Function

Founder Editor G. B. Preston

Volume 16 Part 5

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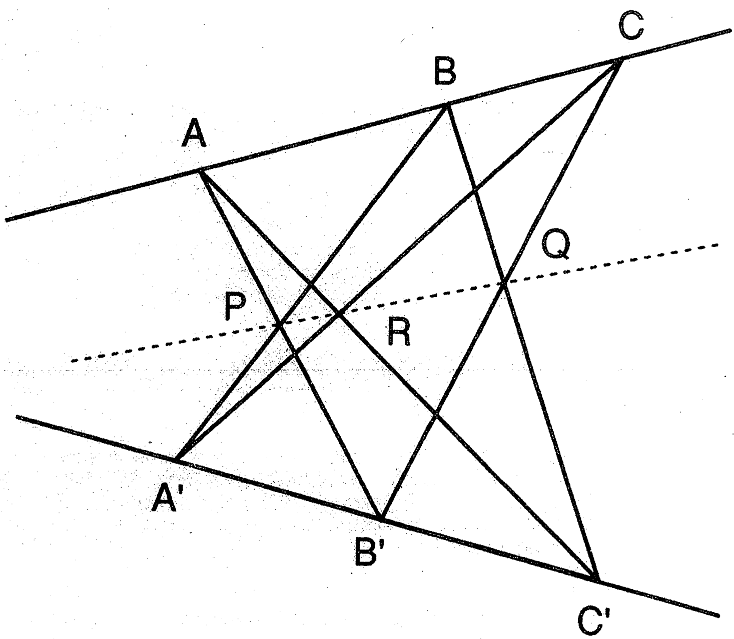

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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of secondary schools.

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Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. Recent issues have carried articles on advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside front cover.

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FUNCTION

Volume 16 *Part 4*

(Founder editor: G.B. Preston)

So much has been written about the Möbius strip; it has become so familiar one would think all its properties well-known and readily accessible. But we have a surprise for you. *The* Möbius strip? – not so, there are two of them and they are distinct! The point is not discussed in any accounts we have seen. John Stillwell makes good the omission in his article on p. 141.

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THE FRONT COVER

The front cover diagram for this issue was sent in by Garnet J. Greenbury, one of our regular contributors. It illustrates a result known as *Pappus' Theorem*. The earliest known account of it is indeed that to be found in Pappus' book *The Collection*.† Pappus was a Greek geometer who lived in Alexandria in the first half of the fourth century. The part of *The Collection* (Book VII) in which the theorem is found is believed to contain material derived from a now lost book by Euclid: *The Porisms*.†† So it is quite possible that Pappus' Theorem was known to Euclid who lived much earlier, and the result may have been discovered even before that.

The theorem is now seen as being fundamental to the branch of geometry known as *projective geometry*, much of which comes from a much later date. A, B and C are three points lying on one straight line and A', B', C' are three points lying on another. Let AB' and $A'B$ meet in P, BC' and $B'C$ meet in Q, CA' and $C'A$ meet in R. Then P, Q and R lie on a straight line. See Figure 1.

The proof is quite difficult and will not be given here – though we concur with Mr Greenbury's remark that the result "is a most beautiful theorem which should be part of everyone's experience".

Pappus' Theorem has close and deep relations with another theorem from projective geometry: *Desargues' Theorem*. Desargues' Theorem was the subject of our cover story for *Vol.* 10, *Part* 3. A simple proof of Desargues' Theorem is known and was indicated there. We can also prove Desargues' Theorem from Pappus' Theorem, but not *vice versa*. Desargues' Theorem requires a further assumption to be made if we are to deduce Pappus' Theorem from it.

Pappus' Theorem has the more modern distinction of being the first geometrical result to be proved by means of a computer algebra package. The work was done at Brown University (U.S.A.) in 1968 using the computer language FORMAC on what was then a medium-sized machine, but now would be considered small (in storage space – not in size!).

The two researchers involved, Elsie Cerutti and Philip Davis, set up co-ordinates for the various points A, B, C, A', B', C' and had the computer work out the co-ordinates of P, Q, R. They then had it calculate the area of the "triangle" PQR, which area the computer discovered to be zero. They also scanned the output and discovered from it several new (previously unknown) theorems. These however are technical and will not be given here. The story is told in the journal *American Mathematical Monthly* (Oct. 1969).

The other point to note is that Pappus' Theorem is now recognised as a special case of a much more general result. The curves known as conic sections are those resulting

† For more on this work, see the article by Winifred Frost in *Function Vol.* 16, *Part* 3 and the article on p. 139 of this issue.

†† The English word "porism" is now archaic, but it was once used to mean "corollary" – a relatively straightforward consequence of a more important theorem. The word derives directly from the Greek, but the precise significance of the term for Euclid and other mathematicians of antiquity has been disputed.

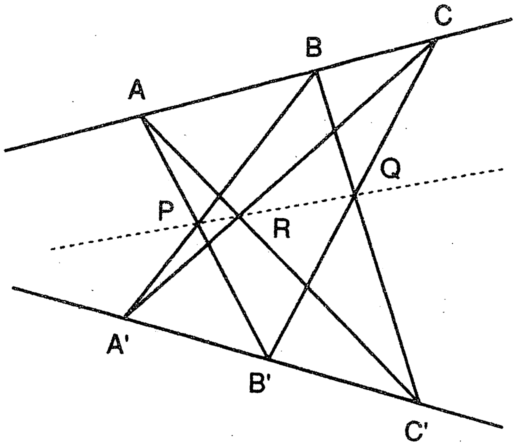

Figure 1

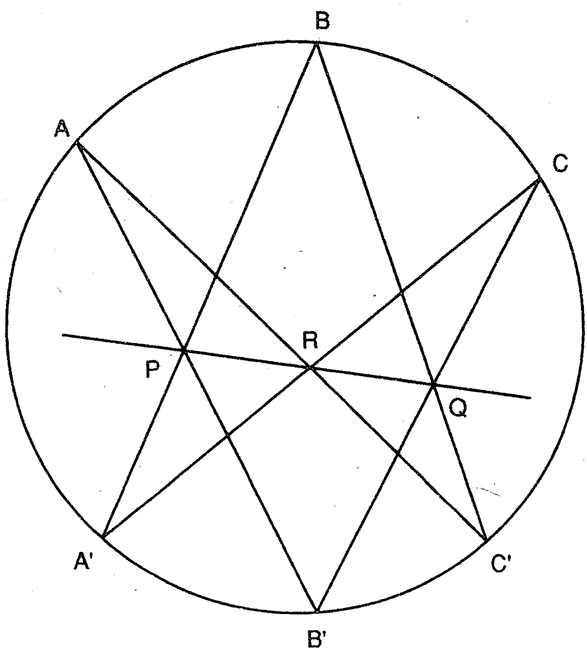

Figure 2

from making a cut (or section) through a cone (a right circular cone formed by all the lines connecting the points of a circle to a point directly above its centre). The conic sections were discussed in the cover story for *Function, Vol.* 10, *Part* 2. Various curves are conic sections: the parabola, the hyperbola, the ellipse (with the circle as a special case), and also either a pair of straight lines or a single straight line.

Pappus' Theorem begins with the initial six points A, B, C, A', B', C' lying on a pair of straight lines. In fact, they may lie on any conic. Figure 2 shows one example – the case in which the conic is a circle.

This fact was discovered by the French mathematician and philosopher, Blaise Pascal (1623-1662). It was one of his very first contributions to mathematics; he made the discovery at the age of sixteen. Nowadays it is usually termed Pascal's Theorem in his honour – although the older term "mystic hexagon theorem" is sometimes employed, as it can be represented as revealing a property of the diagonals of certain special hexagons.

All in all, Pappus' Theorem has been very much in the mathematical news for probably about 2000 years.

* * * * *

More on remembering Pi

Our regular contributor Garnet J. Greenbury in 1991 offered a $40 prize for the best mnemonic for Pi. This was won by Melissa Freudenstein, a Year 12 student at the Henry Lawson High School, Grenfell, N.S.W. Here is her 205-word offering.

War! A loss I can't reimburse to people. Grace the males fighting memorable battles necessary for to mar humanity. Help people to reveal wars mad men struggle for to release oppressed lives! Inhumanity, evidence must, I emphasize, torture a person desperate and foolhardy. Animosity has curious power I can't estimate as necessity demands this hostility. This, with death, treachery is war. Mundane pretence, a deadly mire. Guilty or innocent battle on. Fighting maintains memorable scathing scenes in eternity. The evil memories of greed and hate as intractable despair. Bloody corpses displayed evidence of a grim manifest. Suppress untold greed I say, as conflict is sin – greedy powers will dwindle. Symbolize war insanity with hero homage. Grotesque empty idols! Adept strength go to all a society in peace and potential love. Poignant I am. Securely true partisan. A destruction willing more death. To maintain wars, I am idiotic. A promotion for grievous irony is regrettable. Sneer while human sacrifice suffer loss with relief. So to symbolize evil, continue insolence. Major hurt dispenses out. War memories – a confusing parade with loss of rational thinking.

The *Australian Mathematics Teacher* (*Vol.* 48, *No.* 1, 1992, p.35) notes a simpler and much shorter mnemonic:

May I have a large container of coffee?

For other such mnemonics, see *Function, Vol.* 16, *Part* 4, p. 128.

THE GOLDEN SECTION

Michael A.B. Deakin, Monash University

Figure 1 below shows a regular pentagon $PQRST$ together with all its diagonals. The sides are all equal and may be taken to have length 1. I.e.

$$PQ = QR = RS = ST = TP = 1$$

Similarly all the angles PQR, QRS, etc. are equal and their magnitude is $\frac{3\pi}{5}$ radians or 108°.

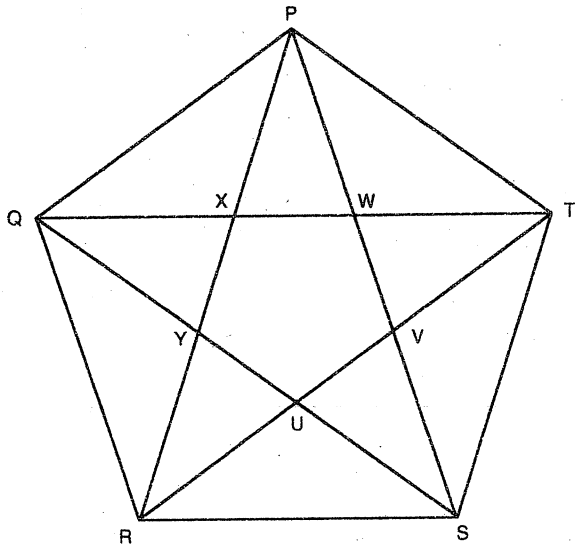

Figure 1

The lengths of the diagonals QT, RP, SQ, TR, PS are also equal to one another. Let the common length be τ.

Now the triangles QUT, SUR are similar (because, as may easily be shown, QT is parallel to RS). Thus

$$\frac{QU}{US} = \frac{QT}{RS} = \frac{\tau}{1}. \tag{1}$$

But $PQUT$ is a parallelogram with $PQ = PT$ (that is to say, a rhombus). Thus $QU = PT = 1$. Thus Equation (1) tells us that

$$\frac{1}{\tau - 1} = \tau, \tag{2}$$

that is to say,

$$\tau - 1 = \frac{1}{\tau}, \tag{3}$$

i.e.

$$\tau^2 - \tau - 1 = 0. \tag{4}$$

This is a quadratic equation with roots

$$\tau = \tfrac{1}{2}(1 \pm \sqrt{5}).$$

In our present context we choose the positive root

$$\tau = \tfrac{1}{2}(\sqrt{5} + 1) = 1.618\ 033\ 988\ 7\ldots\ .$$

This value is called the *golden ratio* (or *golden section*)† and it arises in many other contexts as well. Perhaps the very simplest is that shown in Figure 2.

The rectangle $ABCD$ has

$$AB = CD = \tau,\ BC = DA = 1$$

in suitably chosen units.

The points E, F are so chosen that $AE = FD = 1$, and so $AEFD$ is a square.

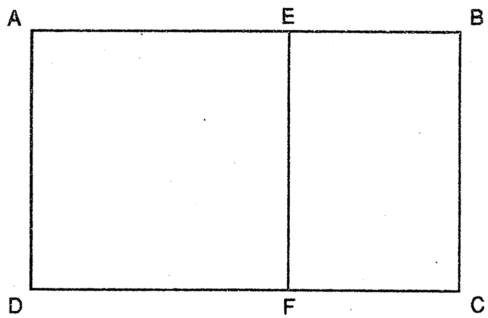

Figure 2

We now have $BE = \tau - 1$, and this, by Equation (3), is $1/\tau$. Thus

$$BC/BE = 1/(\tau-1) = \tau, \text{ by Equation (2)},$$

† The name "golden section" is the subject of a lengthy entry in Volume 1 of the supplement to the Oxford English Dictionary. This states in part: "This celebrated proportion has been known since the 4th century B.C. ... . Of the several names it has received, *golden section* (or its equivalent in other languages) is now the usual one, but it seems not to have been used before the 19th century." The earliest use they cite is an 1835 one in German; the earliest English use a mention in the 1875 edition of the *Encyclopaedia Britannica*. The value here denoted by τ is often represented instead by ϕ.

and the rectangle $BCFE$ is similar to the original $ABCD$. Such rectangles are called *golden rectangles.*

Note another property also from Figure 2:

$$AB \times BE = AE^2 \tag{5}$$

which may very readily be proved using Equation (4). It is under this guise that the golden ratio first appears in Euclid. (It is Proposition 11 of Book II; pp. 402-403 of Heath's English edition.)

Given an interval AB, Euclid was concerned to construct the point E such that Equation (5) held. (Such constructions were to be carried out using only compasses and unmarked straightedge rulers.)

Here is Euclid's construction.

Draw AX perpendicular to AB and such that $AX = AB$. Let Y be the mid-point of AX. Join BY. Draw YZ through A such that $YZ = BY$. On AB, mark off E such that $AE = AZ$. I leave it to the reader to show that $AB/AE = \tau$.

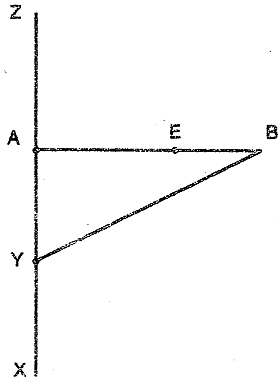

Figure 3

Heath remarks that as this construction is necessary to the construction of a regular pentagon it must have been devised by the Pythagoreans - followers of Pythagoras (who lived about 500 B.C.). They were closely associated with the regular pentagon.

Go back now to Figure 2. The line EF cuts (or "sections", as older usage would have it) the rectangle $ABCD$ into two parts – a square ($AEFD$) and a smaller replica of itself ($BCFE$). This follows directly from Equation (3) which therefore tells us

A golden rectangle comprises a square and a smaller golden rectangle.

This process may be repeated; see Figure 4. From the new golden rectangle $BCFE$, cut (or section) off the square $BHGE$ to form the new golden rectangle $CFGH$. Similarly form the smaller golden rectangles $FGIJ$, $GKLI$, and so on.

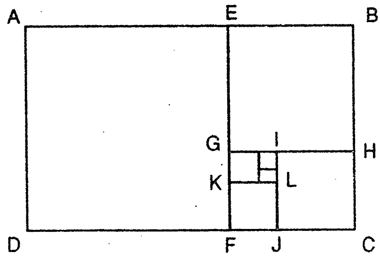

Figure 4

The process clearly may be carried on *ad infinitum* and this observation may have been very important in the history of mathematics. To see why, begin by considering a rectangle whose sides are in the ratio n/m where n and m are integers and $n > m$. We can now divide the rectangle into a square of side m and a rectangle of sides $n - m$, m. Now if $n-m > m$, we may cut off a further square of side m and perhaps even several such. Otherwise, take a square of side $n - m$, etc. Eventually the process will terminate and the smallest square will have a side equal to the greatest common divisor of $n + m$.

See for an illustration Figure 5, which depicts the case $n = 20$, $m = 12$. Remove first a square of side 12, to leave a 12×8 rectangle. Next remove a square of side 8 to leave an 8×4 rectangle. This comprises two squares of side 4, and

$$4 = \text{g.c.d. } (20, 12).$$

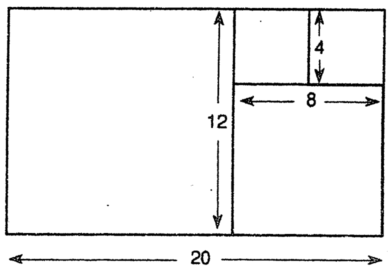

Figure 5

(The construction just given is in fact a geometric version of Euclid's algorithm for finding the greatest common divisor; see *Function, Vol.* 8, *Part* 5.)

But if we begin with a golden rectangle, each removal of a square produces a new golden rectangle. Thus the process can never terminate, but must go on forever as we have seen. It follows that the sides are not in integral ratio to one another. In other words

τ *is irrational.*

The discovery of irrational numbers is generally traced back to the Pythagoreans but we have very few details of it. Generally it is assumed that the number they found to be irrational was $\sqrt{2}$, but there is little direct evidence for this. In 1945, a mathematician named von Fritz suggested that the first number to be seen to be irrational may in fact have been τ. The *pentagram* (the star made up of the diagonals in Figure 1) was very much associated with the Pythagoreans, and von Fritz produced a version of the above argument using pentagrams. (The simpler, rectangular, version given here is due to my colleague Chris Ash.)

The golden section arises in other contexts as well. The connection with the Fibonacci sequence and the phenomena of "phyllotaxis" (or leaf arrangement) were noted by Robyn Arianrhod in *Function, Volume* 16, *Part* 4 and in the cover story of that issue.

In particular, Robyn showed that the ratio of two successive Fibonacci numbers approximated τ and indeed approached arbitrarily close to τ as the numbers got larger. There is more to this. It can be proved that τ is represented by the infinite continued fraction

$$\tau = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}\,. \tag{6}$$

(For articles on continued fractions, see *Function, Vol.* 4, *Part* 4 and *Vol.* 11, *Part* 2; the proof of Equation (6) was Problem 4.2.1) If we truncate the infinite expression (6) we get precisely the ratio of two successive Fibonacci numbers. Successive truncations give $1, 2, \frac{3}{2}, \frac{5}{3}$, etc. These are the so-called "convergents" of τ, each being an improvement on its predecessors and on all other such fractions with denominators less than the one given. Thus the Fibonacci ratios are the best possible rational approximations to τ.

Equation (6) gives the very simplest possible infinite continued fraction and as an infinite continued fraction represents an irrational number, we can see that, in a sense, τ is the very simplest irrational number. It would thus be fitting if von Fritz were right and it was indeed the first to be proved irrational, but historical conclusions cannot be arrived at by such arguments!

Figure 6 reproduces Figure 4 but with a superimposed spiral. This curve homes in on the point at which AC and BF intersect. Any line through this point will cut the spiral at a fixed angle whose value is approximately $76^\circ\ 43'$. It is thus referred to as an *equiangular spiral*.†

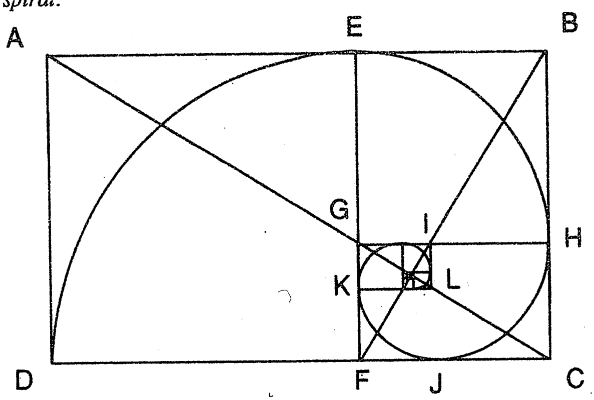

Figure 6

This too arises in nature, again in phyllotaxis – the spirals in sunflower heads approximate equiangular spirals – but perhaps most clearly in nautilus shells (see Figure 7).

A close approximation to the equiangular spiral may be produced by taking circular quadrants DE, EH, HJ, etc., and some people mistakenly believe this to be exact. However, close attention to Figure 5 shows that the spiral actually goes *outside* the rectangle before re-entering it at E.

† It is also sometimes known as the logarithmic spiral.

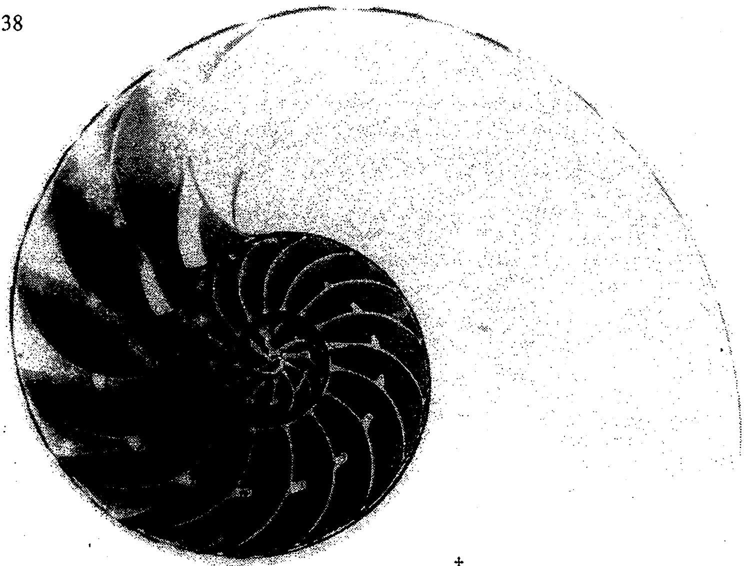

Figure 7†

From time to time there have been claims that the golden rectangle and the equiangular spiral have particular aesthetic appeal. An artificial "nautilus" may be seen outside the National Gallery of Victoria (at the north end of the moat). The Time-Life book *Mathematics* claims to find golden rectangles in the work of several famous painters, notably the Dutch abstractionist Mondrian. *The Age* (17/12/1990) made a similar claim in respect of the Australian painter Jeffrey Smart.

The systematic study of the golden ratio in art goes back to the American author Jay Hamridge, who claimed evidence for its use in ancient Greek pottery. Hamridge analysed the shapes of many Greek vases by drawing cross-sections of them and superposing rectangles on these. Some of the rectangles were in the proportion $\sqrt{n} : 1$ ($n = 1, 2, ..., 6$) and others in the ratio $\tau : 1$ (the golden rectangle, which Hamridge referred to as the "Rectangle of the Whirling Squares" – because of the properties illustrated in Figures 4 and 6).

The trouble with this sort of analysis is that with combinations of 7 standard rectangles and an allowance for error (which Hamridge attributed to shrinkage during firing) many, many shapes can be generated.

Similarly with the pictures. The illustrations in *Mathematics* superpose golden rectangles on various pictures and I have done this too with the Smart picture illustrating the *Age* article. There are golden and non-golden rectangles to be found in all these pictures – however, it is a little hard to assign much significance to all this.

My reading is that Smart does make conscious use of the golden rectangle; similar claims have been made in respect of the American artist George Bellows and of the Renaissance Florentine Piero della Francesca. The golden rectangles in Mondrian's work, however, are thought by the authors of *Mathematics* to be unconscious. It would be interesting to see if *viewers* were conscious of whether or not particular pictures contained golden rectangles and whether or not it mattered.

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† Our thanks to Dr. T.E. Hall for the loan of his nautilus shell.

An example of the non-rectangular use of the golden section in art is provided by the famous painting *The Birth of Venus*. A 1914 account, quoted in the OED supplement referred to earlier, goes: "In Sandro Botticelli's Venus ... the line containing the figure from the top of the head to the soles of the feet is divided, at the navel, into the exact proportion given by .... the 'Golden Section'."

I checked this claim by measurement of the reproduction in the book, *Complete Paintings of Botticelli*. The top of the head is not very well defined and the feet are somewhat apart, with the soles not visible. However, a straight line may be found between the ball of the left foot, the navel and a reasonably prominent tress of hair. Measuring to the nearest millimetre, I found

Tress to ball of foot: 206 mm; Navel to ball of foot: 127 mm.

Then

$$\frac{206}{127} = 1{\cdot}62 \ldots \approx \tau = 1{\cdot}618\ldots .$$

This is really very impressive agreement – the more so as I decided what to measure *before* doing any calculations. However, the questions remain. Did Botticelli intend it? Is this the only such ratio in his work? Is our enjoyment of the painting enhanced by this numerical fact?

Further Reading

There is a great deal of material available on the golden section, the equiangular spiral and the Fibonacci sequence. Very accessible is the Time-Life book *Mathematics* (by David Bergamini and others). A bit more mathematical, but very likely available in many school and municipal libraries, is E.P. Northrop's *Riddles in Mathematics* (now available in a Penguin reprint). See especially pp. 53-60 and the notes on p. 228 of this edition. More technical, but very informative are the accounts in H.M. Cundy and A.P. Rollett's *Mathematical Models* (O.U.P.), pp. 62-64, H.S.M. Coxeter's *Introduction to Geometry* (Wiley), pp. 160-172, and E.H. Lockwood's *A Book of Curves* (C.U.P.), pp. 98-109. These references contain a lot of interesting additional material left out of this article on grounds of space.

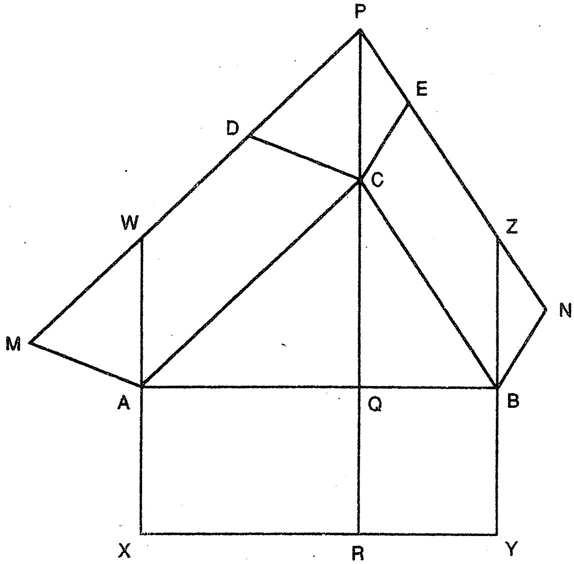
* * * * *

PAPPUS' GENERALISATION† OF PYTHAGORAS' THEOREM

Garnet J. Greenbury, Upper Mt. Gravatt, Queensland

Given a triangle ABC, draw on the sides AC and BC two parallelograms $MDCA$ and $ENBC$; extend the sides MD and NE as shown (see the diagram overleaf), to meet at a point P. Join PC and extend it first to Q where it meets AB and then beyond to R, making QR equal in length to PC.

† Euclid's *Elements* was a collection of geometric results, some of which he derived from earlier mathematicians. Pythagoras' Theorem was known long before Euclid. Euclid is, however, credited with a generalisation in which the squares on the sides of a right-angled triangle are replaced by any similar rectangles (and thus by any similar figures). This is Proposition 31 of Book VI of the *Elements*. Pappus' further generalisation is Proposition 1 of Book IV of his *Collection* – the work described by Winifred Frost in *Function*, *Vol.* 16, *Part* 3.



Then draw a parallelogram on the side AB of the triangle, making the sides AX and BY parallel to QR, and equal to QR in length.

Pappus' generalisation of Pythagoras' theorem is:

$$\text{Area } ABXY = \text{area } MDCA + \text{area } ENBC.$$

Proof: Extend XA to W, and YB to Z as shown. Then

$$\begin{aligned} \text{Area } MDCA &= \text{area } WPCA \quad [\text{both based on } AC \text{ and between the parallels } AC, MP] \\ &= \text{area } AQRX \quad [QR = PC;\ WX \parallel PR] \end{aligned}$$

$$\text{Area } ENBC = \text{area } BZPC = \text{area } BQRY \quad [\text{similarly}].$$

By addition

$$\text{Area } ABYX = \text{area } AQRX + \text{area } BQRY.$$

Pythagoras' theorem can be recovered by making angle C a right angle, and making the parallelograms squares. The proof that $PC = AB$ (necessary for this) is very easy.

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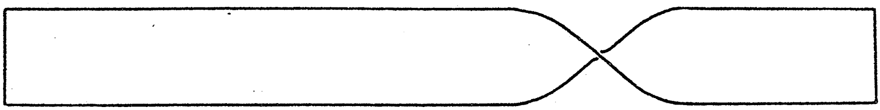
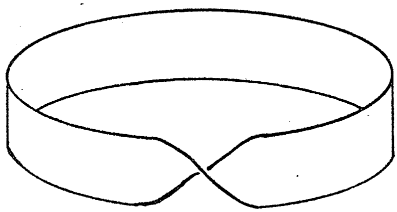
THE TWO MÖBIUS BANDS

John Stillwell, Monash University

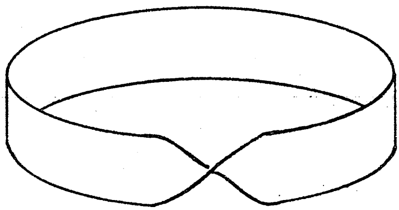
The Möbius band is a surface made from a rectangle (of paper, say) by giving one end a half twist –



and then joining it to the other end –



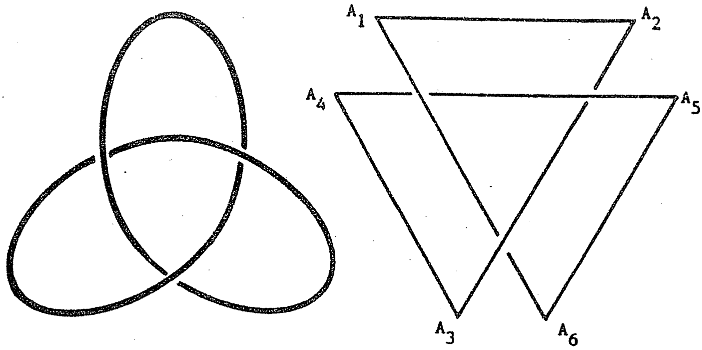
Notice that if the twist is made in the opposite direction the resulting Möbius band is the mirror image of the first –



The half twist in the Möbius band makes it very different from an untwisted band, and more entertaining. For example, it has only one edge, and only one side. A less obvious fact is that the Möbius band is different from its mirror image. If you make two mirror image Möbius bands out of paper you will find that, no matter how you twist and turn them, they never look the same. It seems they *are* different, but how can this be proved?

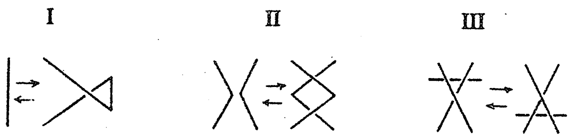
The ideal approach to this problem comes from knot theory, which studies the knotting and linking of closed curves in space. Of course, the Möbius band is a surface, not a curve, but we shall see shortly that we can keep track of its position in space by looking at two curves on it. So let us begin by seeing what knot theory tells us about curves.

The curves we shall study are intended to model closed loops of string, and for this purpose it suffices to consider *polygons* in space. A polygon consists of finitely many line segments ("sides") A_1A_2, A_2A_3, ..., $A_{n-1}A_n$, A_nA_1 with no intersections except the A_i where $A_{i-1}A_i$ meets A_iA_{i+1} (and the A_1 where A_nA_1 meets A_1A_2). The following figure shows the so-called trefoil knot and a polygonal version of it.



A more realistic version of the knot is of course obtainable by increasing the number of sides in the polygon, but the hexagon already captures the knotting of the trefoil perfectly. Moreover, we do not need an actual hexagon in space, but only a planar diagram like the one above. Such a diagram, in which there are finitely many "crossings" where a side of the polygon passes over or under one other side, is called a *knot projection.*

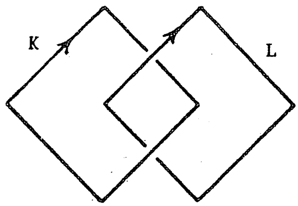
The possible deformations of the knot can be modelled by simple modifications of the knot projection called *Reidemeister moves.* They are represented by the following three diagrams, each showing just the part of the projection which is modified.



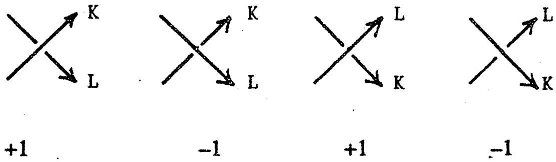
The class of projections obtainable from a given projection by Reidemeister moves represents all possible deformations of a given knot. The main problem of knot theory is to decide whether two given knots are the same (up to deformation), i.e. to decide whether the projection of one can be transformed into the projection of the other by Reidemeister moves.

When the two knots K, K' are in fact the same it is only a question of patience to actually find a sequence of Reidemeister moves which shows $K = K'$. However, when $K \neq K'$ an indirect approach is necessary. One has to find a property $p(K)$ of a knot projection K which is *unaltered* by Reidemeister moves, and hence a property of all deformations of K; then if $p(K) \neq p(K')$ one can conclude that $K \neq K'$. Such properties are quite hard to find for knots (i.e. single polygons), but there is an easy one for *links* (systems of two or more polygons) called the *linking number.*

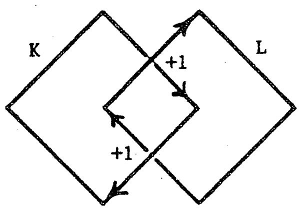
To compute the linking number for a pair of polygons K, L one first assigns a *direction* to each of K and L. For example



It can then be seen that each possible crossing of K with L looks like one of the following four types



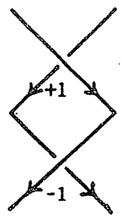
to which we assign a value +1 or −1 as shown. The *linking number* $l(K, L)$ is simply the sum of these values over all crossings of K with L. Roughly speaking, $l(K, L)$ measures "how much K winds around L". In our example we have



hence $l(K, L) = +1 + 1 = +2$.

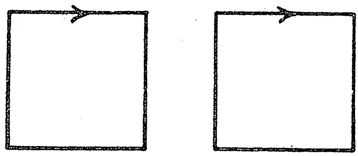
Notice now that $l(K, L)$ is unaltered by Reidemeister moves because:

- Reidemeister I does not involve a crossing of K with L,
- Reidemeister II involves only crossings whose values cancel, if it involves a crossing of K with L at all, e.g.



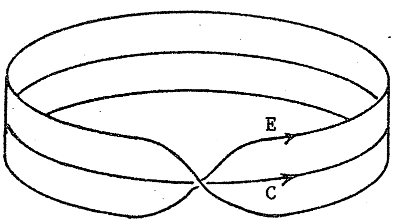
• Reidemeister III only shifts the crossings, without changing their values.

It follows in particular that our example with two crossings is not the same as the following "trivial link"

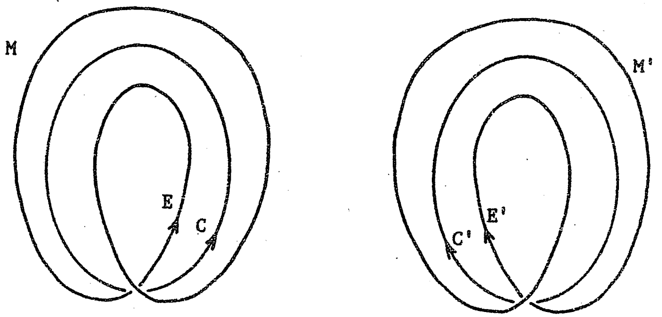


for which $l(K', L') = 0$. Notice also that if we direct any of the polygons differently we still get $l(K', L') = 0$, and $l(K, L) = \pm 2$. Thus the two-crossing link is *not* trivial – its polygons K, L cannot be unlinked by Reidemeister moves.

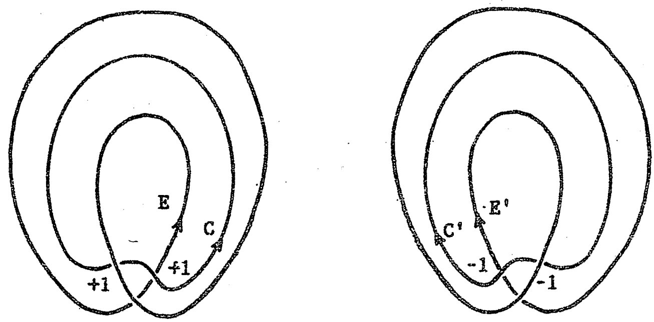
Now we can return to the problem of the two Möbius bands. The key to its solution is the linking number of the edge E of the Möbius band M with its centreline C. A deformation of M merely deforms the link of E with C, and hence does not alter the linking number $l(E, C)$ obtained from chosen directions of E and C. Also, deformation of M preserves the equality, or otherwise, of the directions of E and C around the band. For example, in the figure below they have the same direction around the band, and they cannot end up in opposite directions through deformation of the band.



This prompts us to compute $l(E, C)$ when E and C have the same direction, and to compare it with $l(E',C')$, where E′ and C′ are the edge and centreline of the mirror image Möbius band M′. To minimise the number of crossings we draw the two Möbius bands as follows –



and then deform their centrelines slightly to eliminate the triple crossings:



It is now clear that

$$l(E, C) = +1 + 1 = +2 \quad \text{and} \quad l(E', C') = -1 - 1 = -2.$$

Moreover, if we reverse the direction of E and C (to get the other case where E, C have the same direction) then *the values of the crossings are unaltered.* This follows from the definition of the values of crossings. Thus $l(E, C) \neq l(E', C')$ whenever the directions of E and C are the same, and hence the link of E and C cannot be deformed into the link of E′ and C′ by deformation of M. In particular, there is no deformation of M into its mirror image M′.

Remarks

1. A similar argument can be used to show that the figure on the cover of *Function Vol.* 15, *No.* 3 cannot be deformed into its mirror image.

2. The most famous result of this type is that the trefoil knot cannot be deformed into its mirror image. This was first proved by the German mathematician Max Dehn in 1914. The title of his paper, *Die beiden Kleeblattschlingen* (the two trefoil knots), suggested the title of this one.

In 1984 the New Zealand mathematician Vaughn Jones found a new way to show the difference between the two trefoil knots, using what is now called the *Jones polynomial.* His approach was simplified by Lou Kauffman of the University of Illinois so that the Jones polynomial can now be seen quite easily to be unaltered by Reidemeister moves. It is, however, considerably harder to compute than the linking number.

* * * * *

LETTER TO THE EDITOR

Updating Two Stories via "the Net"

Some items have appeared on the electronic news network ("the net") in the past few months which have a bearing on two articles published in recent issues of *Function.*

In "Rooks and Multi-dimensional Chess Boards" (*Function, Vol.* 14, *Part* 5, October 1990), Mark Kisin investigated the problem of placing the smallest possible number of rooks on a d-dimensional chessboard of size n, in such a way that every cell is either occupied or threatened by at least one rook. (A normal chessboard corresponds to the case $d = 2$ and $n = 8$.) The solution for $d = 2$ is straightforward, and the solution for $d = 3$ was presented in that article. An unpublished part of Mark Kisin's article contained some partial results for higher dimensions.

Around last April, Jim Propp from the Massachusetts Institute of Technology posed the rook problem on the net. Several people responded with partial answers. Dan Hoey, from the Naval Research Laboratory in Washington D.C., gave a partial solution to the case $d = 3$, and claimed that his method could be generalised to yield a (rather complicated) lower bound for the number of rooks in higher dimensions. Geoff Bailey (who gave his organisation as the "Venusian Glee Club", but his e-mail address suggests that he is actually from the University of Sydney), gave the exact answer for $d = 3$, and stated that he had solved this a while ago in a competition.

Some further information on Fermat's Last Theorem, which was discussed in the History of Mathematics section of *Volume* 16, *Part* 4 of *Function* (August 1992), appeared in a regular item on the net which provides answers to frequently asked questions. (A lot of people must keep asking about the status of Fermat's Last Theorem!) According to the net, it has been proved, using a combination of theory and checking exceptional cases by computer, that $x^n + y^n = z^n$ has no positive integer solutions for any integer n in the range $3 \leq n \leq 2\,000\,000$. It is also interesting (and frustrating!) to note that there are many unproved conjectures with a large body of evidence to support them, any one of which would imply Fermat's Last Theorem ... if only we could prove just one of them.

Peter Grossman
Monash University

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COMPUTERS AND COMPUTING

The idea for this issue's column was provided by *Locus*, a counterpart to *Function*, published in the African state of Malawi. Item 13 in their combined issue 7 and 8 (1992) was supplied by D. Mundy of Chancellor College, Zomba, Malawi, under the title "Real Mathematics on a Computer or Computers can't add up!". This column reproduces the thrust of Dr. Mundy's article.

Let

$$S_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} .$$

Then $S_n = 1$, $S_2 = 1.25$, $S_3 = 1.3611...$, $S_4 = 1.423611...$, etc.

It can be proved that

$$\lim_{n\to\infty} S_n = \frac{\pi^2}{6} \simeq 1.6449341... .$$

Thus if we compute S_n for larger and larger values of n, then we should approach this limit ever more closely as n gets larger.

Let us see what happens.

The following BASIC program will compute S_n .

```
READ n
Sn = 0
FOR I = 1 TO n
     LET Tn = I^ - 2
     LET Sn = Sn + Tn
NEXT I
PRINT n; Sn
DATA < value of n >
END.
```

This readily gives the following table. (Details may differ according to the machine and the software package employed. What follows is from Microsoft Quick Basic on a MacPlus.)

Table 1

n	S_n
100	1.634 984
1000	1.643 935
10 000	1.644 725
100 000	1.644 725

Now, something has gone horribly wrong. The machine took about two minutes to calculate $S_{10\ 000}$ and nearly 20 minutes to get the same answer for $S_{100\ 000}$. The extra time was entirely wasted. What the computer was doing all this time we'll see in a minute.

But note that because the computer has given the same value for $S_{10\ 000}$ and $S_{100\ 000}$ it will give the same answer for any S_n for $n \geq 10\ 000$. The computer's estimate of $\pi^2/6$ is permanently out by 0.000209... . Not a very impressive record at all!

Now our program computed S_n by starting at $S_1 = 1$, $S_2 = S_1 + 1/2^2$, etc. Let us now add the terms in reverse order. This, slightly modified, form of the program will do this for us.

```
READ n
Sn = 0
FOR I = 1 to n
     LET J = n + 1 - I
     LET Tn = J^ - 2
     LET Sn = Sn + Tn
NEXT I
PRINT n; Sn
DATA < value of n >
END
```

We now find:

Table 2

n	S_n
100	1.634 984
1000	1.643 934
10 000	1.644 834
100 000	1.644 924

Now, comparing this result with the known limit $\pi^2/6$, we are clearly much better off with Program No. 2 than with Program No. 1. Indeed, it can be proved that, for large n,

$$S_n \approx \frac{\pi^2}{6} - \frac{1}{n},$$

and on this approximation, Program No. 2 has done very well.

But why the discrepancy? Surely the two programs merely add the same numbers, but in reverse order. Program No. 1 computes

$$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}$$

while Program No. 2 computes

$$\frac{1}{n^2} + \frac{1}{(n-1)^2} + \dots + \frac{1}{1^2} .$$

To explain where the difference comes from, we quote from Dr. Mundy's article.

> "The discrepancy is caused by the representation for real numbers inside a computer. The computer representation of real numbers is based on the notation known as *floating point*. This representation uses a fixed amount of space to represent a real number. For each real number, six digits are set aside to represent the fractional part (it is always non-zero except when the number is zero) and two digits to represent the exponent. E.g. 314.156 could be represented by $.314156 \times 10^3$, whilst 0.000314156 would be represented by $.314156 \times 10^{-3}$. So the largest and smallest non-negative numbers we can have are
>
> $$.999999 \times 10^{99} \quad \text{and} \quad .100000 \times 10^{-99}$$
>
> respectively.
>
> "Now consider what happens when a very small number is added to a very big number. For example, in evaluating
>
> $$.123455 \times 10^{24} + .678901 \times 10^{-1}$$
>
> the expected result would be
>
> $$123456000000000000000000.0678901,$$
>
> which would require 31 digits to represent the fractional part. But since the computer uses floating point, it will only take the first six significant digits for the fractional part. Thus the answer will only appear as
>
> $$.123456 \times 10^{24}.$$
>
> "Returning to our calculation we notice that when we do the first calculation, we are bound to run into a problem we have just described. We shall be adding a very small fraction at some point and the computer will lose that number for it takes only the first six significant figures for its floating point representation. In contrast, in our backward calculation at no point in the calculation will the sum be larger with respect to the new term being added.
>
> "In conclusion, the discrepancy observed in the calculation was due to the representation of real numbers in the computer using the floating point notation which limits the number of digits in the fractional part. However, by changing the order in which terms are added a more accurate result is obtained. So, although the computer can't add up, it can add down!"

So, for most of the long time the computer took to calculate $S_{100\,000}$ using the first program, it might as well have been adding zeroes to S_n.

* * * * *

HISTORY OF MATHEMATICS

EDITOR: M.A.B. DEAKIN

More on Napoleon's Theorem

In this column (*Vol.* 16, *Part* 2), I drew attention to a result known as Napoleon's Theorem and gave what I could of its history. Rather more has since come to light and I would like to pass this on to readers as well.

First to recapitulate. Let *ABC* be any triangle and on each of its sides erect equilateral triangles (either all facing outward or all facing inward). Let these be *ABR*, *BCS* and *CAQ* as shown in Figure 1 which depicts the "facing outward" case. If 0_1 is the centre of the equilateral triangle *BCS*, 0_2 that of *CAQ* and 0_3 that of *ABR*, then the theorem states that the triangle $0_1 0_2 0_3$ is equilateral.

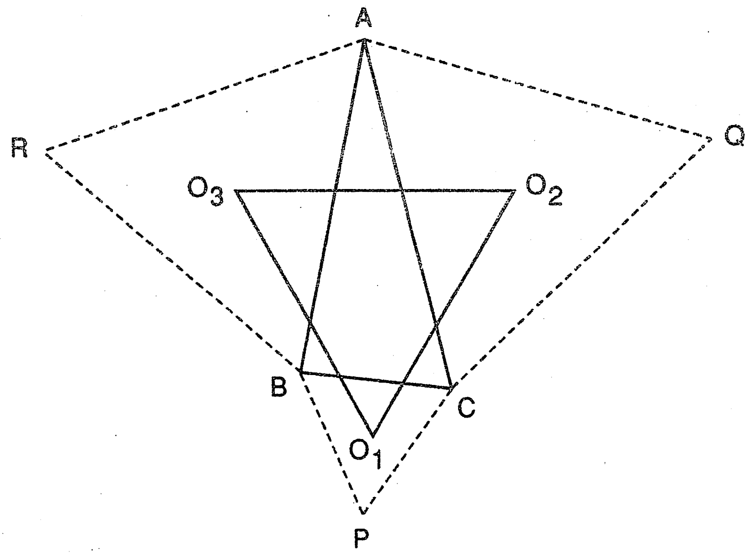


Figure 1

The theorem is discussed in Coxeter & Greitzer's book *Geometry Revisited* and two paragraphs of their discussion were reproduced in the earlier article (due to a regrettable oversight, without quotation marks).

Since publication of that article, two further discussions have come to my attention. Emeritus Professor Bernhard Neumann of A.N.U. has sent a copy of a 1982 article he wrote, entitled "Plane Polygons Revisited". It is in fact a transcript of a lecture Professor

Neumann delivered on a number of occasions in various countries, usually under the title "Napoleon, my father and I".

Neumann's father, Richard Neumann, was an electrical engineer, who, in the course of research into the theory of transformers and 3-phase current, came upon electrical applications of the theorem. It was this that interested his son in the problem and led Bernhard to produce generalisations (different from those mentioned in the earlier article). Independently and more or less at the same time, similar generalisations were produced by the American mathematician Jesse Douglas.

Professor Neumann's paper gives some more references to Napoleon's Theorem itself and we will return to these. But first let me introduce the other source that reached me. The magazine *Alpha* is a German counterpart of *Function* and there is an exchange agreement operating between the two. The latest issue of *Alpha* to reach us was the April 1992 issue and this contains an article on Napoleon's Theorem.

Here, in translation,† is what Dr. Wolfgang Dörband, author of the *Alpha* article, has to say.

> What is the connection between Napoleon's Theorem and Napoleon I (Bonaparte, 1769-1821), the French emperor? In the 18th edition of his Geometry text *Elementi di geometria* (Venice, 1912), the Italian author Faifofer claimed that Napoleon left Lagrange to prove this theorem. However, in the current state of History of Mathematics, there are no sources for this claim. The theorem itself was first mentioned (as far as has been ascertained to date) by Turner in his *Elementi di geometria* (Palermo, 1843) but without mention of Napoleon. (See Joachim Fischer's *Napoleon und die Naturwissenschaften* - Franz Steiner Verlag Wiesbaden GmbH; Stuttgart, 1988.)
>
> When mathematical concepts or theorems bear the names of historical figures, one should not automatically identify the name with the originator. This even applies to such well-known names as (e.g.) Pythagoras or Thales. The spectrum of such dedication is very broad, containing many gradations; to name but a few: "guaranteed priority", "first to publish", "intellectual partnership", "absolutely unconnected", "suspected of plagiarism". In Mathematics such names act as technical terms. Their historical relevance often cannot be conclusively determined, as documented sources have been lost or were never available. On the other hand, new sources may turn up by a stroke of good fortune. As far as Napoleon is concerned, we may note that contemporary witnesses and documents are known attesting to his talent for and interest in Mathematics.

I rather imagine that, failing some new source turning up "by a stroke of good fortune", this is the best we are likely to do. Reading between the lines, I take it that Dörband got his information from Fischer's book whose title translates as "Napoleon and the Natural Sciences". Presumably it was Fischer who found the references to Faifofer and to Turner. I am unable to find any details of Faifofer, but if his "Elements of Geometry"

† Our thanks to Anne-Marie Vandenberg.

ran to 18 editions, clearly it was a very popular text. Turner, despite his English sounding name, was an Italian: Guglielmo Turner (1809-1852). He was born, lived and died in Palermo, the capital of Sicily. He was a mathematics teacher and a member of the Jesuit order; his text on Geometry was his only published work.

The textbooks by Turner and Faifofer seem to be the earliest known mentions of the theorem itself and of its attribution to Napoleon, respectively. Dörband's account of what Faifofer says is not quite clear. What I take it to mean is that Napoleon suggested the theorem and that, acting on this suggestion, Lagrange proved it.

Such a story is quite plausible. Napoleon most probably did know Lagrange and if he came up with such a result, it is the sort of thing he would mention to the mathematician, who would be able quite easily to supply a proof.

Bernhard Neumann's paper tells us that the triangle $0_1 0_2 0_3$ of the picture (the so-called "outer Napoleon triangle") and the other (undrawn) triangle that could be constructed with Q on the other side of CA, etc. (the so-called "inner Napoleon triangle") have the same centroid and this point is also the centroid of the original triangle ABC. Furthermore, the area of the triangle ABC is equal to the difference in areas between its two Napoleon triangles. Neumann and others were able to generalise these results to arbitrary plane polygons.

As to the theorem itself, he notes an attribution to Napoleon recorded in a 1938 paper and he remarks that according to the French author Laisant, the result was well-known by 1877.

Dörband devotes some time to a special case in which A, B, C are collinear: the triangle has collapsed into a line segment. Readers may care to investigate this case for themselves.

* * * * *

PROBLEMS AND SOLUTIONS

EDITOR: H. LAUSCH

Function *wishes to thank all readers who have contributed problems and solutions in the course of this year. The problems editor looks forward to problems and solutions readers will send in 1993.*

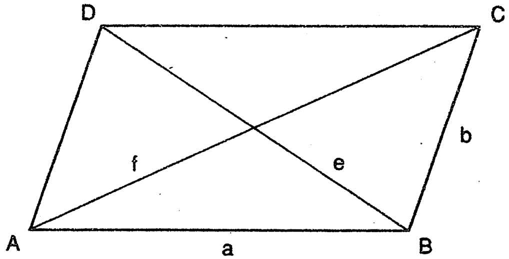
SOLUTIONS

Problem 15.4.2 (K.R.S. Sastry, Addis Ababa, Ethiopia) A parallelogram $ABCD$ (with diagonals AC and BD) is called self-diagonal if the sides are proportional to the diagonals, i.e. $AB : AC = BC : BD$. Prove that the parallelogram $ABCD$ is self-diagonal if and only if $AC + BD = \sqrt{2(AB+BC)}$.

Dieter Bennewitz (Koblenz, Germany), whose proof follows below, showed that the definition of "self-diagonal" requires the following modification, as the statement above is incorrect as it stands:

A parallelogram $ABCD$ (with diagonals AC and BD) is called self-diagonal if the sides are proportional to the diagonals, i.e. $AB : AC = BC : BD$ or $AB : AC = BD : BC$.

Solution (by Dieter Bennewitz, Koblenz, Germany).



Claim: $\frac{a}{f} = \frac{b}{e}$ or $\frac{a}{e} = \frac{b}{f}$ if and only if $f + e = \sqrt{2(a+b)}$.

Proof. 1. "Only if": We use the "parallogram property", *viz.* $e^2 + f^2 = 2a^2 + 2b^2$. We have either $\frac{a}{f} = \frac{b}{e}$ or $\frac{a}{e} = \frac{b}{f}$. By symmetry, we may assume without loss of generality that $\frac{a}{f} = \frac{b}{e}$. Then $e = \frac{b}{a}f$. Hence $\left[\frac{b}{a}f\right]^2 + f^2 = 2a^2 + 2b^2$, implying that $f^2\left[\frac{b^2}{a^2} + 1\right] = 2a^2 + 2b^2$ and further $f = \sqrt{2}\,a$; similarly, $e = \sqrt{2}\,b$, so that $e + f = \sqrt{2(a+b)}$.

2. "If": Now

$$e + f = \sqrt{2(a+b)} \tag{1}$$

is given. Squaring both sides of this equation, we obtain

$$e^2 + f^2 + 2ef = 2a^2 + 2b^2 + 4ab.$$

This, together with the parallelogram property, yields $2ef = 4ab$, thus

$$ef = 2ab. \tag{2}$$

By (1), $e = \sqrt{2(a+b)} - f$, and, by (2), $(\sqrt{2(a+b)} - f)f = 2ab$. Hence

$$f^2 - \sqrt{2(a+b)}f + 2ab = 0$$

$$f = \frac{\sqrt{2}}{2}(a + b) \pm \sqrt{\tfrac{1}{2}(a+b)^2 - 2ab}$$

$$f = \frac{\sqrt{2}}{2}(a + b) \pm \frac{\sqrt{2}}{2}\sqrt{(a-b)^2}$$

$$f = \frac{\sqrt{2}}{2}[(a + b) \pm (a - b)]. \tag{3}$$

We distinguish three cases:

1) $a > b$:

If, in (3), the sign $+$ is taken, then $f = \sqrt{2}\ a$, $e = \sqrt{2}\ b$, which implies $ea = fb$. If the sign $-$ is taken, then $f = \sqrt{2}\ b$, $e = \sqrt{2}\ a$, which implies $fa = eb$.

2. $a < b$: similarly, if the sign $+$ is taken in (3), then $ea = fb$ follows; if the sign $-$ is taken, $fa = eb$ follows.

3. $a = b$: in this case, $e = f = \sqrt{2}\ a$, which implies $ea = fb$.

Problem 16.4.3 (P.A. Grossman, Caulfield) Prove that every triangle is "approximately isosceles", which is to say that in every triangle there are two sides whose lengths are in a ratio that is less than $(1 + \sqrt{5}) : 2$.

Solution (by Bécsi János, Hódmezövásárhelykútasipuszta near Hódmezövásárhely, Hungary). Let a, b and c be the sides. We may assume that $a \geq b \geq c$. Put $\tau = \frac{1+\sqrt{5}}{2}$ and note that

$$\tau^2 - \tau - 1 = 0. \tag{*}$$

Suppose that $a : b \geq \tau$ and $b : c \geq \tau$. Then $a \geq b\tau$ and $b \geq c\tau$, which implies $a \geq c\tau^2$. Since the largest side in a triangle is shorter than its other two sides put together, we have $a < b + c$. Hence $\tau^2 a < \tau^2 b + \tau^2 c \leq \tau a + a$, implying that

$$\tau^2 - \tau - 1 < 0, \tag{**}$$

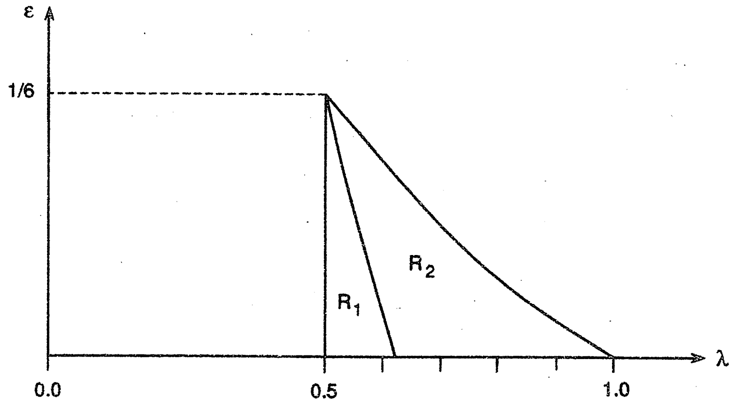
which contradicts (*). Therefore there are two sides whose lengths are in a ratio that is less than $(1 + \sqrt{5}) : 2$, q.e.d.

This, however, does not exhaust the matter, comments the Chief Editor, M.A.B. Deakin, who writes:

"Problem 16.4.3 intrigued me – the more so as I had been working on the golden ratio and not known of this occurrence of it. We may as well assume that $a + b + c = 1$ and set $r_1 = \frac{a}{b}$, $r_2 = \frac{b}{c}$. Put

$$\alpha = \tfrac{1}{2} - \varepsilon,\ b = \lambda(\tfrac{1}{2} + \varepsilon),\ c = (1 - \lambda)(\tfrac{1}{2} + \varepsilon).$$

We are seeking $\max(\min(r_1, r_2))$ over the allowable region of (λ, ε)-space.

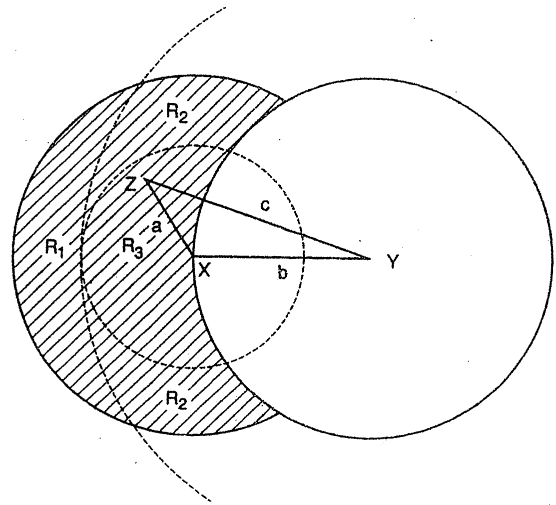


"The allowable region is a subset of the square $\frac{1}{2} \le \lambda \le 1,\ 0 \le \varepsilon \le \frac{1}{2}$. But as $a > b$, we also have $\frac{1}{2} - \varepsilon > \lambda(\frac{1}{2} + \varepsilon)$, i.e. $\varepsilon < \frac{1}{2}\left[\frac{1-\lambda}{1+\lambda}\right]$. This region further decomposes into R_1, R_2 where $r_1 > r_2$, $r_2 > r_1$ respectively (see diagram). In R_1, $\varepsilon < \frac{1}{2}\left(\frac{1-\lambda-\lambda^2}{1-\lambda+\lambda^2}\right)$ and here we require $\max\limits_{R_1} r_2$. But as r_2 increases with λ, this maximum occurs at $\varepsilon = 0$, $\lambda = 1/\tau$; i.e. we get $r_2 = \tau$. We now investigate $\max\limits_{R_2} r_1$. But r_1 decreases as ε and λ increase. Thus r_1 is maximised on the common boundary of R_1, R_2 where $r_1 = r_2$. But we have already seen that this maximum occurs at $(0, \frac{1}{\tau})$.

"If the degenerate case is allowed, the maximum is achieved and of course $\frac{a}{b} = \frac{b}{c} = \tau$,

as it should be".

Peter Grossman, who proposed the problem, further comments:



Let the sides be $a \le b \le c$. Let XY be the side with length b. Assume the shortest side is incident to X, and the longest side is incident to Y. Then the third vertex Z of the triangle lies in the shaded region bounded by the circles with radius b centred at X and Y. Now draw the (dashed) circles, one central at Y with radius $b\tau = \left(\frac{1+\sqrt{5}}{2}\right)$, and one centred at X with radius $\frac{b}{\tau}$. Then:

$$Z \in R_1 \Rightarrow \frac{b}{a} < \tau$$

$$Z \in R_2 \Rightarrow \frac{b}{a} < \tau \text{ and } \frac{c}{b} < \tau$$

$$Z \in R_3 \Rightarrow \frac{c}{b} < \tau.$$

PROBLEMS

The 33rd International Mathematical Olympiad

The 1992 International Mathematical Olympiad (IMO) took place in July at Moscow, the capital of the Russian Federation. Official teams of up to six students from 56 countries sat the contest. It consisted of two four-and-a-half-hour examinations held on subsequent days. Altogether 26 gold, 55 silver and 74 bronze medals were awarded. Each student who missed out on a medal but obtained a perfect score for the solution to at least one problem received an honourable mention. Six students headed the list with perfect scores (42 marks – 7 marks for each of their solutions to the six problems), three of whom were from the People's Republic of China. In the unofficial ranking of countries, the People's Republic of China came first (240 out of 252 possible points), second were the United States of America (181), and Romania ended up third (177). The team from the Commonwealth of Independent States took fourth position (176). The list continues with: 5. United Kingdom (167), 6. Russian Federation (158), 7. Germany (149), 8. Hungary and Japan (142 each), 10. France and Vietnam (139 each). Scoring only nine points less than the strong team from Bulgaria, where mathematical olympiads have had a very long tradition, the Australian team did very well in achieving 19th place (118). Function *congratulates*

gold medal winner:	*Benjamin Burton (39 points), year 12, John Paul College, Queensland,*
silver medal winner:	*Anthony Henderson (25 points), year 11, Sydney Grammar School, NSW,*
bronze medal winners:	*Frank Calegari (15 points), year 11, Melbourne Church of England Grammar School, Victoria,*
	Rupert McCallum (15 points), year 11, North Sydney Boys' High School,

winner of an honourable mention:

Adrian Banner (13 points), year 12, Sydney Grammar School, NSW.

Here are the problems of the 1992 IMO:

FIRST DAY

1. Find all integers a, b, c with $1 < a < b < c$ such that $(a-1)(b-1)(c-1)$ is a divisor of $abc - 1$.

2. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = y + (f(x))^2 \quad \text{for all } x, y \text{ in } \mathbb{R}.$$

3. Consider nine points in space, no four of which are coplanar. Each pair is joined by an edge (that is, a line segment) and each edge is either coloured blue or red or left uncoloured. Find the smallest value of n such that whenever exactly n edges are coloured, the set of coloured edges necessarily contains a triangle all of whose edges have the same colour.

SECOND DAY

4. In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exist two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

5. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|$$

where $|A|$ denotes the number of elements in the finite set A.
(Note: the orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)

6. For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive square integers.

a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.

b) Find an integer n such that $S(n) = n^2 - 14$.

c) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

During the IMO members of the Australian delegation were presented with problems from numerous national mathematics contests of 1991/92. Here is a sample for Function *readers. Please send in your solutions:*

16.5.1 (Swedish Mathematical Competition 1991 – final round, Question 1. Determine all positive integers m and n such that

$$\frac{1}{m} + \frac{1}{n} - \frac{1}{mn} = \frac{2}{5}.$$

16.5.2 (Republic of Slovenia – 36th mathematics competition for secondary school students, first class, Question 2.) For natural numbers $a_0, a_1, \ldots, a_{1992}$,

$$a_0^{a_1} = a_2^{a_2} = a_2^{a_3} = \ldots = a_{1991}^{a_{1992}} = a_{1992}^{a_0}$$

holds. Prove that $a_0 = a_1 = a_2 = \ldots = a_{1992}$.

16.5.3 (1st Mathematical Olympiad of the Republic of China [Taiwan], Question 3.) If $x_1, x_2, \ldots, x_n$ are n non-negative numbers, $n \geq 3$, and $x_1 + x_2 + \ldots + x_n = 1$, prove that $x_1^2x_2 + x_2^2x_3 + \ldots + x_n^2x_1 \leq \frac{4}{27}$.

16.5.4 (28th Spanish Mathematical Olympiad – First Round, Question 8.) Let ABC be any triangle. Two squares $BAEP$ and $ACDR$ are constructed, externally to ABC. Let M and N be the midpoints of BC and ED, respectively. Show that AM and ED are perpendicular and AN and BC are perpendicular.

16.5.5 (43rd Mathematical Olympiad in Poland, Final Round, Question 5.) The regular $2n$-gon $A_1A_2 \ldots A_{2n}$ is the base of a regular pyramid with vertex S. A sphere passing through S cuts the lateral edges SA_i in the respective points B_i $(i = 1, 2, \ldots, 2n)$. Show that

$$\sum_{i=1}^{n} SB_{2i-1} = \sum_{i=1}^{n} SB_{2i}.$$

The TELECOM Mathematics Contests of 1992

On 14 August these two mathematics competitions were held in Australian schools. Here are the problems. Time allowed for either contest paper was four hours.

Junior Contest

(For students in Year 10 or less)

1. Mark and Paul played a game of Bulls and Cows, using the digits 1, 2, 3, 4, 5 and 6 to make a four-digit number. Mark made such a number and Paul proceeded to guess it by writing down a four-digit guess. Mark awarded his guess with a bull for each digit guessed in its correct position and a cow for each digit guessed not in its correct position. For example, if Mark's secret number was 3352 and Paul guessed 3124 then he would score one bull and one cow (note that the 3 only scores a bull – not a bull and a cow; each guessed digit can score at most once).

In a particular game, Paul made five guesses in turn, as follows:

1	2	3	4	Two bulls
3	2	4	1	Two cows
5	1	4	2	One bull, one cow
3	3	6	1	One bull, one cow
5	1	4	1	One bull, one cow.

What was Mark's secret number?

2. O is a point in a triangle ABC. OP, OQ and OR are the perpendiculars from O to BC, CA and AB respectively. Squares are drawn on PC, QA and RB. If $BP = 5$, $CQ = 6$ and $AR = 7$, find the sum of the areas of the three squares.

3. Let the area of ΔXYZ be denoted by $|XYZ|$. $ABCD$ is a parallelogram. P, Q, R and S are points on the sides AB, BC and DA respectively, with PR parallel to AD and SQ parallel to AB. Prove that $|QPA| + |QPR| = |QPD|$.

4. Show that, for n a positive integer, $5^{2n} - 600n - 1$ is divisible by 576.

5. Determine all 20-digit perfect squares whose leftmost 9 digits (in ordinary base-ten representation) are nines.

Senior Contest

(For students in Year 11)

1. Identical with Problem 5 of Junior Contest.

2. Prove that there exists a positive integer n such that the sum of all integers that are greater than 1992 and less than $1992 + n$ is a perfect square.

3. Let ABC be an isosceles triangle with base BC and $\angle BAC = 100^\circ$. The bisector of $\angle ABC$ intersects AC in P. Prove that $BC = AP + PB$.

4. Let S be a set consisting of n elements. If (A, B) is an ordered pair of subsets of S, let $f(A, B)$ be the number of elements that A and B have in common. Determine the sum of the numbers $f(A, B)$ where (A, B) ranges over all ordered pairs of subsets of S including pairs (A, A). [Note: by definition of *ordered pairs*, (A, B) and (B, A) are counted as different pairs if $A \neq B$.]

5. Prove that from every convex pentagon one can select three diagonals that have lengths which occur as side lengths in some triangle.

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