is Jim, and the game continues as follows:

- Jim takes 3 counters from heap 3; new position: (2, 3, 4, 5).
- Kim takes 1 counter from heap 1; new position: (1, 3, 4, 5).
- Jim takes all 3 counters from heap 2; new position: (1, 0, 4, 5).
- Kim takes 2 counters from heap 4; new position: (1, 0, 4, 3).
- Jim takes 2 counters from heap 3; new position: (1, 0, 2, 3).
- Kim takes 1 counter from heap 4; new position: (1, 0, 2, 2).
- Jim takes the remaining counter from heap 1; new position: (0, 0, 2, 2).
- Kim takes 1 counter from heap 4; new position: (0, 0, 2, 1).
- Jim takes one counter from heap 3; new position: (0, 0, 1, 1).
- Kim takes the counter from heap 3; new position: (0, 0, 0, 1).
- Jim becomes the winner by removing the last counter.

After this encounter you could feel tempted to query: "Should Kim have prevented Jim from winning?" Well, she should have, but could she have by sticking to the rules?

Charles Bouton (1901/2) was the first to provide an analysis of Nim. Besides Nim, though, there is a wealth of impartial games.

Consider the game of Kayles (Dudeney 1910), or Rip Van Winkle's Game (Loyd 1914). Again, Kayles can be played on a tabletop with heaps of counters. A move consists of taking 1 or 2 counters from any *one* heap and then exercising the option whether to split this heap into two smaller heaps or not. Then there is Lasker's Nim, invented by the mathematician, philosopher and world chess champion Emanuel Lasker (1868-1941): positions are the same as in Nim, i.e. heaps of counters. A move consists of choosing one heap and either reducing its size (as in an ordinary Nim move) or splitting it into two smaller heaps. Or, try the games of Prim and Dim. In Prim you may remove m counters from a heap of size n provided m and n are relatively prime (i.e. no whole number larger than 1 divides both m and n). In Dim you may take d counters from a heap of size n provided d divides n. In Welter's Game, positions are heaps, no two different ones of which have the same size. A move consists of choosing one heap and reducing its size such that its new size differs from all the other heap sizes. And so on and so forth.

Roland Percival Sprague

In 1936 an article on "mathematical battle games", written by R P Sprague, appeared in a Japanese mathematical journal, showing that *every* impartial game is, basically, a Nim game. Each position and each move of an arbitrary impartial game is "translatable" into a Nim position and into a Nim move, respectively.[1] Hence, if one knows how to win Nim and has full control over the "translation" processes, then one will know how to win *any* impartial game. After reading this *Function* article, you will be able to win Nim, whenever possible.[2] However, certain impartial games are quite difficult to "translate" into Nim. Readers interested in this much more demanding problem should consult Berlekamp, Conway and Guy (1982).

Roland Percival Sprague was born one hundred years ago, on July 11, 1894, in Unterliederbach near Hoechst (Germany) where his father worked

[1] P M Grundy discovered the same result independently three years later.

[2] Clearly, you cannot always win. Take e.g. two heaps of size 1 and have the "privilege" of moving first.

as a research chemist for a famous industrial company. Having served as President of the Edinburgh Mathematical Society, Sprague's paternal grandfather Thomas Bond Sprague (1830-1920) published articles on mathematical games. Sprague's maternal grandfather was the mathematician Hermann Amandus Schwarz (1843-1921), whose most widely known result is the Schwarz inequality, a far-reaching generalisation of the so-called Cauchy inequality.[3] Looking for further mathematicians in Sprague's family tree, one discovers his great-grandfather Eduard Ernst Kummer (1810-1893); Kummer had made notable progress in his attack on Fermat's Last Theorem. One generation further into the past, one finds Nathan Mendelssohn (1782-1852) among Sprague's ancestors; Mendelssohn was a maker of mathematical instruments and had workshops in Berlin and London. A son of the philosopher and mathematician Moses Mendelssohn (1728-1786), he was an uncle of the composer Felix Mendelssohn Bartholdy and of Rebecka Dirichlet, wife of the mathematician Peter Gustav Lejeune Dirichlet (1805-1859).

Sprague went to school in Berlin and then studied mathematics, physics and chemistry at the Georgia Augusta University, Göttingen, where he finished with a diploma of education. Between 1925 and 1949 he taught at several Berlin grammar schools. In 1944 he completely analysed a special case of Welter's Game. In 1949 Sprague began teaching at the Pedagogical University of Berlin. In 1950, at the age of 56, he submitted a thesis and was awarded the degree of Doctor of Philosophy. In 1961 his delightful book of mathematical problems was published which, among many other delights, contains a complete analysis of Lasker's Nim; two years later an English translation under the title *Recreation in mathematics: some novel problems* was prepared. Sprague died in Berlin on August 1, 1967.[4]

How to win Nim if you can

From the definition of impartial games it follows that each position is either a "winner" position or a "loser" position, but never both. Winner positions are those from which the player to move next has a winning strategy available, i.e. can force a win by playing "cleverly". The remaining positions are loser positions. In particular, the final Nim position, the

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[3] The Cauchy inequality tells us that if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are real numbers, then $a_1b_1 + \ldots + a_nb_n \leq \sqrt{a_1^2 + \ldots + a_n^2}\sqrt{b_1^2 + \ldots + b_n^2}$.

[4] On 11 July 1994, exactly one hundred years after Sprague's birth, the 1994 International Mathematical Olympiad was officially opened in Hong Kong. The fourth problem in this contest, submitted by Australia, is one of Sprague's inventions (see p. 32 of this issue).

empty tabletop, is a loser position. A winner position is such that there is always at least one move that turns it into a loser position, whereas every move from a loser position leads to a winner position. How does one recognise whether a given Nim position is a winner or a loser position?

Write down the heap sizes, one beneath the other, in base two notation. In our sample Nim game, Jim initially had to move from the position (2, 3, 7, 5). Jim wrote down:

heap sizes: in decimal notation	in base two notation
2	10
3	11
7	111
5	101

In your base two notation array, count the ones within each column. If at least one column has an odd number of ones, your position is a winner position, otherwise it is a loser position. Jim had three ones in the first column (and also in the second column) from the right. Therefore he knew he could force a win.

Why is that so? When a positive whole number is diminished, at least one digit 1 of its base two representation must change to 0. If, in your base two notation array, each column has an even number of ones, this situation will thus change after every possible Nim move. On the other hand, suppose there is a column in your base two notation array that has an odd number of ones. Then there is always at least one Nim move which produces an even number of ones in each column (why?). Hence, at each position having at least one column with an odd number of ones in its corresponding base two notation array, you can find a move that will force your opponent to create another such position. If you choose your moves according to this strategy, your opponent will invariably be unable to create the position $(0, 0, \ldots, 0)$. It will be you who takes the last counter.

Now verify that Jim is a knowledgeable Nim strategist! Poor Kim! From the initial position (2, 3, 7, 5), Jim moved to (2, 3, 4, 5). Note that there were two more winner moves for Jim, namely to (1, 3, 7, 5) and to (2, 0, 7, 5).

If you like these kinds of games and wish to get deeper into their play mechanics, there are several ways of learning more about them:

(i) you could try to analyse some of the impartial games mentioned in this article;

(ii) you could study literature on impartial games, e.g. Berlekamp, Conway and Guy (1982) or Sprague (1963);

(iii) you could invent further impartial games and work out their strategies.

Have fun!

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MULTI-DIGIT NUMBERS

Alan R Boyd, Dunedin, New Zealand

Some properties of numbers depend on the base in which the number is written. For example, a number of 3 or more digits in base 10 is divisible by 4, if and only if the number formed by its last 2 digits is divisible by 4. This test does not apply in all other bases. Thus 344_{10} is divisible by 4 since 44 is divisible by 4. However,

$$344_{10} = 1 \times 7^3 + 0 \times 7^2 + 0 \times 7 + 1 = 1001_7$$

but $01_7 = 1_{10}$ is not divisible by 4.

Other properties of numbers are independent of the number system used. For example, 13 is a prime number whether it is written in base 10, 13_{10}, or in base 7, 16_7. In this article we are concerned with some properties of numbers of the former kind in base 10.

Consider the product of the digits of the number 24. Notice that there exists a *multiple* of the *digit* product, $(2 \times 4) \times 3$, which is equal to 24. We say that 24 is an example of a *multi-digit* number. On the other hand, there is no integral multiple of the digit product, 2×5, of 25 which is equal to 25. Hence 25 is not a multi-digit number. A number of 2 or more digits is called a multi-digit number if there exists a multiple of the digit product which is equal to the number, i.e. if and only if the number divided by its digit product is an integer.

Multi-digit numbers appear to be rare: there are only 1210 multi-digit numbers less than 10 000 000. Table 1, below, shows the distribution of multi-digit numbers (mdn's) up to 10 000 000, and some features of the distribution.

Some properties of multi-digit numbers

(a) The number of multi-digit numbers is infinite. A proof of this can be constructed using the pattern of column 4, Table 1.

(b) No multi-digit number has a zero digit. For example, 208 cannot be a multi-digit number because, whatever n is chosen,

$$(2 \times 0 \times 8) \times n \neq 208.$$

N (no of digits)	n_1 (No of mdn's with N digits)	n_2 (No of mdn's with N or fewer digits)	Smallest mdn with N digits	Largest mdn with N digits
2	5	5	11	36
3	20	25	111	816
4	40	65	1111	9612
5	117	182	11111	93744
6	285	467	111111	973728
7	743	1210	1111111	9939915

N (no of digits)	Approximate median mdn with N digits	No of mdn's with digits all different	Ratio of successive values of n_1	Ratio of successive values of n_2
2	15	4	-	-
3	224	14	4	5
4	2916	8	2	2.6
5	24112	9	2.925	2.8
6	234432	4	2.436	2.566
7	2282112	1	2.607	2.591

Table 1

(c) An odd multi-digit number cannot have any even digit because the product of an even number and any other numbers is even, e.g., 273 cannot be a multi-digit number because, for each n, $(2 \times 7 \times 3) \times n$ is even, and so cannot equal 273. Consequently, if a multi-digit number has an even digit (not zero) before its last digit, the last digit must also be even (2 or 4 or 6 or 8). In this case, since two digits (at least) are even, the digit product, and hence the multi-digit number, is divisible by 4.

(d) Any even multi-digit number is divisible by 4. The case where the last digit and at least one other are even has been considered in (c). The only remaining case is that where the last digit is even and the rest are all odd.
If the last digit is 4 or 8, the digit product and hence the multi-digit number has a factor 4.
If the last digit is 2 or 6 and the second-last is 1, the number formed by the last 2 digits (12 or 16) is divisible by 4, and hence the multi-digit

number is divisible by 4. Similarly, if the second-last digit is 3 or 5 or 7 or 9. Hence any even multi-digit number is divisible by 4.

(e) If a number is a multi-digit number, the number is divisible by each of its digits, but the converse is false. The smallest counterexample is 48, which is divisible by each of its digits, but is not a multi-digit number. Other counterexamples are: $124 = 2^2 \times 31$, $324 = 2^2 \times 3^4$, $728 = 2^3 \times 7 \times 13$ and $936 = 2^3 \times 3^2 \times 13$.

(f) A multi-digit number which has a 5 before its last digit must also end in 5. The reason is that the digit product has a factor 5, so the multi-digit number is divisible by 5 and hence must end in 5 or 0. By (b) above, no multi-digit number can have a zero digit, so it must end in 5. Hence the digit product of the multi-digit number, and the number itself, are divisible by 25. Numbers divisible by 25 end in 25, 50, 75, 00. But 50 and 00 are excluded by (b) above and 25 is excluded by (c) above. Hence a multi-digit number with a 5 before the last digit must end in 75, and so must be divisible by 7 as well as 25. The smallest multi-digit number beginning with 5 is 51 975, and the only other multi-digit number beginning with 5 and less than 1 000 000 is 511 175.

Some curiosities and conjectures

From Table 1, the 'middle' multi-digit number with 2 digits begins with 1, and for any other number of digits up to 7, the 'middle' multi-digit number begins with 2. It appears that, for numbers with a fixed number of digits, the larger the initial digit, the less likely it is to be a multi-digit number. Among 2-digit numbers, there is no multi-digit number with first digit larger than 3, and for 3-digit numbers, there is no multi-digit number with first digit 9. The smallest multi-digit number having 9 as a digit is $1197 = (1 \times 1 \times 9 \times 7) \times 19$.

Some multi-digit numbers have all their digits different. 1 687 392 is the largest such multi-digit number, and 3276 is the largest which does not contain 1.

It appears that, as multi-digit numbers increase, the more likely they are to contain the digit 1 at least once. Nevertheless, there are some large multi-digit numbers which do not contain 1, e.g., 2 322 432 2 322 432 is a 14-digit multi-digit number. Try to make some other conjectures, e.g., using the last 2 columns of Table 1.

Some subsets of the set of multi-digit numbers

(a) Perfect figure multi-digit numbers.

If the sum of the digits of a multi-digit number is equal to the digit product, the number is called a *perfect figure* multi-digit number. For example, 132 is a perfect figure multi-digit number because $(1 + 3 + 2) \times 22 = 132$ is the same as $(1 \times 3 \times 2) \times 22 = 132$. Interchange of the first two digits doesn't affect the sum and the product, so 312 is also a perfect figure multi-digit number.

A number is divisible by 9 if and only if the sum of its digits is divisible by 9. This result can be used to show that 11 133 is a perfect figure multi-digit number: $1+1+1+3+3 = 1 \times 1 \times 1 \times 3 \times 3 = 9$ and since the digit sum 9 is divisible by 9, so is 11 133. The test is still satisfied by numbers formed by rearranging the digits of 11 133, so nine additional perfect figure multi-digit numbers can be constructed from 11 133. These are 11 313, 11 331, 13 113, 13 131, 13 311, 31 113, 31 131, 31 311, 33 111.

(b) Repetitions of sets of digits.

The number 1112 is a multi-digit number. If this set of digits is repeated, we obtain 1112 1112 which again is a multi-digit number. Repetition again of the original digits gives 1112 1112 1112 which is also a multi-digit number, but a further repetition gives 1112 1112 1112 1112, which is not a multi-digit number. In this case, the maximum number of successive repetitions (including the original set of digits) all forming multi-digit numbers is 3. Some other such sets of digits are shown in Table 2 below.

Maximum number of successive repetitions	Original multi-digit numbers
2	12, 175, 1116, 11172, 11232, 11711, 12112, 32112, 114112, 122112, 218112, 1111712, 1113912, 1122112, 1161216, 1242112, 2138112, 2322432, 9716112
3	1112, 2112, 11112, 21312, 111112, 111312, 113112, 13112, 311112, 1111112, 1111113, 1111131, 1111311, 1113111, 1131111, 1311111, 3111111
4	112, 1111111911
5	1114112

Table 2

(c) Powers of numbers.

Successive powers of 6 are: 36, 216, 1296, 7776, 46 656, 279 936, … . The first three, *viz.* $6^2, 6^3, 6^4$, are multi-digit numbers, but $6^5, 6^6, 6^7$ are not. It is not known whether there is a higher power of 6 which is a multi-digit number.

The following multi-digit numbers which are powers have been found:

$36 = 6^2$, $144 = 12^2$, $2916 = 54^2$, $11\,664 = 108^2$, $41\,616 = 204^2$,
$82\,944 = 288^2$, $186\,624 = 432^2$, $1\,218\,816 = 1104^2$, $2\,214\,144 = 1488^2$
$216 = 6^3$, $13\,824 = 24^3$, $4\,741\,632 = 168^3$
$1296 = 6^4$, $49\,787\,136 = 84^4$
$248\,832 = 12^5$, $8\,349\,416\,423\,424 = 384^5$
$764\,699\,349\,893\,278\,334\,976 = 3024^6$
$128 = 2^7$
$429\,981\,696 = 12^8$

Notice that in all cases except 2^7, the base is a multiple of 6.

As far as I know, it remains unknown whether a multi-digit number exists which is a higher power of another number than the 8^{th} power.

Some related problems have been posed:

(a) Prove or disprove that 128 is the only power of 2 with two or more digits, each of which is a power of 2. (M Walker, 1949 [1], 1972 [2].)

(b) Find an integer, in any base, such that its digit product is equal to the square root of the integer. (A solution [3] of this problem has been found, but no solution has yet been found in base 10.)

References

1. Walker M, *American Mathematical Monthly*, Vol 56, 1949, p 39.

2. Walker M, *Tomorrow's Math*, C Ogilvy, OUP, New York, 1972, pp 85-124.

3. *New Zealand Mathematics Magazine*, Vol 21, No. 4, Oct. 1984, Problem 45.

Alan Boyd is not a trained mathematician but has a natural interest in mathematics, especially number theory. He has published his work in several other journals. His hobbies are Scrabble and classical music.

* * * * *

Quick estimates

The Nobel prize-winning physicist Richard Feynman developed the ability to do arithmetic fast by employing various tricks. One lunchtime he boasted to his colleagues that he could work out in sixty seconds the answer to any numerical problem that they could state in ten seconds, within an accuracy of ten percent.

In fact, problems that might appear difficult can be often solved quite readily using series or other approximation methods, if accuracy only within ten percent is required. For example, given the task of evaluating the integral of $1/(1 + x^4)$ over a short interval on which the function varied only a little, all Feynman would have needed to do was to evaluate the function at a few points (probably two or three at most), and use a simple numerical integration technique. The hardest problem he was given was to find the binomial coefficient of x^{10} in $(1+x)^{20}$, which he only just managed to do in the sixty seconds available.

Feynman's downfall came when someone walking past the lunchroom heard his boast, and set him the problem of finding the tangent of 10^{100} radians. A little thought reveals that there is simply no way of obtaining even an approximate answer except by dividing 10^{100} by π accurately enough to be able to estimate the remainder of the division, an impossible task in the time available.

(This last example notwithstanding, the habit of obtaining a rough estimate of the answer in order to provide an independent check on a calculation is a useful one to cultivate.)

* * * * *

HISTORY OF MATHEMATICS

Two Updates

Michael A B Deakin

1. The witch of Agnesi

The curve

$$y = \frac{a^3}{x^2 + a^2} \tag{1}$$

is known as the "witch of Agnesi", and it formed the basis for the cover story of *Function, Volume* 10, *Part* 4. (See also *Volume* 16, *Part* 2.) But why does this curve have such a bizarre name? The name *Agnesi* is that of Maria Gaetana Agnesi, one of the few female mathematicians to achieve fame and recognition in the years before our present century.

Her story was told in the course of the article mentioned above. However, perhaps a brief résumé is in order for the benefit of new readers. Agnesi was the daughter of a wealthy merchant. (Not as is sometimes wrongly stated a professor of mathematics.) Born in 1718, she was the first noteworthy female mathematician in modern times, although at least two women of antiquity may make such claims to fame. (See *Function, Volume* 16, *Parts* 1, 3.)

Her principal achievement was the production of a text on the then very recent and difficult topic of calculus. (Agnesi's life overlapped with Newton's by some nine years.) She was a child prodigy and excelled in many other areas as well as mathematics – languages in particular. She was indeed fluent in seven languages by the age of seven!

Her calculus book was very well regarded and was, in English translation, a standard text in Britain early last century. She was made a professor of mathematics at the University of Bologna, although the post was a purely honorary and nominal one (she continued to live in Milan). She was thus, at least in a formal sense, the first woman to become a professor of mathematics in a modern university.

However, her involvement with mathematics was rather short-lived. Her father, who seems to have pushed her unmercifully, died in 1752, and after this Maria, freed from his ambitions for her, devoted herself to works of piety and charity as well as to the care of her numerous younger brothers and sisters. From time to time there have been suggestions that she be canonised in recognition of this aspect of her life.

Until quite recently there was no really satisfactory biography of Maria Agnesi, but in 1989 the scholarly journal *Archive for History of Exact Sciences* carried an account by the eminent mathematician and historian Clifford Truesdell. It is a 1992 addendum to this article that gives the best account of the curve (1).

The first mathematician to study it was the gifted amateur Fermat (1601-1665) after whom Fermat's Last Theorem is named. Later it was also investigated by Guido Grandi (1671-1742), for whom see *Function, Volume* 8, *Part* 2. Grandi gave a mechanical construction of the curve by rolling, or turning, a circle along the x-axis. For details, see the *Function* cover story referred to above. Grandi, using as his base a rather obscure Latin word *versoria*, created the Italian word *versiera*, and applied it to his curve (Equation (1) in other words). The original Latin means "rope that turns a sail". The Italian meant roughly "curve resulting from the turning". In her book, which was much more influential and more widely read than Grandi's, Maria used the term *versiera* with this same meaning.

Agnesi's book was translated into English by a man called Colson, who learned Italian for the purpose, but did not learn it very well. At this point of the work, he misread *la versiera* as *l'avversiera*, which means "the witch" (or, more accurately, "she-devil"). The name has stuck.

The word *versiera* has no other meaning in Italian than the name of the curve (1), and, as the English name for that curve is "witch", then that must be given as the English equivalent of the Italian, and some dictionaries follow this convention. However, the word *versiera* does not mean "witch" in our usual sense!

2. The first atom bomb

In *Function, Volume* 10, *Part* 1, I published an article on the mathematics of measurement. It discussed a rather wonderful technique for deriving physical formulae from surprisingly little information. One example I gave concerned the energy released by the very first atomic explosion in 1944. The US military (not unnaturally) wanted to keep this piece of information secret. However, they allowed the release of a film of the explosion and from this G I Taylor in the UK and perhaps L I Sedov (in the then USSR) were able to calculate the supposedly secret energy.

I recently came across a translation from a Russian text that may throw a little more light on this story. [See G I Barenblatt's *Dimensional Analysis* (Trans. P Makinen; New York: Gordon & Breach, 1987).] I quote from this account.

> In an atomic explosion, a rapid (one might say, instantaneous) release of a significant amount of energy E occurs within a small region (one might say, at a point). A strong spherical shock-wave ... develops at the point of detonation; in the early stages, the pressure behind the wave front is several thousand times the initial air pressure, whose influence may be neglected in the early stages of the explosion.

Barenblatt then goes on to give the details of a calculation very like the one in my earlier article. His result is

$$\frac{5}{2}\log r_f = \frac{5}{2}\log(Const) + \frac{1}{2}\log(E/\rho_0) + \log t \qquad (2)$$

where r_f is the radius of the fireball, ρ_0 is the normal density of air, t is the time since detonation and $Const$ is a constant to be determined. He then continues:

> [This equation] is due to G I Taylor, who processed data from a movie film of a fireball taken during an American nuclear test by J Mack ... [Taylor was able to show] that the constant $Const$ has a value close to unity. Knowing this, it was possible to determine the energy of the explosion from the experimental dependence of the radius of the front [i.e. fireball] on time ... At the time, Taylor's publication of this value (which turned out to be approximately[1] 10^{21} erg) caused, in his words, "much embarrassment" in American governmental circles: this figure was considered top secret, even though Mack's film was not classified.

This last statement, however, may not be strictly true; Taylor himself stated that the film was declassified in 1947. Working for the British war effort, he produced in 1941 a secret report in which Equation (2) was derived and which further went on to give a theoretical value (which he also checked by experiments with powerful chemical explosives) for the

[1] An erg is a unit of energy in the centimetre-gram-second system.

term *Const*. That value was very nearly 1. Thus Equation (2) simplifies in that the first term on the right vanishes. So in theory, if merely a single pair of values r_f and t is known, then E may be calculated as ρ_0 is also known. The use of a film allows the comparison of many such pairs, and also a check of the underlying theory (which is very good indeed).

What I would guess is that Taylor, subsequent to his 1941 report, and through US-UK co-operation, got hold of a copy of Mack's film while it was still classified, and surprised the US authorities by deducing a figure for which he had not been given clearance.

How Sedov got the result (if indeed he did) is less clear. His textbook on dimensional analysis appeared after Taylor's paper and indeed reprints a diagram from it. He certainly derived Equation (2) independently and published his work (in Russian) in 1945 and 1946. An English translation (of which I have not been able to get a copy) of some of this appeared in 1946. But if indeed Sedov got hold of data such as Mack's film, it would have to have been by means of espionage and so he would have been less forthcoming about that aspect of his work.

Quite how far the Germans got is also in doubt. It seems that they too derived Equation (2). (The topic of explosions, atomic or otherwise, is of great interest to all parties in a war!) But they had fewer opportunities for espionage than did the Soviets (who were allies, though hardly trusted ones, of the US at the relevant time). It is thus unlikely that they deduced the value of E. Moreover, *their* war was rapidly coming to an end.

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MATHEMATICAL NURSERY RHYME

Humpty Dumpty sat on a wall,
Starting from rest to have his great fall,
The distance he fell, the king's men will swear,
measured exactly $\frac{1}{2}gt^2$.

From: *The Surprise Attack in Mathematical Problems*
by L A Graham (Dover, 1968)

* * * * *

COMPUTERS AND COMPUTING

Predictable "Random" Sequences

Cristina Varsavsky

In our last issue we simulated the rolling of a die by using electronically generated random numbers. In this issue we will have a closer look at how these numbers are produced by computers.

We will present here one of the most widely used algorithms for generating sequences of random numbers. The sequence $x_0, x_1, x_2, x_3, \ldots$ is generated by the iterative linear formula

$$x_{n+1} = ax_n + b \pmod{m} \tag{1}$$

where x_{n+1}, the next random number in the sequence, is calculated from the previous one, x_n, by multiplying it by a and then adding b using modulo m arithmetic.[1] To put it simply, x_{n+1} is the remainder after $ax_n + b$ is divided by m. As this is an iterative process, a starting point x_0 – usually called the *seed* – must be chosen. Computers would normally use a variable, for example the system time, to produce the seed.

Let us illustrate with an example how this works. Take $a = 7, b = 3$, and $m = 13$, i.e. the random number generator formula is

$$x_{n+1} = 7x_n + 3 \pmod{13} \tag{2}$$

Let $x_0 = 5$ be the seed. The sequence generated is 12, 9, 1, 10, 8, 7, 0, 3, 11, 2, 4, 5, ... as shown below:

$$\begin{array}{ll}
x_0 = 5 & \\
x_1 = 7 \times 5 + 3 = 12 & \pmod{13} \\
x_2 = 7 \times 12 + 3 = 9 & \pmod{13} \\
x_3 = 7 \times 9 + 3 = 1 & \pmod{13} \\
x_4 = 7 \times 1 + 3 = 10 & \pmod{13} \\
x_5 = 7 \times 10 + 3 = 8 & \pmod{13} \\
x_6 = 7 \times 8 + 3 = 7 & \pmod{13} \\
x_7 = 7 \times 7 + 3 = 0 & \pmod{13} \\
x_8 = 7 \times 0 + 3 = 3 & \pmod{13} \\
x_9 = 7 \times 3 + 3 = 11 & \pmod{13} \\
x_{10} = 7 \times 11 + 3 = 2 & \pmod{13} \\
x_{11} = 7 \times 2 + 3 = 4 & \pmod{13} \\
x_{12} = 7 \times 4 + 3 = 5 & \pmod{13}
\end{array}$$

[1] An article about modular arithmetic appeared in *Function Vol.* 17, *Part* 1.

From this point on, these twelve numbers will repeat again and again. Similarly, the seed $x_0 = 9$ will generate the sequence 1, 10, 8, 7, 0, 3, 11, 2, 4, 5, 12, 9, ... , and the seed $x_0 = 8$ produces 7, 0, 3, 11, 2, 4, 5, 12, 9, 1, 10, 8, ... , always cycling through the same twelve numbers. This is obviously a very limited random number generator as only twelve different numbers can be produced. In practice, a much larger value of m must be chosen in order to generate numbers without repetition. The size of m is surely limited by the number of digits the computer can handle.

So random numbers can be generated by a repetitive process where an iterative formula is applied to a seed. But care must be taken in the choice of the particular seed. This becomes clear when in our example (2) we take $x_0 = 6$ as the seed. In that case we have

$$\begin{aligned} x_0 &= 6 \\ x_1 &= 7 \times 6 + 3 = 6 \quad \pmod{13} \end{aligned}$$

with 6 being the only number produced with the formula (2). For this reason, 6 is called the *fixed point* for (2). But how can we avoid choosing the fixed point as the seed? Well, the fixed point is the solution to:

$$x = 7x + 3 \pmod{13} \tag{3}$$

In order to solve (3) using arithmetic modulo 13, first subtract $x + 3$ from both sides:

$$-3 = 6x \pmod{13}$$

Then, since $-3 = 10 \pmod{13}$, we have

$$10 = 6x \pmod{13}$$

Now, to make x the subject we have to divide by 6, or in other words, multiply by the inverse of 6. So we need to find a number which when multiplied by 6 gives 1 in arithmetic modulo 13. By trial and error we discover that 11 works. Thus

$$11 \times 10 = 11 \times 6x \pmod{13}$$

And finally, we express $11 \times 10 = 110$ as 6, the remainder after 110 is divided by 13, to get

$$x = 6 \pmod{13}$$

The only tricky part of the previous derivation was to find the inverse of 6 modulo 13, that is, a number which when multiplied by 6 using arithmetic modulo 13 gives 1. In *Function Vol.* 17, *Part* 1, we explored inversion in modular arithmetic: the conclusion was that it is possible to find the inverse of a number b modulo m if b and m do not have common factors greater than 1. In addition, if m is a prime number, the inverse of b can be calculated as $b^{-1} = b^{m-2} \pmod{m}$. In our example $6^{-1} = 6^{11} = 362797056 = 11 \pmod{13}$.

To explore the meaning of a in equation (1), try the example

$$x_{n+1} = 5x_n + 3 \pmod{13} \tag{4}$$

You can check that 9 is the fixed point, so we take any number but 9 as the seed, say 7. The sequence generated is 12, 11, 6, 7, ... , cycling through these four numbers again and again, leading to the conclusion that the choice of a will determine the length of the cycle. There is a result, which we will not prove here, saying that if

$$a^k = 1 \pmod{m} \tag{5}$$

then the sequence will repeat every k numbers. The smallest positive k to attain (5) is usually called the *order* of a in arithmetic modulo m. In our example (4), $5^4 = 1 \pmod{13}$ and we get only four different random numbers. In example (2), $7^{12} = 1 \pmod{13}$, and twelve different random numbers are generated. Therefore care must be taken also with the choice of a; a value with the largest possible order must be chosen.

Here is a slightly more realistic example: take $m = 239$, $a = 21$, $b = 1$. I used the computer algebra package *MAPLE* to find the fixed point 227, and also the order 238 for $a = 21$. Therefore the formula

$$x_{n+1} = 21x_n + 1 \pmod{239}$$

will produce 238 different random numbers. The sequence, produced in *MAPLE* with seed 1, goes as follows: 22, 224, 164, 99, 168, 183, 20, 182, 238, 219, 59, 45, 229, 30, 153, 107, 97, 126, ... ; every value but 227 occurs in the sequence.

Computers usually generate sequences of random numbers in the interval [0,1]. To achieve this we simply need to divide the numbers generated above by 238. So our sequence corresponding to the previous example would be 0.0924, 0.9412, 0.6891, 0.4160, 0.7059, 0.7689, 0.0840, 0.7647, 1.0000, 0.9202, 0.2479, 0.1891, 0.9622, 0.1261, 0.6429 , ...

A question any statistician would ask is whether these numbers are uniformly distributed within the interval [0,1], testing the randomness of the sequence with various techniques. A simple test we can perform is to take the mean of the first few numbers. I did this, again with *MAPLE*, and for the first 30 numbers it gave me 0.559, for the first 100 numbers, 0.519, and for the first 150, 0.499; clearly converging to 0.5 as we take a larger portion of the sequence. You can perform a similar test on the random generator of the programming language you use or any application such as spreadsheets that produces them. You can also write your own code to produce random sequences with formula (1), and try out different values for a and b.

By now you might have already concluded that random numbers generated using an iterative linear formula are not strictly random: if a random number is given to you and if you knew the formula your computer uses, you would be able to determine the following one. For this reason, these numbers are more correctly called *pseudo-random numbers.*

* * * * *

The Etymology of "Sine"

The English word "sine" comes from a series of mistranslations of the Sanskrit *jya-ardha* (chord half). The fifth-century Indian mathematician Āryabhata frequently abbreviated this term to *jya* or *jiva*. When some of the Hindu works were later translated into Arabic, the word was simply transcribed phonetically into an otherwise meaningless Arabic word *jiba*. But since Arabic is written without vowels, later writers interpreted the consonants *jb* as *jaib*, which means bosom or breast. In the twelfth century, when an Arabic trigonometry work was translated into Latin, the translator used the equivalent Latin word *sinus*, which also meant bosom, and by extension, fold (as in a toga over a breast) or a bay or gulf. This Latin word has now become our English "sine".

From *A History of Mathematics*,
by V Katz (Harper Collins, New York, 1993)

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PROBLEM CORNER

SOLUTIONS

PROBLEM 18.4.1 (Ian Collings, Deakin University)

What scores are possible in an Australian Rules football match in which the number of goals multiplied by the number of behinds equals the number of points (e.g. 7, 7, 49)? (1 goal = 6 points, 1 behind = 1 point.)

More generally, what scores satisfying this condition are possible in a game in which each goal is worth p points?

SOLUTION by Ian Collings (modified by the editors)

Let x be the number of goals and y the number of behinds. Then $6x + y = xy$ where x and y are non-negative integers. If $x = 1$, the equation has no solution. If $x \neq 1$ then

$$y = \frac{6x}{x-1}$$

Therefore $6x$ is divisible by $x-1$. Since x and $x-1$ have no common factors greater than 1, $x-1$ must be a divisor of 6. Thus the possible values of x are 0, 2, 3, 4 and 7. The solutions are shown in Table 1.

Goals	Behinds	Points
0	0	0
2	12	24
3	9	27
4	8	32
7	7	49

Table 1

If each goal is worth p points, then similar reasoning leads to the equation

$$y = \frac{px}{x-1}$$

where $x-1$ must be a divisor of p.

Derek Garson, of Lane Cove, NSW, provided the answer in a different form: either $x = y = 0$, or $y = p + t$ and $x = y/t$, where t is a divisor of p.

PROBLEM 18.4.2

The fraction $\frac{16}{64}$ is unusual among the fractions with two-digit numerators and denominators, because it can be simplified to the correct answer, $\frac{1}{4}$, by "cancelling" the 6's, even though the cancellation is not a mathematically valid operation. Find all fractions with this property.

SOLUTION

Assume that the *second* digit of the numerator is cancelled with the *first* digit of the denominator. (If the reverse is true, the fraction is just the reciprocal of one of the solutions we will obtain.) Then we are looking for integers x, y and z in the range 1 to 9 such that

$$\frac{10x+y}{10y+z} = \frac{x}{z}$$

This equation can be rearranged to give $9xz + yz = 10xy$. There are now several ways we could proceed. One is to write the equation as $(9x+y)z = 10xy$, from which we deduce that either $z = 5$ or $9x + y$ is divisible by 5. If $z = 5$, it is a simple exercise to list the potential values of x (1 to 9) and calculate the corresponding y values. If $9x + y$ is divisible by 5, each x value from 1 to 9 yields at least one y value between 1 and 9 (namely $y = x$), and in most cases two (e.g. $x = 1$ yields $y = 1$ or $y = 6$). The corresponding values of z can then be found. The answers are $\frac{16}{64}, \frac{19}{95}, \frac{26}{65}, \frac{49}{98}$, and all fractions with the same repeated-digit number in the numerator and the denominator (e.g. $\frac{11}{11}$).

S I B Ayeni, from Prahran, Vic., investigated the more general problem in which the numbers are not restricted to 2 digits. He provided many examples of such fractions, perhaps the most interesting of which are those in the pattern $\frac{16}{64} = \frac{166}{664} = \frac{1666}{6664} = \ldots = \frac{1}{4}$.

PROBLEM 18.4.3

The guests at a party are each asked how many of the other guests they know. Every guest gives a different answer. Prove that at least one of the guests must be lying. (Assume that "knowing someone" is symmetric, i.e. if A knows B then B also knows A.)

SOLUTION

Three readers provided solutions to this problem: Ben McMillan (Year 10, Geelong Grammar School), Derek Garson (Lane Cove, NSW), and Sani Susanto (Monash University). The solution below is Ben McMillan's.

Let n be the number of guests. There are n possibilities for the number of guests any given person may know: $0, 1, 2, \ldots, n-1$ (where 0 corresponds to knowing nobody, and $n-1$ corresponds to knowing everybody). Suppose that each guest knows a different number of other guests. Then for each of the n possibilities listed there must be exactly one guest who knows that number of other guests. In particular, one guest, say A, must know everyone, including the guest, B, who knows no-one. But by symmetry, B must also know A, which is a contradiction.

This solution is essentially correct, but a careful examination of the argument shows that it breaks down if $n = 1$ (and the result is in fact false in this case). However, no mathematician would describe an event with only one person present as a "party"!

More on some earlier problems

PROBLEM 15.2.1

The problem asked for the values of $\omega_p^{f(n)}$, where ω_p is a p-th root of unity (i.e. a complex number whose p-th power equals 1), and $f(n) = 1 + 2 + 3 + \ldots + n$. This is a problem from a few years back for which we have not published a solution. Keith Anker of Monash University has now produced one, but it is too long to give here. He showed that the sequence of values of $\omega_p^{f(n)}$ is periodic, with period p if p is odd, and period $2p$ if p is even.

PROBLEM 18.3.2

The problem asked for the unique 5-digit number which, when multiplied by 4, yields the number formed by writing the digits of the original number in the reverse order. A solution was provided in *Vol.* 18, *Part* 5. S I B Ayeni (Prahran, Vic.) investigated the general problem of seeking k-digit numbers which, when multiplied by an integer m, result in the number formed by reversing the digits of the original number. By running a Basic program for $2 \leq k \leq 7$ and $2 \leq m \leq 9$, he obtained the sequence of numbers 1089, 10989, 109989, 1099989, ... with $m = 9$, and the sequence 2178, 21978, 219978, 2199978, ... with $m = 4$. It is easy to show that the patterns continue.

PROBLEMS

PROBLEM 19.1.1 (Originally from the *New Zealand Mathematics Magazine*, 30(1), 1993; reproduced in *Mathematical Digest*, 94, January 1994, University of Cape Town).

It is possible to choose four lattice points in the plane (i.e. points (x, y) for which x and y are both integers), and connect each of these points to each of the other three points with a line segment, in such a way that none of the line segments pass through any lattice point (see Figure 1).

Suppose we now choose *five* lattice points in the plane, and connect each of these points to each of the other points with a line segment. Prove or disprove: there must be at least one line segment that goes through a lattice point.

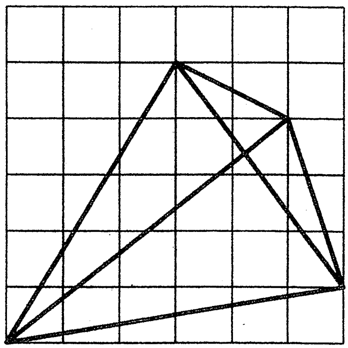

Figure 1

PROBLEM 19.1.2 (from IX Mathematics Olympiad "Thales", in *Epsilon*, 26, 1993, p 107).

What is the minimum number of cubes required to construct the building in Figure 2?

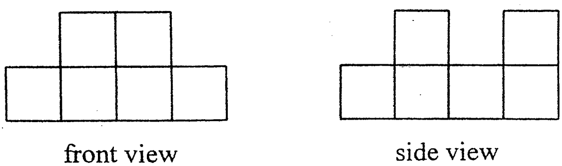

Figure 2

PROBLEM 19.1.3 (K R S Sastry, Dodballapur, India)

$ABCDE$ is a convex pentagon in which each diagonal is parallel to a side. The sizes of the angles EAB, ABC, BCD, CDE and DEA form an increasing arithmetic progression. Show that the angles BCA, ACE and ECD are also in arithmetic progression.

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The problems of the XXXV International Mathematical Olympiad (IMO)

Hans Lausch

Results of last year's IMO have already been reported in *Function, Vol* 18, *Part* 5 (October 1994). About one hundred problems were submitted from all over the world to a committee set up by the organising country, Hong Kong, which shortlisted two dozen of them. From the shortlist, the international IMO jury finally selected six problems. They had been submitted by Armenia (the "only if" part of Problem 2), Australia (Problem 4 and the "if" part of Problem 2), Finland (Problem 6), France (Problem 1), Romania (Problem 3), and the United Kingdom (Problem 5). Here are the problems:

Problem 1

Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \leq n$ for some $i, j, 1 \leq i \leq j \leq m$, there exists $k, 1 \leq k \leq m$ with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \ldots + a_m}{m} \geq \frac{n+1}{2}.$$

Problem 2

ABC is an isosceles triangle with $AB = AC$. Suppose that

(i) M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F lies on the line AC such that E, Q and F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

Problem 3

For any positive integer k, let f_k be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation has precisely three 1s.

(a) Prove that for any positive integer m, there exists at least one positive integer k such that $f(k) = m$.

(b) Determine all positive integers m for which there exists exactly one k with $f(k) = m$.

Problem 4

Determine all ordered pairs (m, n) of positive integers such that $\frac{n^3+1}{mn-1}$ is an integer.

Problem 5

Let S be the set of real numbers greater than -1. Find all functions $f : S \to S$ satisfying the two conditions:

(i) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y in S;

(ii) $f(x)/x$ is strictly increasing for $-1 < x < 0$ and for $0 < x$.

Problem 6

Show that there exists a set A of positive integers with the following property: For any infinite set S of primes, there exist positive integers $m \in A$ and $n \notin A$, each of which is a product of k distinct elements of S for some $k \geq 2$.

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