Function

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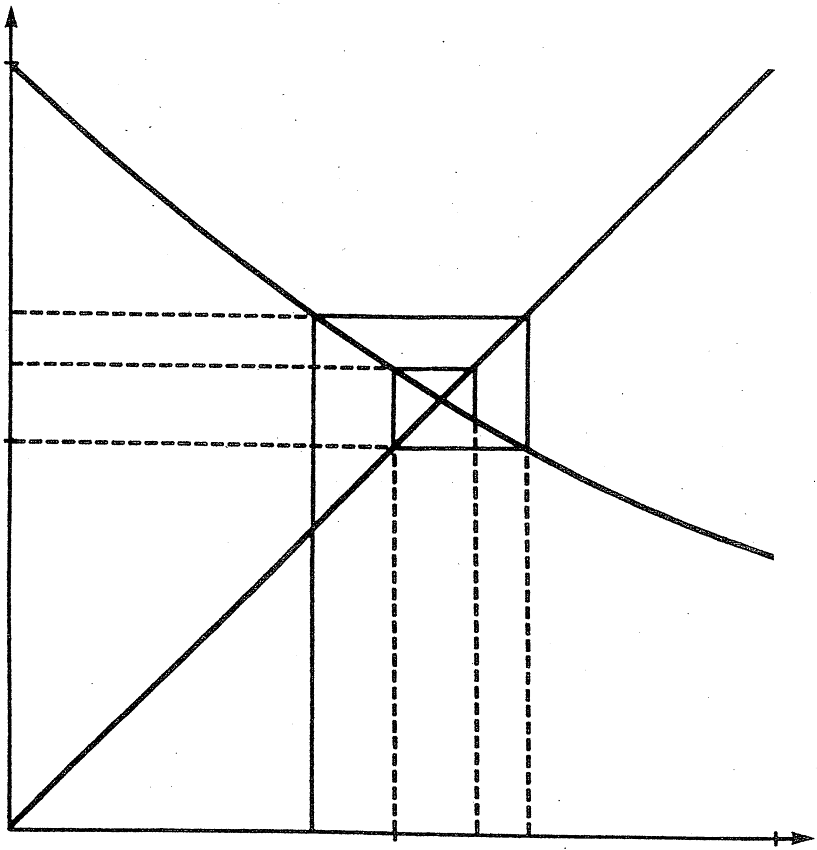

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FUNCTION

Volume 16 *Part 3*

(Founder editor: G.B. Preston)

EDITORIAL

Kent Hoi's article "One Good Turn" (*Function*, *Vol.* 16, *Part* 1) arose out of a VCE project and, as we wrote at the time, "very ably looked at the problem set by the examiners". Unfortunately, as two correspondents have pointed out (see pp. 83-87), the problem set by the examiners bears little relation to reality. We too deplore the phony "relevance" of such artificial questions and think they have no place. Nonetheless, Kent's article was a nice piece of work.

CONTENTS

THE FRONT COVER

Michael A.B. Deakin, Monash University

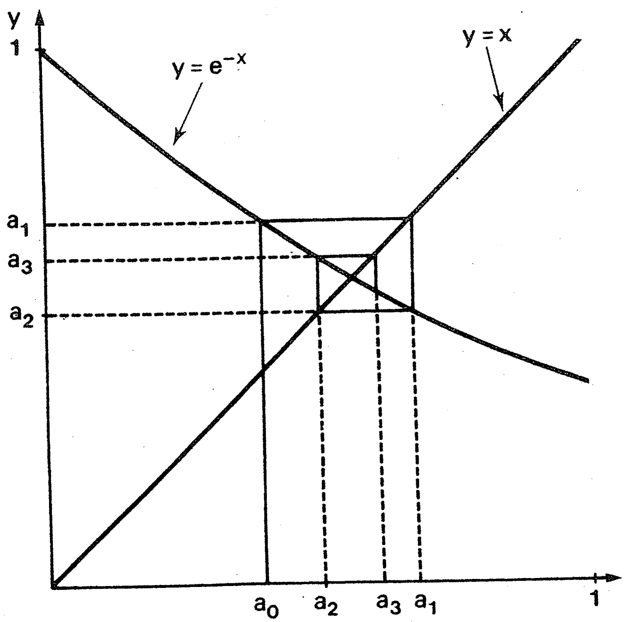
The Front Cover diagram, reproduced below, is based on a method for solving equations numerically. In this case the equation to be solved is

$$e^{-x} = x.$$

The two functions e^{-x}, x are graphed and it is seen that the graphs intersect for $x \simeq 0.57$. The graph is mainly illustrative, however. We do not actually use it in the computation.

Begin with an approximation a_0, say, to the correct value of the solution. In this case, I chose $a_0 = 0.4$. Then

$$e^{-a_0} = 0.67032 = a_1 \quad \text{(say)}.$$



The value of a_1 is represented on the vertical axis and, by reflecting it in the line $y = x$, on the horizontal axis also. This gives a value rather closer than the earlier value. Continue in this way

$$e^{-a_1} = a_2$$

$$e^{-a_2} = a_3, \text{ etc.}$$

The solid spiral on the diagrams indicates the way that the successive values $a_0, a_1, a_2, \ldots$ better and better approximate the true value.

I first implemented the algorithm on a small calculator. I entered 0.4 and then pressed in order +/–, INV and e^x. This gave 0.67032; repeating these three keystrokes gave 0.5115448; and so I continued. Eventually I found

$$a_{28} = a_{30} = a_{32} = \ldots = 0.5671\ 433$$

$$a_{29} = a_{31} = a_{33} = \ldots = 0.5671\ 432$$

as approximations to the exact value.

I next used a spreadsheet on my Macintosh to find

$$a_{39} = a_{41} = a_{43} = \ldots = 0.5671\ 4329\ 03$$

$$a_{40} = a_{42} = a_{44} = \ldots = 0.5671\ 4329\ 04.$$

The following BASIC programme solves the equation by the same method.

```
DIM A(30)
A(0) = 0.4
FOR I = 1 TO 30
    A(I) = EXP(-A(I-1))
    PRINT I; A(I)
    NEXT I
END
```

On my machine, this gave

$$a_{27} = a_{28} = a_{29} = \ldots = 0.5671\ 433.$$

* * * * *

WHEN CATTLE FARMING BECOMES AN ECOLOGICAL DISASTER

Hans Lausch, Monash University

Only 11 km south of Brunswick in Lower Saxony, where the eminent mathematicians Carl Friedrich Gauss (1777-1855) and Richard Dedekind (1831-1916) were born, lies Wolfenbüttel, a small Northern Renaissance city with a famous library: the Herzog August Bibliothek. It houses literary treasures that rulers of the former Duchy of Brunswick-Lüneburg-Wolfenbüttel and their capable librarians accumulated over the centuries. Most noteworthy among the Wolfenbüttel librarians are: Gottfried Wilhelm Leibniz (1646-1716) and Gotthold Ephraim Lessing (1729-1781). However, not the mathematician Leibniz but the writer and dramatist Lessing is known for having unearthed from the ducal library a manuscript with a delightful mathematical problem.

As a boarder at St. Afra Grammar School in Meissen, Saxony, Lessing had delivered a speech on the mathematics of the barbarians (*De mathematica barbarorum*). His interest in mathematics declined suddenly after he had come across a book that combined geometry with chiromancy, the "art" of palmistry. Nonetheless, Lessing's mathematical acumen allowed him to decide that his find deserved publication. The manuscript is based on an anthology by the Byzantine monk and diplomat Maximos Planudes (*c*.1260-*c*.1310), who also wrote an article about the Indian (= Arabic) numerals, then new to his countrymen from Constantinople. The manuscript consists of a number-theoretical problem formulated in 22 Greek elegiac couplets, i.e. 44 lines with hexameter and pentameter alternating. By its headline we are informed that the Sicilian mathematician Archimedes of Syracuse (284-212 B.C.) had composed it and sent it to his North-African colleague Eratosthenes of Cyrene as a puzzle for the geometers in Alexandria. The text is accompanied by a scholium, i.e. an explanatory comment. While Lessing himself and later scholars doubted Archimedes' authorship, the Danish Archimedes expert, J.L. Heiberg, and the French mathematician P. Tannery did believe the headline.

The problem is about the sacred cattle herds belonging to Helios, the sun god (or Hyperion, the sun titan), which he kept on a Sicilian cattle station. There were four herds, each of a different colour, *viz.* while, blue, checkered and yellow, and each having both oxen and cows. In Lessing's notation, the number of white, blue, checkered and yellow oxen is W, X, Y and Z respectively; the number of white, blue, checkered and yellow cows is w, x, y and z respectively.

Here is an English translation of the original.

"Tell me, friend, precisely the number of Helios' cattle.

"Carefully calculate, if wisdom has not abandoned you, how many there were that once were grazing on Sicily's pastures, divided into four herds. Each herd had a different colour, the one was milky white, but the second was shining in darkest black. The third, however, was brown, the fourth was checkered; in each one the bulls numerically outweighed the cows. And such was the relation: the white ones numerically equalled the brown ones and a third taken together with half of the black ones, o friend. Further the black set was equal to the fourth part and the fifth of the checkered augmented by all the brown ones. Finally you have to put equal the number of checkered bulls to the sixth and also the seventh part of the white ones plus all of the brown set.

"With the female cattle the story is different: Those with whitish hair were equal to the third part and the fourth part of the blackish cattle, cows and bulls. Further the black cows were equal to the fourth and the fifth part of the herd of the checkered ones, counting cows as well as bulls. Also, the checkered cows were one fifth plus one sixth of all those with brown hair, as they went to the pasture. Finally, the brown cows were one sixth and one seventh of the whole herd with whitish hair.

"If you can tell me precisely, my friend, how many of the cattle were joined there together, also how many cows there were of each colour and how many well-fed bulls, then one will call you quite proficient in arithmetic.

"But you will still not be counted among the sages; well now, come and tell me, how the story continues: If all white bulls and black bulls united and stood in an orderly fashion, the number of ranks equalled the number of files; the wide land of Sicily would be completely filled by the set of bulls. If, however, you put together the brown and the checkered ones, then they would form a triangle, one standing in front, and none of the brown and checkered bulls would be missing, while none of any other colour would be found among them.

"If you have found also this and conceived by your mind, and tell me the ratio, friend, in each herd, then you may walk around as a proud winner; for your fame in scholarship will shine brightly."

Lessing re-wrote the original problem, which we call a system of Diophantine equations, in "the now common notation":

$$W = \tfrac{1}{2}X + \tfrac{1}{3}X + Z = \tfrac{5}{6}X + Z \tag{1}$$

$$X = \tfrac{1}{4}Y + \tfrac{1}{5}Y + Z = \tfrac{9}{20}Y + Z \tag{2}$$

$$Y = \tfrac{1}{6}W + \tfrac{1}{7}W + Z = \tfrac{13}{42}W + Z \tag{3}$$

$$w = (\tfrac{1}{3} + \tfrac{1}{4})(X + x) = \tfrac{7}{12}(X + x) \tag{4}$$

$$x = (\tfrac{1}{4} + \tfrac{1}{5})(Y + y) = \tfrac{9}{20}(Y + y) \tag{5}$$

$$y = (\tfrac{1}{5} + \tfrac{1}{6})(Z + z) = \tfrac{11}{30}(Z + z) \tag{6}$$

$$z = (\tfrac{1}{6} + \tfrac{1}{7})(W + w) = \tfrac{13}{42}(W + w) \tag{7}$$

$$W + X = \square \tag{8}$$

$$Y + Z = \Delta \tag{9}$$

where $\square$ stands for "a perfect square", i.e. a number of the form n^2, n being an integer, and Δ stands for "a triangular number", i.e. a number of the form $\frac{m(m+1)}{2}$, m being a positive integer.

By the copyist we are told that $W = 829\ 318\ 560, X = 596\ 841\ 120, Y = 588\ 644\ 800$ and $Z = 331\ 950\ 960$ and are given values for w, x, y and z as well. The grand total of oxen and cows turns out to be 4 031 126 560. "Truly a herd befitting Sicily",† noted Lessing. When substituting W, X and Y, Z into equations (8) and (9) respectively and taking the square root of $W + X = 1\ 426\ 159\ 680$, Lessing discovered that $W + X$ is not a perfect square after all. And he noticed that $Y + Z = 920\ 595\ 760$ "multiplied by 8 and augmented by 1" is not a perfect square, either, which it would have to be if $\frac{m(m+1)}{2} = Y + Z$ had an integer solution m (check this statement!). Lessing considered the possibility that the author of the scholium had erred because "extracting roots in Greek numerals might not have been an easy job".

Wolfenbüttel had been endowed by its dukes with a grammar school, the so-called Great School. In 1773 its conrector (= vice-principal) was the mathematician Christian Leiste, who had been praised by the mathematician Leonhard Euler (1707-1783) for having improved the construction of air-pumps. Working in the immediate neighbourhood of Lessing, the conrector read the problem and attempted a solution. The problem seemed too hard for Leiste, yet he reduced it to a more manageable problem; here is Leiste's approach:

Equations (1), (2) and (3) imply

$$6W - 5X = 6Z;\ 20X - 9Y = 20Z;\ 42Y - 13W = 42Z, \text{ i.e.}$$
$$6\frac{W}{Z} - 5\frac{X}{Z} = 6;\ 20\frac{X}{Z} - 9\frac{Y}{Z} = 20;\ 42\frac{Y}{Z} - 13\frac{W}{Z} = 42.$$

Solving this system of three linear equations in three unknowns, we obtain

$$\frac{W}{Z} = \frac{742}{297}, \frac{X}{Z} = \frac{178}{99}, \frac{Y}{Z} = \frac{1580}{891}, \text{ so that}$$

$$W = \frac{742}{297}Z,\ X = \frac{178}{99}Z,\ Y = \frac{1580}{891}Z.$$

Since Y must be a positive integer and 891 and 1580 are relatively prime, 891 is a factor of Z. Thus we have $Z = 891i$, for some positive integer i, and hence

$$W = 2226i,\ X = 1602i,\ Y = 1580i,\ Z = 891i. \tag{10}$$

We substitute these expressions into equations (4)-(7):

$$12w - 7x = 11214i,\ 20x - 9y = 14220i,$$

$$30y - 11z = 9801i,\ 42z - 13w = 28938i.$$

After solving this system of four linear equations in the four unknowns w, x, y, z we multiply the solutions by their common denominator 4657:

$$\begin{aligned} 4657w &= 7\ 206\ 360i,\ 4657x = 4\ 893\ 246i, \\ 4657y &= 3\ 515\ 820i,\ 4657z = 5\ 439\ 213i. \end{aligned} \tag{11}$$

† Here Lessing uses a pun: the sentence can also be translated as "Truly a rather large crowd, considering Sicily".

We abbreviate $p = 4657$ and observe that p is a prime. As none of the coefficients of i in equations (ii) is divisible by p, we have $i = p \times j$ for some positive integer j. Substituting this into (10) and (11) yields:

$$\begin{aligned} W &= 10\ 366\ 482j,\ X = 7\ 460\ 514j,\\ Y &= 7\ 358\ 060j,\ Z = 4\ 149\ 387j,\\ w &= 7\ 206\ 360j,\ x = 4\ 893\ 246j,\\ y &= 3\ 515\ 820j,\ z = 5\ 439\ 213j. \end{aligned} \tag{12}$$

What we have achieved so far is having solved the system of equations (1)-(7). As j can be any positive integer, we have infinitely many solutions. Now we have to select those which, in addition, satisfy (8) and (9).

Clearly, the next step is to substitute our solutions W, X, Y, Z into (8) and (9), respectively, using (10) and the fact that $i = p \times j$:

$$3828pj = n^2 \quad \text{and} \quad 4942pj = m^2 + m$$

for some integers m and n. Putting $k = 957 = 3 \times 11 \times 29$ and $l = 4942$, the last two equations read:

$$n^2 = 4kpj \quad \text{and} \quad m^2 + m = lpj. \tag{13}$$

It follows that n has $2, 3, 11, 29, 4657$ among its prime factors, and so

$$n = 2 \times 957 \times 4657b = 2kpb$$

for some positive integer b. We see that

$$n^2 = 4k^2p^2b^2.$$

Going back to (13) we find that

$$4k^2p^2b^2 = 4kpj,$$

and so $j = kpb^2 = 4\ 456\ 749b^2$.

Substituting this value into (13) leads to

$$m^2 + m = klp^2b^2,$$

so that, after multiplying by 4 and adding 1,

$$(2m+1)^2 = 4klp^2b^2 + 1.$$

Defining $a = 2m+1$, $u = 4klp^2$, we have to solve

$$a^2 - ub^2 = 1, \tag{14}$$

with a and b being the unknowns and $u = 410\ 286\ 423\ 278\ 424$.

Equations such as (14) (with u being any positive integer) have been referred to in the literature by the name "Pellian equations" or "Pellians", after John Pell (1611-1680), Oliver Cromwell's "resident" in Zurich. Since Pell's connection with the "Pellians" is rather loose, various authors have used the name "Fermat equations" instead, linking them to the famous French mathematician Pierre de Fermat (1601-1665), whose Diophantine equation $x^n + y^n = z^n$ still awaits a general solution. On reaching (14), vice-principal Leiste gave up. For further advice Leiste referred his readers to Euler's *Algebra* (St. Petersburg 1770), where Pellians are solved.

In 1880 the German mathematician Amthor ventured to solve the cattle problem. The leftmost four decimal digits in his number of sacred Sicilian oxen and cows owned by the sun god are 7766; and these are succeeded by – holy cow! – another 206541 digits.

Exercise: suppose the sun god wants to improve the run for his Sicilian cattle. He buys all the land on earth in a bid to consolidate the planet into a single station. How many oxen and cows will, on average, occupy one square kilometre? No doubt, it will be an ecological disaster – for cattle that is, as no place will be left on earth for human beings after the sun god's business transaction.

* * * * *

PEGS IN HOLES

Karl Spiteri, Student, University of Melbourne

Which fits better: a round peg in a square hole or a square peg in a round hole?

To answer this question consider a circle of radius r inside a square of side $2r$ and a square fitted into a circle of radius r. The side of this square will be $r\sqrt{2}$. (See diagram.)

In the first case:

$$\text{Area of circle} = \pi r^2$$
$$\text{Area of square} = 4r^2.$$

So the circle-to-square ratio is $\pi/4 \simeq 0{\cdot}785$.

In the second case

$$\text{Area of square} = 2r^2$$
$$\text{Area of circle} = \pi r^2.$$

So the square-to-circle ratio is $2/\pi \simeq 0{\cdot}637$.

Because $0{\cdot}785 > 0{\cdot}637$, we can conclude that the circular peg in the square hole is the better fit.

I thought to explore generalisations of this result in two directions.

First, in place of squares, I looked at regular n-gons. If the circle is placed inside the n-gon, the circle-to-n-gon ratio is

$$\frac{n}{2\pi}\sin\left[\frac{2\pi}{n}\right] = R_1 \quad \text{(say)}.$$

Conversely if the n-gon is placed inside the circle, the n-gon-to-circle ratio is

$$\frac{\pi}{n}\cot\left[\frac{\pi}{n}\right] = R_2 \quad \text{(say)}.$$

It may be proved that for all n, $R_2 > R_1$. However, as n becomes large R_1, R_2 both get closer and closer to 1. Thus for $n = 100$

$$R_2 = 0{\cdot}99967$$
$$R_1 = 0{\cdot}99934.$$

Secondly, I considered what happens in higher dimensions. This was an extension of my earlier work on hyperspheres. (See *Function, Vol.* 15, *Part* 5.) Let n now refer to the dimensionality of the space. The circle and square refer to the case $n = 2$. When $n = 3$, we refer to the fitting of a cube inside a sphere and the fitting of a sphere inside a cube. In this case, the cube-to-sphere ratio is about $0{\cdot}368$. Call this R_1. The sphere-to-cube ratio is about $0{\cdot}524$. Call this R_2.

Note that we still have $R_2 > R_1$.

However, as I went to higher values of n, I discovered that this pattern did not persist. As long as $n < 9$, we find $R_2 > R_1$, but for $n \geq 9$, the inequality is reversed.

Another interesting feature is that, as n increases, both ratios approach zero! Indeed, if $n = 10$, we have already

$$R_1 \simeq 0{\cdot}0040; \quad R_2 \simeq 0{\cdot}0025.$$

[*Editorial note: We have left out quite a lot of technical detail, but Karl supplied a number of explicit formulae for both his generalisations. In his second case, this enables the proofs of the two properties he points out. We felt, however, that this was too specialised and technical for inclusion in* Function.]

* * * * *

HOW BIG IS A FUNCTION?

Peter Kloeden, Deakin University

At first it may seem just a matter of silly mathematical speculation to ask

How big is a function $f : [a, b] \to \mathbf{R}$?

The answer is not too difficult once we understand or denote exactly what the question means. It does, however, have profound consequences both within mathematics and for the application of mathematics. To see why, consider the closely related question:

What is the distance between two functions $f, g : [a, b] \to \mathbf{R}$?

For example, the numerical values of trigonometric functions that are listed in Mathematical Tables or produced by a pocket calculator are obtained by evaluating suitably chosen polynomials. Consequently the results are not exact, but are only accurate up to a certain number of decimal places. The choice of an appropriate polynomial to approximate a given function requires our knowing how close the polynomial and the function are, that is, the distance between them. As another example, consider the task of controlling a rocket to remain on a pre-assigned trajectory. The rocket will inevitably be buffeted by fluctuations in both wind direction and speed, so the best we can hope to achieve is a perturbed path that remains sufficiently close to the desired one. Here too, we need to say what we mean by the distance between two paths, that is, between the functions describing the paths. There are, in fact, many different, non-equivalent ways of defining such a distance.

To help us understand and answer the questions above, it is useful to remind ourselves of what we mean by the size or magnitude of a real number and by the distance between real numbers. The magnitude or absolute value $|x|$ of a real number $x \in \mathbf{R}$ is just its distance from the origin on the real line and is thus a positive number unless $x = 0$ itself. See Figure 1 and remember that the negative of a negative number is a positive number, e.g. $-(-6) = +6$.

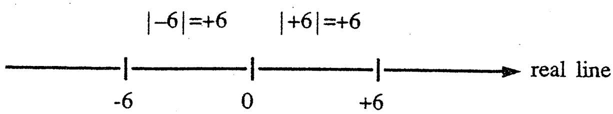


Figure 1. Measuring distance on the real line

Expressed mathematically we have

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +x & \text{if } x > 0 \end{cases}$$

which is the same as $|x| = \sqrt{x^2}$ for all x in $\mathbf{R}$.

The following basic properties of absolute values lie at the heart of what we mean intuitively when we talk about magnitude or size.

(1) $|x| \geq 0$ for all x in $\mathbf{R}$

(2) $|x| = 0$ if and only if $x = 0$

(3) $|\alpha x| = |\alpha|\,|x|$ for all α, x in $\mathbf{R}$

(4) $|x+y| \leq |x| + |y|$ for all x, y in $\mathbf{R}$.

An absolute value is either positive or zero. Of course $|0| = 0$, which is the "if" part of (2), but also $|x| = 0$ holds "only if" $x = 0$. Also, if we multiply a number x by another, α, then we change its magnitude accordingly, that is, by the factor $|\alpha|$ (since α could be negative). Finally, the magnitude of a sum of two numbers can never exceed the sum of their magnitudes. However, the numbers could have opposite signs, e.g.

$$|2 + (-1)| = |1| = 1.$$

But

$$|2| + |-1| = 2 + 1 = 3.$$

So here we have inequality, as $1 < 3$. Equality holds in (4) when one of the numbers is zero or when they both have the same sign.

Things become more complicated when we consider vectors instead of real numbers. The idea of a vector is familiar from analytical geometry, where we specify a point in the plane or space in terms of its coordinates relative to a given coordinate system. For any integer $n \geq 1$ a real n-dimensional vector $\underset{\sim}{x} \in \mathbf{R}^n$ (real n-dimensional space) has the form $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ where its ith-component ("coordinate") x_i, for $i = 1, 2, ..., n$, is a real number.† A 2-dimensional vector $\underset{\sim}{x} = (x_1, x_2) \in \mathbf{R}^2$ is sometimes conveniently represented as a directed line segment from the origin to the point in the Cartesian plane with coordinates (x_1, x_2), and similarly for 3-dimensional vectors. Mathematically it is simpler just to consider a vector $\underset{\sim}{x}$ as a point in the appropriate n-dimensional space $\mathbf{R}^n$. Then, a 1-dimensional vector $(x_1) \in \mathbf{R}^1$ is just a point x_1 on the real line $\mathbf{R}$, a 2-dimensional vector $(x_1, x_2) \in \mathbf{R}^2$ is a point in the plane, and a 3-dimensional vector $(x_1, x_2, x_3) \in \mathbf{R}^3$ is a point in "everyday" space. Higher dimensional vectors may not have such a direct geometric interpretation, but they still arise quite naturally. For example, in relativity theory $\mathbf{R}^4$ represents "space-time" and a 4-dimensional vector $\underset{\sim}{x} = (x_1, x_2, x_3, x_4)$ describes the location (x_1, x_2, x_3) of a particle in everyday space at the time-instant x_4.

Note that when we add or subtract two vectors $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbf{R}^n$ we do so *componentwise*, that is,

† Note that the zero vector $\underset{\sim}{0} = (0, 0, ..., 0)$ has all of its components equal to zero.

$$\underset{\sim}{x} + \underset{\sim}{y} = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$$

and

$$\underset{\sim}{x} - \underset{\sim}{y} = (x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n) = (x_1-y_1, x_2-y_2, \ldots, x_n-y_n).$$

Compare this with what you do in analytical geometry in $\mathbf{R}^2$ and $\mathbf{R}^3$. Similarly, we also multiply a vector $\underset{\sim}{x} \in \mathbf{R}^n$ by a *scalar* $\alpha \in \mathbf{R}$ componentwise, that is,

$$\alpha\underset{\sim}{x} = \alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n).$$

It is quite obvious that the distance between two vectors $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbf{R}^n$ should be the magnitude of their difference $\underset{\sim}{x} - \underset{\sim}{y}$, which is also a vector in $\mathbf{R}^n$. But what do we mean by the magnitude of a vector in $\mathbf{R}^n$ when $n > 1$? Once again, analytical geometry provides us with some clues. By Pythagoras' theorem the length of the line segment joining the origin in $\mathbf{R}^2$ to a point (x_1, x_2) is $\sqrt{x_1^2+x_2^2}$ and that joining the origin in $\mathbf{R}^3$ to a point (x_1, x_2, x_3) is $\sqrt{x_1^2+x_2^2+x_3^2}$. This suggests that the magnitude $\|\underset{\sim}{x}\|$ of an n-dimensional vector $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ should be defined as

$$\|\underset{\sim}{x}\| = \sqrt{x_1^2+x_2^2+\ldots+x_n^2}.$$

This not only equals the geometric distance of the point $\underset{\sim}{x}$ from the origin in $\mathbf{R}^2$ or $\mathbf{R}^3$, but also on the real line $\mathbf{R}^1$ since $\|(x_1)\| = \sqrt{x_1^2} = |x_1|$, the absolute value of the real number x_1. See Figure 2.

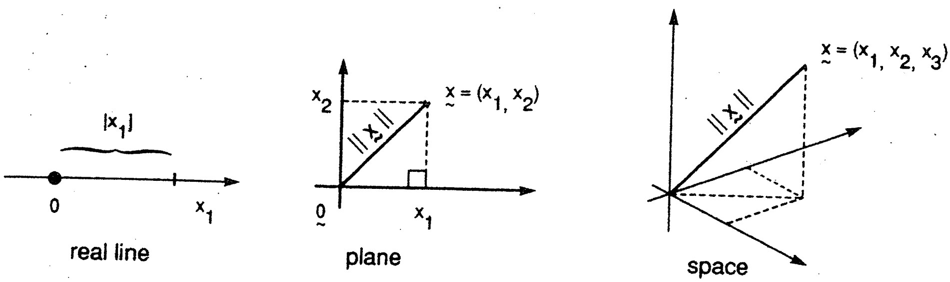


Figure 2. Geometric distances

The *Euclidean norm* $\|\underset{\sim}{x}\| = \sqrt{x_1^2+x_2^2+\ldots+x_n^2}$, as it is called, also satisfies analogous properties to (1)–(4) for the absolute value, that is, (1*) $\|\underset{\sim}{x}\| \geq 0$ for all $\underset{\sim}{x} \in \mathbf{R}^n$

(2*) $\|\underset{\sim}{x}\| = 0$ if and only if $\underset{\sim}{x} = \underset{\sim}{0}$

(3*) $\|\alpha\underset{\sim}{x}\| = |\alpha|\ \|\underset{\sim}{x}\|$ for all $\underset{\sim}{x} \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$

(4*) $\|\underset{\sim}{x}+\underset{\sim}{y}\| \leq \|\underset{\sim}{x}\| + \|\underset{\sim}{y}\|$ for all $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbf{R}^n$.

The last of these is called the *triangle inequality* (why?). It is quite tricky to prove.

The Euclidean norm is a natural generalisation of our everyday experience with the size or magnitude of real numbers and vectors, as is the corresponding *Euclidean distance*

$$\|\underset{\sim}{x} - \underset{\sim}{y}\| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2}$$

between two vectors $\underset{\sim}{x}, \underset{\sim}{y}$ in $\mathbf{R}^n$. There are, however, other possibilities which are sometimes easier to use, for example

(i) the max-norm $\|\underset{\sim}{x}\|_m = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$,

that is, the largest of the numbers $|x_1|, |x_2|, \ldots, |x_n|$,
and

(ii) the 1-norm $\|\underset{\sim}{x}\|_1 = |x_1| + |x_2| + \ldots + |x_n|$.

(We sometimes call the Euclidean norm the 2-norm and denote it by $\|\underset{\sim}{x}\|_2$ to distinguish it from others.) Note that both the max-norm and the 1-norm satisfy properties (1*)-(4*) above.

The three norms $\|\underset{\sim}{x}\|_1$, $\|\underset{\sim}{x}\|_2$ and $\|\underset{\sim}{x}\|_m$ will usually have different numerical values for a given vector. For example,

$$\|(-4, 3, 0)\|_1 = |-4| + |3| + |0| = 7$$

$$\|(-4, 3, 0)\|_2 = \sqrt{(-4)^2+3^2+0^2} = \sqrt{25} = 5$$

$$\|(-4, 3, 0)\|_m = \max\{|-4|, |3|, |0|\} = 4.$$

The shapes of the "unit circle" $\{\underset{\sim}{x} : \underset{\sim}{x} \in \mathbf{R}^n, \|\underset{\sim}{x}\| = 1\}$ show interesting variations when the three norms are used. See Figure 3 overleaf.

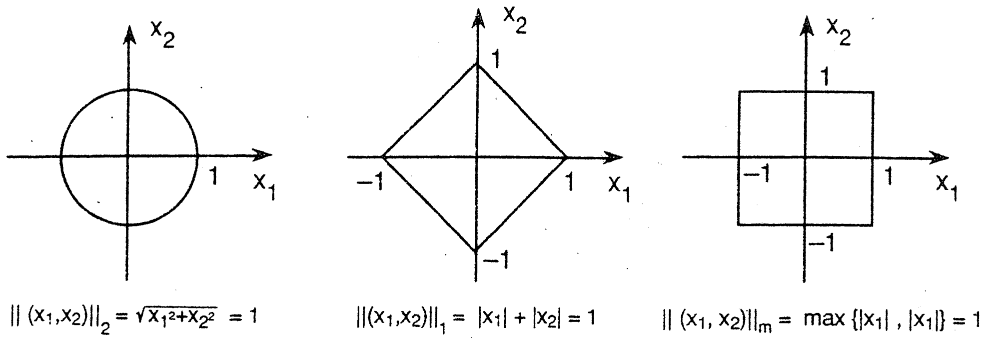


Figure 3. The unit circle in $\mathbf{R}^2$ for different norms

The 1-norm and max-norm also provide an indication of the magnitude or size of a vector, and are much easier to use than the usual Euclidean norm. Fortunately, they are all related to each other in the sense that

$$\|\underset{\sim}{x}\|_m \le \|\underset{\sim}{x}\|_1 \le n\|\underset{\sim}{x}\|_m,$$

$$\|\underset{\sim}{x}\|_m \le \|\underset{\sim}{x}\|_2 \le \sqrt{n}\|\underset{\sim}{x}\|_m,$$

$$\frac{1}{\sqrt{n}}\|\underset{\sim}{x}\|_2 \le \|\underset{\sim}{x}\|_1 \le n\|\underset{\sim}{x}\|_2,$$

for all $\underset{\sim}{x} \in \mathbf{R}^n$. Consequently, any vector in $\mathbf{R}^n$ is always large or always small in all three norms, up to a scaling factor which does not depend on the particular vector. Mathematically, the norms are said to be "equivalent". The equivalence of all norms is a characteristic of any finite dimensional space like $\mathbf{R}^2$.

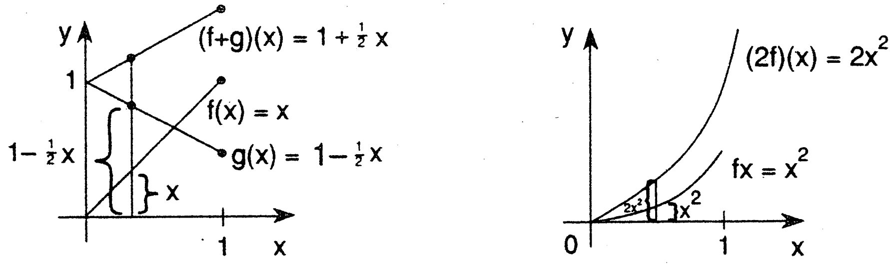
Let us now return to the original problem of measuring how big a function is. Without loss of generality we can restrict our attention to functions $f : [0, 1] \to \mathbf{R}$, in particular to *continuous* functions, i.e. those functions which, roughly speaking, are those with graphs that can be drawn without lifting the pencil from the paper. The space $C([0, 1], \mathbf{R})$ of all continuous functions $f : [0, 1] \to \mathbf{R}$ is an *infinite*-dimensional space. We can think of the value $f(t)$ of a function f at t, $t \in [0, 1]$ as its tth-coordinate, in much the same way that $f(t_i)$ is the ith-component of an n-dimensional vector $(f(t_1), f(t_2), \ldots, f(t_n)) \in \mathbf{R}^n$ formed by evaluating f at only a finite number of points $0 \le t_1 < t_2 < \ldots \le t_n \le 1$. This is a useful analogy, as we can use each of the norms that we considered on $\mathbf{R}^2$ to suggest possibilities for $C([0,1], \mathbf{R})$.

Before we proceed, we need to say what we mean by addition, subtraction and scalar multiplication of functions. These are natural generalisations of their namesakes for vectors in $\mathbf{R}^n$, i.e. being performed "componentwise" or *pointwise* as we usually say, i.e.

$$h = f \pm g : \qquad h(t) = f(t) \pm g(t) \qquad \text{for all } t \in [0, 1]$$

$$h = \alpha f : \qquad h(t) = \alpha f(t) \qquad \text{for all } t \in [0, 1]$$

where $\alpha \in \mathbf{R}$ is called a scalar. See Figure 4.



adding functions

scalar multiplying functions

Figure 4

The max-norm is the easiest of the three norms on $\mathbf{R}^n$ to translate over to the setting of the function space $C([0, 1], \mathbf{R})$. We shall use the same notation and name for it, which we define as

$$\|f\|_m = \max_{0 \le t \le 1} |f(t)|,$$

i.e. the maximum of the absolute values $|f(t)|$ as t varies over the interval $0 \le t \le 1$. This definition makes sense because a continuous function on $[0, 1]$ has a maximum value which it attains at t^*, $t^* \in [0, 1]$. (Compare this with the function f defined by $f(t) = t$ for $0 \le t < 1$ with $f(1) = 0$.) It can be shown that this max-norm satisfies properties (1*)–(4*) of the norms on $\mathbf{R}^n$ (but with $f \in C([0, 1], \mathbf{R})$ rather than $\underset{\sim}{x} \in \mathbf{R}^n$), so has the intuitively necessary properties of a magnitude.

The corresponding distance between two functions $f, g \in C([0, 1], \mathbf{R})$ is then

$$\|f - g\|_m = \max_{0 \le t \le 1} |f(t) - g(t)|,$$

i.e. the largest of the magnitudes of their pointwise differences over the interval $0 \le t \le 1$. For example, if $f(t) = t$ and $g(t) = t^2$ for $0 \le t \le 1$, then

$$\|f - g\|_m = \max_{0 \le t \le 1} |f(t) - g(t)| = |f(1/2) - g(1/2)| = 1/4$$

since $|f(t) - g(t)| = t - t^2 \ge 0$ attains its maximum value of $1/4$ on $[0, 1]$ at $t^* = 1/2$. See Figure 5.

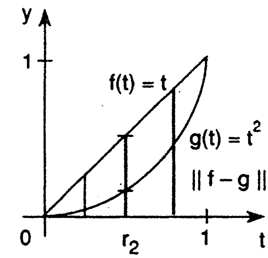


Figure 5

What are the counterparts of the 1-norm and 2-norm of $\mathbf{R}^n$ on the function space $C([0, 1], \mathbf{R})$? Consider a function $f \in C([0, 1], \mathbf{R})$ and a finite number of points $0 \le t_1 < t_2 < \dots < t_n < t_{n+1} = 1$. Then $\underset{\sim}{x} = (f(t_1), f(t_2), \dots, f(t_n)) \in \mathbf{R}^n$ has 1-norm

$$\|(f(t_1), f(t_2), \ldots, f(t_n))\|_1 = |\mathrm{f}(t_1)| + |f(t_2)| + \ldots + |f(t_n)|.$$

A slight variation is to multiply each term by $\Delta t_i = t_{i+1} - t_i > 0$ for $i = 1, 2, \ldots, n$, the difference between successive t values:

$$(|f(t_1)| + |f(t_2)| + \ldots + |f(t_n)|)\Delta t_i.$$

Readers who are familiar with calculus will recognise that as $n \to \infty$ with $|\Delta t_i| \to 0$, this sum converges to the integral of $|f(t)|$ from $t = 0$ to $t = 1$, that is,

$$\lim_{n\to\infty}(|f(t_1)| + |f(t_2)| + \ldots + |f(t_n)|)\Delta t_i = \int_0^1 |f(t)|\,dt,$$

which is the area under the graph of $y = |f(t)|$ for $0 \le t \le 1$. It can be shown that

$$\|f\|_1 = \int_0^1 |f(t)|\,dt,$$

which we shall call the 1-norm on $C([0, 1], \mathbf{R})$, also satisfies properties (1*)–(4*) and is thus an intuitively acceptable measure of magnitude. So too is the corresponding 2-norm on $C([0, 1], \mathbf{R})$, which is defined by

$$\|f\|_2 = \sqrt{(\int_0^1 |f(t)|^2 dt)}.$$

The 1-norm and 2-norm on $C([0, 1], \mathbf{R})$ provide us with additional measures of distance between functions in $C([0, 1], \mathbf{R})$. For example, with $f(t) = t$ and $g(t) = t^2$ on $0 \le t \le 1$ we have

$$\|f - g\|_1 = \int_0^1 |t - t^2|\,dt = \int_0^1 (t - t^2)dt \qquad (\text{as } t - t^2 \ge 0 \text{ for } 0 \le t \le 1)$$

$$= \left[\tfrac{1}{2}t^2 - \tfrac{1}{3}t^3\right]_{t=0}^{1} = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6},$$

whereas

$$\|f - g\|_2 = \sqrt{(\int_0^1 |f(t)|^2 dt)} = \sqrt{(\int_0^1 (t^2 - 2t^2 + t^4)dt)}$$

$$= \sqrt{([\tfrac{1}{3}t^3 - \tfrac{2}{4}t^4 + \tfrac{1}{5}t^5]_0^1)} = \sqrt{(\tfrac{1}{3} - \tfrac{2}{4} + \tfrac{1}{5})} = \sqrt{1/30}.$$

Not unexpectedly, we usually obtain different numerical values for the different norms. They are related in the sense that

$$\|f\|_1 \le \|f\|_m, \quad \|f\|_2 \le \|f\|_m, \quad \|f\|_1 \le \|f\|_2$$

for all $f \in C([0, 1], \mathbf{R})$. However, unlike their counterparts in $\mathbf{R}^n$, these function space norms do not satisfy general inequalities in the other direction, so what is small in one norm need not be small in another. For example, for $0 < \varepsilon < 1$ consider the function $f_\varepsilon \in C([0, 1], \mathbf{R})$ defined by

$$f_\varepsilon(t) = \begin{cases} \dfrac{1}{\varepsilon}\left[1 - \dfrac{t}{\varepsilon^2}\right] & \text{for } 0 \le t \le \varepsilon^2 \\ 0 & \text{for } \varepsilon^2 \le t \le 1. \end{cases}$$

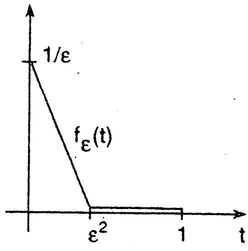
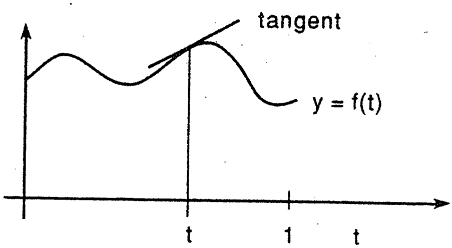


Figure 6

Then $\|f_\varepsilon\|_1 = \frac{1}{2}\varepsilon$ (the area of the triangle in Figure 6), whereas $\|f_\varepsilon\|_m = 1/\varepsilon$. As $\varepsilon \to 0$, one of these becomes very small, whereas the other becomes infinitely large!

For an infinite-dimensional space such as $C([0, 1], \mathbf{R})$ different norms are usually not equivalent. Which one we use will depend very much on the nature of the problem that we are investigating. For example, the max-norm is appropriate if we require two paths to be close in geometric space for every time instant, whereas an integral norm like the 1- and 2-norms are useful if we want to minimise an accumulative effect such as energy or fuel consumption.

The problem of guiding a rocket over a pre-assigned flight-path raises another interesting issue. Here we usually require the speed as well as the position of the rocket to be close to the planned values. Now, speed is just the derivative of the position as a function of time, that is, the slope of the tangent to the graph of the position as a function of time. See Figure 7.



$$f' = f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Figure 7

A continuous function need not be differentiable, that is, have a derivative, for each t. However, if a function is differentiable then it must be continuous. Let C ([0, 1], **R**) be the space of functions $f : [0, 1] \to \mathbf{R}$ which are differentiable everywhere on [0, 1], with both f and its derivative f' being continuous functions on [0, 1]. Then

$$\|f\|_d = \|f\|_m + \|f'\|_m = \max_{0 \le t \le 1} |f(t)| + \max_{0 \le t \le 1} |f'(t)|$$

is a norm (i.e. it satisfies properties (1*)–(4*)) on the function space $C_d([0, 1], \mathbf{R})$ which takes into account both the magnitude of the function and its time-rate of change. The corresponding distance $\|f - g\|_d$ is small if both the values of the functions and of their derivatives are close. To emphasise this point, let $f(t) = \frac{1}{10} \sin(1000t)$ and $g(t) \equiv 0$ for $0 \le t \le 1$. Then

$$\|f - g\|_m = \max_{0 \le t \le 1} \left| \frac{1}{10} \sin(1000t) \right| = \frac{1}{10}.$$

The corresponding derivatives are $f'(t) = \frac{1000}{10} \cos(1000t)$ and $g'(t) \equiv 0$, so

$$\|f' - g'\|_m = \max_{0 \le t \le 1} \left| \frac{1000}{10} \cos(1000t) \right| = 100.$$

The functions f and g are close, but their rates of change differ dramatically. See Figure 8.

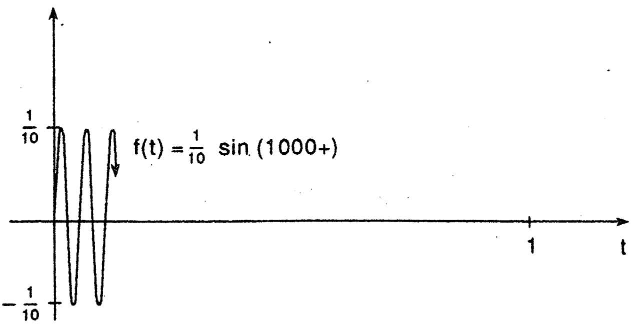


Figure 8. Schematic only – very rapid oscillations!

The study of function spaces such as $C([0, 1], \mathbf{R})$ and $C_d([0, 1], \mathbf{R})$ and their properties norms included, is a major part of modern mathematics, which is known as *Functional Analysis*. The infinite-dimensionality of typical function spaces certainly gives rise to many complications, which are much of the fascination of the subject.

* * * * *

LETTERS TO THE EDITOR

The Bend in the Road

Kent Hoi's article "One Good Turn" (*Function, Vol.* 16, *Part* 1, p.14) prompts a number of comments.

In the first place, the interpolated remark (b), referring to the point where the road starts to curve, is incorrect. This is *not* a point of inflection.

At a point of inflection, the curvature, *if it exists*, is zero; but a point where the curvature is zero is not necessarily a point of inflection. For example, the curve $y = x^4$ has zero curvature at the origin (0,0), but there is no point of inflection there. The author's "that is" is wrong.

An example of a point of inflection where the curvature does not exist is the point (0,0) on the graph

$$y = \begin{cases} -x^2 & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$$

(see Figure 1).

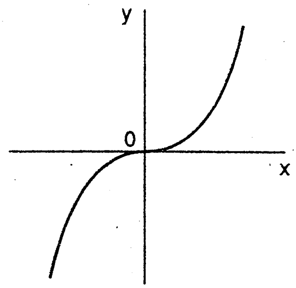


Figure 1

For negative x near zero the curvature is near -2; for positive x near zero the curvature is near $+2$. Hence the curvature does not exist at $x = 0$. Yet, in terms of the dictionary definition, there is a point of inflection at $x = 0$, because the curvature changes from negative to positive, or from convex to concave, as we pass that point. Put another way, the curve crosses its tangent at $x = 0$.[†]

Exercise: Show, mathematically, that there is, indeed, a tangent to this graph at $x = 0$.

The second point to make is to extend the editorial remark at the end of the article. It neither went far enough nor did it attend to the principal issue, namely, this is not the way surveyors go about their work. Surveyors join straight sections of road to circular arcs by means of "transition curves" which give smooth changes in the curvature as defined in the editorial note. (See Figure 2 overleaf.)

[†] For a careful analysis of the concept "point of inflection", see *The Mathematical Gazette*, December 1991.

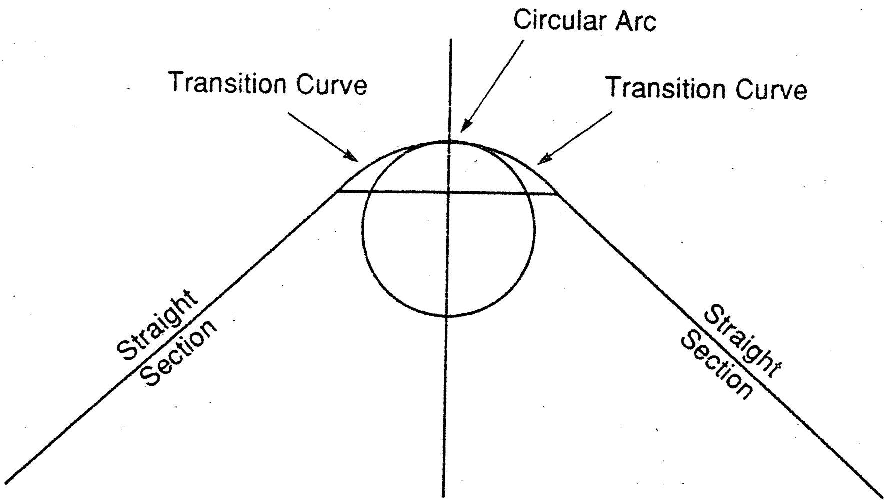


Figure 2

[Clearly we could not use a curve such as shown in Figure 1 as a railway or roadway transition curve, because of the discontinuous acceleration at $x = 0$, giving rise to a large impulsive force on a vehicle passing this point. But the cubic curve, which looks similar, could be used. Half the cubic curve, say $y = x^3$ for $x \geq 0$, is called, by surveyors, the cubic semi-parabola, and is one of several curves which they commonly use as transition curves. They do not use high-degree polynomials or cosine curves.]

The desirable properties of transition curves may be understood by reference to Figure 3.

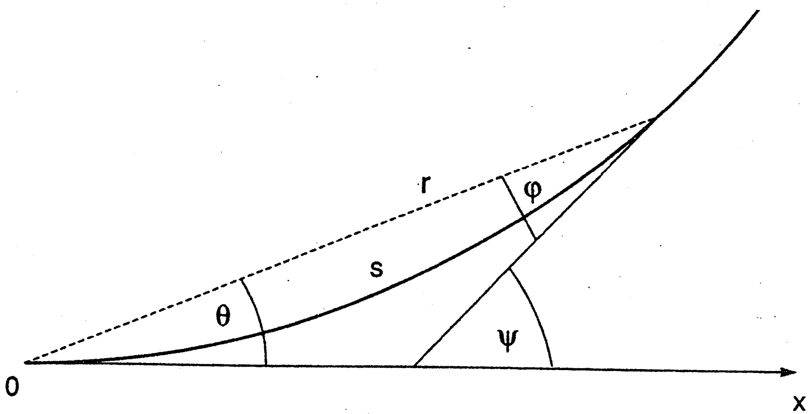


Figure 3

In the notation of Figure 3, we may write formulae for some transition curves used by surveyors.

(i) $\theta = \frac{1}{2}\phi$, which is the lemniscate[†], and is ideal for pegging by polar coordinates. Its polar equation is

$$r^2 = a^2 \sin 2\theta.$$

(ii) $\tan\theta = \frac{1}{3}\tan\psi$, which is the cubic semi-parabola referred to above. Its cartesian equation

$$a^2 y = x^3$$

is well adapted to pegging by cartesian coordinates. The curvature increases with x to a maximum at $x = 0.4a$ and then decreases as x increases. This curve was proposed in 1960 by the late Jim Thornton-Smith[††] who wrote:

> "The cubic semi-parabola is not a true spiral, its curvature attaining a maximum and then becoming very flat again as the slope angle increases; however, in railway design in cases where the transition length is small in comparison with the radius of the circular curve which it joins, it has advantages of simplicity and symmetry which make it especially convenient for pegging it by cartesian coordinates".

(iii) $s^2 = a^2\psi$, which is the clothoid[†††]. This has no simple equation in polar or cartesian coordinates. Its virtue is that the curvature is proportional to arc-length along the curve.

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* * * * *

Two Good Turns

The problem of designing a bend in a road or railway line was considered by Kent Hoi in *Function, Vol.* 16, *Part* 1. However, the curves actually used in such designs are not to be found by such means. As the editorial note appended to the article indicated, we need first to use a correct definition of curvature, but secondly the curves required need not (and often do not) have simple cartesian or polar equations.

[†] The lemniscate was used as the cover of *Function, Vol.* 1, *Part* 4.

[††] In the technical journal *Empire Survey Review.*

[†††] Also known as Cornu's spiral. See the cover story for *Function, Vol.* 4, *Part* 4.

Let κ be the curvature. This is $1/\rho$ where ρ is the radius of a best-approximating circle at the relevant point on the curve. Let s measure arc-length along the curve. The simplest and most natural functional relationship is that curvature be proportional to arc-length:

$$\kappa = \frac{1}{\rho} \propto s. \tag{1}$$

This equation uses two *intrinsic* variables, which are not referred to an external framework, as are the *extrinsic* cartesian (x, y) or polar (r, θ).

The curve defined by Equation (1) is known by various names: klothoid, clothoid, Cornu's spiral, Euler's spiral and the Railway Transition Curve. It has also been found useful in the science of Optics.

If we relate intrinsic variables κ (or ρ), s and the tangent angle† ψ, we may write Equation (1) in its standard form

$$\kappa = 2Ks$$

which is also known as Cesaro's Intrinsic Equation. Note that

$$\kappa = \frac{d\psi}{ds}$$

and this measures the rate of turn experienced by a driver travelling along the curve.

We thus have

$$\frac{d\psi}{ds} = 2Ks$$

and so deduce

$$\psi = Ks^2 + L$$

where L is a constant of integration.

But now if we choose $\psi = 0$ when $s = 0$, it follows that $L = 0$ and so

$$\psi = Ks^2. \tag{2}$$

This is called the Whewell Intrinsic Equation.

From (2),

$$s = \sqrt{\psi/K}$$

and we can deduce that

$$\rho = \frac{1}{K} = \frac{ds}{d\psi} = \frac{1}{2\sqrt{K\psi}}. \tag{3}$$

This is referred to as the Euler Intrinsic Equation.

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† See Figure 3 of the previous letter [*Ed.*].

If we wish to refer this to *extrinsic* cartesian coordinates, we need to solve the equations

$$\frac{dx}{ds}\ (= \cos\ \psi) = \cos(Ks^2)$$

$$\frac{dy}{ds}\ (= \sin\ \psi) = \sin(Ks^2).$$

The solution of these equations is beyond the scope of *Function*, but it can be done on a computer. It leads to the study of Fresnel integrals and the theory of Cornu spirals.

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* * * * *

SQUARE ROOTS USING MATRICES

If we multiply the column vector $\begin{bmatrix}1\\1\end{bmatrix}$ by the matrix $\begin{bmatrix}1&2\\1&1\end{bmatrix}$ the result is $\begin{bmatrix}3\\2\end{bmatrix}$. Multiplying this again by $\begin{bmatrix}1&2\\1&1\end{bmatrix}$ gives $\begin{bmatrix}7\\5\end{bmatrix}$, and continuing in this way we may generate from the matrix the sequence

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}7\\5\end{bmatrix}, \begin{bmatrix}17\\12\end{bmatrix}, \begin{bmatrix}41\\29\end{bmatrix}, \begin{bmatrix}99\\70\end{bmatrix}, \begin{bmatrix}239\\169\end{bmatrix}, \dots .$$

The ratios of the two entries in these vectors are respectively

$$1.0,\ 1.5,\ 1.4,\ 1.417,\ 1.414,\ 1.4143,\ 1.4142,\ \dots$$

values which oscillate about but also get closer and closer to $\sqrt{2}$.

More generally

$$\begin{bmatrix}1&p\\1&1\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix} = \begin{bmatrix}x+p\\x+1\end{bmatrix}$$

and it may be proved that if x is an approximation to $\sqrt{p}$, then $(x+p)/(x+1)$ is a better one. If $x < \sqrt{p}$, then $(x+p)/(x+1) > \sqrt{p}$ and *vice versa*. Equilibrium occurs when

$$x = (x+p)/(x+1),$$

and this quadratic equation reduces to $x^2 = p$, i.e. $x = \pm\sqrt{p}$.

When $p = 3$, the sequence (beginning with $x = 1$) goes:

$$1,\ 2,\ 1.67,\ 1.75,\ 1.727,\ 1.733,\ \dots$$

converging to $\sqrt{3}$ (= 1.732 ...). If we put $p = 4$, the sequence is

$$1,\ 2.5,\ 1.86,\ 2.05,\ 1.984,\ 2.005,\ 1.998,\ \ldots$$

and the limit is 2.

I was reminded of this material by reading J.B. Miller's "Square Roots of Matrices" in *Function, Vol.* 16, *Part* 1.

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[The proof that this algorithm works is rather tedious. Mr. Greenbury supplied a rather nice graphical illustration coincidentally rather like the cover picture for this issue, but it needs concepts outside the scope of Function. *Ed.]*

* * * * *

HISTORY OF MATHEMATICS

EDITOR: M.A.B. DEAKIN

Function, Vol. 16, *Part* 1 told the story of Hypatia, usually seen as the world's first woman mathematician. A footnote to that story, however, foreshadowed the claims of an earlier woman. Her name was *Pandrosion* and Winifred Frost of the University of Newcastle has very kindly sent us her story. What I particularly like about it is its flavour of the techniques, difficulties, minutiae and rewards of research in the history of mathematics.

PAPPUS AND THE PANDROSION PUZZLEMENT

Winifred Frost, University of Newcastle

Pappos of Alexandria (Pappus in the Latinised version) was a Greek mathematician of the 4th century A.D., whose work comes some six centuries after Euclid, when the great geometrical tradition was kept alive by teachers like him who wrote commentaries on the works of their famous predecessors. He was mentioned by the later writers Marinus and Proclus at Athens in the 5th century, and Eutocius at Alexandria in the 6th, as a commentator on Euclid and Ptolemy. There are also two later references to him, one placing him at the beginning of the 4th century, the other at the end of it. He is known to us as the author of "The Collection", which consists of eight books of which the first, half the second, and perhaps the end of the eighth are missing. Pandrosion is the person to whom he addressed the third book, and thereby hangs a tale.

Nothing would be known of the "Collection" if it were not for the survival of a single manuscript, now known as *Vaticanus Graecus 218*, which had reached the Vatican Library by the 16th century, and was probably copied in Byzantium (today's Constantinople) in the 9th or 10th century. How many removes it is from Pappus's original manuscript cannot be known unless another earlier manuscript is found, but it may be only one or two, including that of the copyist who prepared the work for publication. It is disorganised, contains repetitions and mistakes, and could not have been prepared for publication by Pappus himself. Eleven other manuscripts of the remnants of it, mostly 16th century, exist, as well as several with only some or one of the books, but they have all been proved to be copied from the *Vaticanus* itself or its descendants.

Until the first printed edition of Federicus Commandinus was published posthumously in Pesaro, Italy, in 1588, very few had access to Pappus's work, but Commandinus extended its readership with a Latin translation, omitting the part of Book II. No other edition was printed until 1875, when Fridericus Hultsch's edition was published in Berlin, with Greek on the left page and Latin on the right. This is still the standard edition. The only edition in a modern language is the French translation of Hultsch's text by Paul Ver Eecke, published in 1933. These are the only complete texts available to a modern scholar, sixteen centuries after Pappus wrote.

Hultsch's edition was based largely on a secondary manuscript, *Paris 2440*, though he was aware that the *Vaticanus* was superior, and had been able to consult Books II to VI. He also consulted two other manuscripts, but the advantages of photography and modern means of travel were not available to him. So it is only quite recently that further progress has been made to make the "Collection" more accessible to those who have no Latin or Greek. (The outline of its contents may be found in "A Manual of Greek Mathematics" by Sir Thomas Heath, the famous 19th century translator of Euclid's "Elements", and of Archimedes, Apollonius and Diophantus.)

Australia has its own Pappus scholar, Professor A. Treweek, former Professor of Greek at Sydney University. He obtained photocopies of the *Vaticanus* in 1938 and, comparing Hultsch's text with it, "realised that all was not well with it". After the war he copied out the whole *Vaticanus* (a truly modern scribe!) which is difficult to read, as it is in a minuscule hand and some letters have quite a different form from today's script. He spent study leaves in 1949-50 and 1955-56 collating all the other extant manuscripts, either by personal inspection or from photostats, noting all variations of the texts on the left-hand pages of his manuscript. He greatly contributed to the advance of the study of Pappus by restoring many illegible passages where water damage had occurred. He did this by deciphering the mirror image offset visible on the opposite page. In the course of his work, he proved from relationships between the manuscripts that they all stem from the *Vaticanus*, and that its date was 10th century, or possibly 9th. By reading what was visible after rebinding of numbers marking the beginning of a quaternion or quire (of four sheets folded once), he deduced that the missing quires must number either two or six. His doctoral thesis contained his transcription of Books II to VI and his report on the manuscripts and their relationships. The latter was published in the scholarly journal *Scriptorium* in 1957.

Further research continues to be done. In 1986, A. Jones' translation of Book VII was published. This is the most historically interesting book, listing the books that made up the "domain of analysis", works by Euclid, Apollonius and Aristaeus (many of which we would otherwise know nothing about) and showing the method of proof of propositions by analysis of what must precede if the proposition is assumed to be true, until something given or known is reached, and then by synthesis, that is by the reverse process, or building up of the proof.

More recently a very scholarly work, "Textual Studies in Ancient and Medieval Geometry", by W.R. Knorr examines the texts, their contents, history and relationships, of eight commentators on the works of the ancient mathematicians, from Heron of Alexandria in the first century, to Eutocius and John Philoponus in the sixth, and including Pappus, Theon and Hypatia.

My interest in Pappus began when I was looking for a long-term project that would combine my interest in mathematics and classics, and Bob Berghout of Newcastle University told me that there was no English translation of the "Collection". In the way of fools who rush in, I began at once with Book II, which shows how to multiply any quantity of single letter numbers together by taking out all factors of ten, and multiplying only the base numbers together. Factors of ten are again taken out, their total number is divided by four, and the number of myriads (10000s) is obtained. He quotes Apollonius as his source.

Book III is more interesting. There are four unrelated sections containing 45 propositions and 15 lemmas. The first section is concerned with the ancient problem of the duplication of the cube (equivalent to finding two mean proportionals between two given quantities); the second shows how, in certain cases, a triangle may be constructed on the base of, and inside another triangle, yet have its other two sides together greater than, equal to, or in given ratio to the remaining sides of the outer triangle, and extends to polygons with four or more sides. The third section shows how ten different means between two given quantities may be obtained, and how, given an extreme quantity and the mean, to find the other extreme quantity; the fourth shows how to inscribe in a given sphere the five regular polyhedra – pyramid, cube, octahedron, dodecahedron and icosahedron ("Sale of the Century" symbol). His method differs from Euclid's in Book 13 of the Elements, and is probably based on Theodosius's "Spherics".

At the very beginning of Book III one gets a taste of Pappus's style. He addresses it to Pandrosion ("dedicates" is not the right word), only to show his low opinion of her students. "Certain people", he says, "who claim to have learned mathematics from you, set out the enunciation of the problems in what seemed to us an ignorant manner". In his account of the duplication of the cube, he gives no credit to the unknown student ("an important person, reputed to be a geometer"), not recognising that this person's method of approximating the two means† is at least as good as the other ancient methods, which of course were sanctified by age. In fact it is an iterative procedure, obtaining successive improving approximations, and it may be proved by modern methods that the approximations converge to the exact solution. Again in the propositions on finding the means, he scorns the effort of "a certain other person" to exhibit the arithmetic, geometric and harmonic means on a semi-circle, which he does using only four lines, and says: "but how BZ is a mean of the harmonic mediety, or of which straight lines, he does not say", even though it is not difficult to prove that it is, and of the same two lines as for the other two means. So we begin to see a conceited irascible old gent, fallible himself, but intolerant of the faults of others. (Perhaps his name, Pappos – grandfather, is indicative of something other than reverence?) There are several other examples in Books VI and VII of his critical attitude to other scholars.

One wonders whether his criticism would have been quite so terse if Pandrosion had not been a woman. Yet she certainly was; all the manuscripts have the feminine vocative†† form of the adjective translated as "most excellent", which Hultsch has changed from the long *e* (eta) to the short *e* (epsilon) of the masculine vocative case. Emendations should

† The geometric method gives ways of approximating $2^{1/3}$, $2^{2/3}$ (the "two means") by ruler and compass construction. It is now known that no such construction can give them *exactly*.

†† The Greek words inflect according to their function in the sentence and also with gender. The "vocative" case is used when a person is being directly addressed. The vocative feminine is used when that person is female. The feminine form has η, whereas the masculine has ε, the other form of the Greek letter *e*.

never be undertaken without careful thought, and there are clues in the name itself to warn against it. The *-ion* ending is a diminutive or pet name ending, as we say Jimmy for James, and the original, Pandrosos, was the name of one of three sister-goddesses, daughters of Cecrops and Aglauros in Greek myth. It means "all dewy", not a likely male epithet. Following Hultsch, Ver Eecke calls her Pandrosio, using the modern male termination. Three other books have dedications: Books VII and VIII to "my son Hermodorus" (no doubt of his sex, though he may not actually have been Pappus's son), and Book V to Megithion, otherwise unknown, but all the manuscripts have the masculine adjective.

So we have in Pandrosion a female teacher of mathematics at Alexandria, probably a younger contemporary of Pappus, perhaps even at the same institution, the Museum (or University). Pappus's date was determined in the 1930s by Professor A. Rome from the fact of his observation of an eclipse of the sun in 320 A.D., mentioned in his commentary on Ptolemy's "Almagest", part of which survives. Since he therefore pre-dates Theon,† father of Hypatia, there is at least one, and perhaps two generations between Pandrosion and Hypatia. This means that it is Pandrosion and not Hypatia for whom we may make the claim "first known woman mathematician". And perhaps more may have been heard of her if she, like Hypatia, had been the daughter of a famous mathematician.

Since completing the translation of Book III, Bob Berghout and I have visited Professor Treweek at his home. He is now eighty, and was so pleased that two more Australians were taking up his "*vitai lampada*" that he has given us his precious manuscript copy of the *Vaticanus*, his volumes of Hultsch's and Ver Eecke's translations, the *Scriptorium* article and other valuable books, for which no thanks would be adequate. Our aim is to continue with the work of a complete English text and commentary, of which the mathematical and historical aspect is Bob's field, while I translate and revise with his invaluable help.

* * * * *

PROBLEMS AND SOLUTIONS

EDITOR: H. LAUSCH

SOLUTIONS

1. The Steiner-Lehmus Theorem

Here is one more solution to the

Steiner-Lehmus Theorem (Problem 15.1.5). If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

Solution (provided by Garnet J. Greenbury, who proposed the problem). Let AD and BE be the equal angle bisectors. Make $\angle BEF = \angle BAD$ and $\angle FBE = \angle ADB$. Extend FB to H and draw the perpendiculars FG and AH. Join AF.

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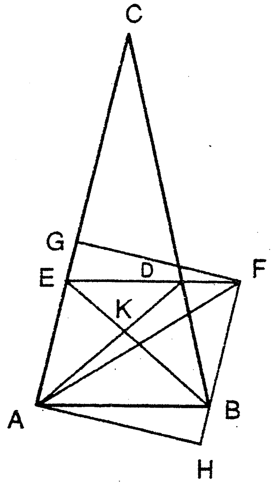
† Theon observed two later eclipses in the year 364.



Since $\angle BAD$ and $\angle ADB$ are two angles of the same triangle, they are together less than two right angles, and hence EF and BF will meet. Therefore ΔABD, ΔEFB are congruent, and $BD = BF$, $AB = EF$.

$$\angle AKB = \angle KAE + \angle AEK = \angle BEF + \angle AEK = \angle AEF.$$

But $\angle AKB = \angle KDB + \angle DBK = \angle FBK + \angle KBA = \angle FBA$. Hence

$$\angle AEF = \angle FBA,$$
$$\angle FEG = \angle ABH,$$
ΔFGE and ΔAHB are congruent,
$GE = HB$ and $GF = HA$,
ΔFGA and ΔAHF are congruent,
$$GA = HF,$$
$$AE = FB = BD,$$
ΔABE and ΔBAD are congruent,
$$\angle EBA = \angle BAD,$$
$$\angle CBA = \angle BAC,$$
$CA = CB$, q.e.d.

Function *also thanks the following readers for their contributions in connection with this problem: Seung-Jin Bang (Seoul, Republic of Korea), John Barton (North Carlton), J.A. Deakin (Shepparton) for two more solutions (including one by M.E. Richards), and Garnet J. Greenbury for more material.*

2. Solutions to other problems

It's Beetham's Problem

In Function, *Volume 13, Part 3, June 1989, p. 96, a problem was published that had been on a paper sat by candidates for membership in the team to represent Australia at the 1989 International Mathematical Olympiad (Braunschweig, Germany). A considerable number of* Function *readers sent in solutions, so that the problem became our "problem of the year 1990". Here it is:*

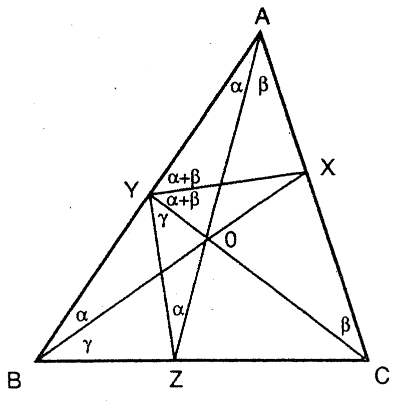
Problem. Let O be the circumference of the triangle ABC, and let X and Y be the points on AC and AB respectively such that BX intersects CY in O. Suppose $\angle BAC = \angle AYX = \angle XYC$; determine the size of this angle.

Function *thanks John Barton, North Carlton, for pointing out a comment in* Mathematical Gazette, *December 1991, p. 458, where we learn of the problem's origin. The problem was originally published in* Mathematical Gazette, *December 1969, p. 403 by Richard Beetham, who asked solvers to find the angle and prove the result by elementary geometry, and re-published in* Mathematical Gazette, *June 1991, as Problem 75.D.* Function *reader Francisco Bellot, Valladolid, Spain, then spotted the duplication, whereupon* Mathematical Gazette *informed its readers of the problem creator. Meanwhile Andy Liu, Edmonton, Alberta, Canada, sent* Function *one more solution to the problem, which was also used in a Canadian high school competition. The solution is due to Kevin Kwan, Ontario:*

Solution. Join AO and extend it to meet BC at Z. By Ceva's Theorem,

$$\frac{AY}{YB} \cdot \frac{BZ}{ZC} \cdot \frac{CX}{XA} = 1.$$

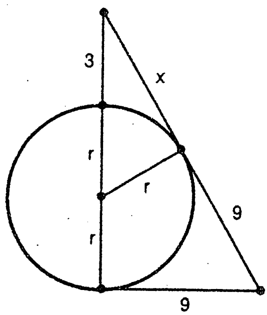
Since YX bisects AYC, $\frac{AY}{YC} = \frac{AX}{XC}$. Hence $\frac{CY}{YB} = \frac{BZ}{ZC}$, so that YZ bisects BYC. Let $\angle BAO = \alpha$, $\angle CAO = \angle ACO = \beta$ and $\angle CBO = \gamma$.



Then $\alpha + \beta + \gamma = 90^\circ$. Since $\angle AYX = \angle XYC = \alpha + \beta$, $\angle CYZ = \gamma$. It follows that $BYOZ$ is a cyclic quadrilateral, so that $\angle YZO = \angle ABO = \alpha$. The sum of the interior angles of triangle AYC is $3\alpha + 4\beta = 180^\circ$, while that of triangle AYZ is $4\alpha + 2\beta + \gamma = 180^\circ$. The second equation reduces to $3\alpha + \beta = 90^\circ$. Solving this system of equations, we have $\beta = 30^\circ$ and $\alpha = 20^\circ$, so that $\angle BAC = 50^\circ$.

Problem 15.4.5 (from ancient China; in *Presek* 17, part 1, 1989/90). A city has a circular wall. We do not know its circumference or its diameter. The city has four gates. Outside, a tall tree grows, 3 *li* north of the city. If we leave the city through its southern gate and then walk eastward, we have to walk 9 *li* before we can see the tree. Calculate the circumference and the diameter of the fortress. (1 *li* = 612 metres.)

Solution (Dieter Bennewitz, Koblenz, Germany).



$$(3 + 2r)^2 + 9^2 = (x + 9)^2 \tag{1}$$
$$(3 + r)^2 = r^2 + x^2. \tag{2}$$

Equation (2) implies

$$9 + 6r = x^2$$
$$3(3 + 2r) = x^2$$
$$3 + 2r = \frac{x^2}{3}.$$

Substituting this in (1) yields

$$\left(\frac{x^2}{3}\right)^2 + 9^2 = (x + 9)^2$$
$$x^4 - 9x^2 - 162x = 0$$
$$x(x^3 - 9x - 162) = 0$$
$$x(x - 6)(x^2 + 6x + 72) = 0.$$

The only positive solution is $x = 6$. It follows that

$$r = \frac{x^2 - 9}{6} = 4.5$$

$$d = 2r = 9.$$

Therefore the diameter is 9 *li* and the circumference is $2\pi \cdot 4.5$ *li* ≈ 28.3 *li*.

PROBLEMS

Problem 16.3.1 (from Switzerland). The number representing the year of birth of a famous Swiss mathematician has the following properties:

(i) the number of its digits is a perfect square;
(ii) the sum and the difference of its third and its fourth digit (counted from the left) as well as its rightmost digit ar perfect non-zero squares;
(iii) the two leftmost digits, read as a decimal integer, form a perfect square.

In which year was this mathematician born? Who was he?

Problem 16.3.2 (Juan Bosco Romero Marquez, Valladolid, Spain). Let ABC and $A'B'C'$ be two right-angled triangles with sides a, b, c and a', b', c' respectively. Suppose that $a > b \geq c$ and $a' > b' \geq c'$ and that $\angle ABC > \angle A'B'C'$. Let $A''B''C''$ be the triangle with sides a'', b'', c'' such that $a'' = aa'$, $b'' = bb' + cc'$ and $c'' = bc' - b'c$. Prove that $\Delta A''B''C''$ is right-angled, and evaluate its area, circumradius and inradius as well as $\angle A''B''C''$.

Olympiad News

The Asian Pacific Mathematics Olympiad of 1992

Since 1989 the Asian Pacific Mathematics Olympiad has taken place every year. This year twelve countries from the Pacific Rim entered students for the contest in March: Australia, Canada, Colombia, Hong Kong, Indonesia, Mexico, New Zealand, the Philippines, the Republic of China, the Republic of Korea, Singapore and Thailand.

Time allowed for the paper was four hours, no calculators were to be used, and each question was worth seven points. Here is the paper – please send in your solutions:

Question 1 (proposed by Canada):

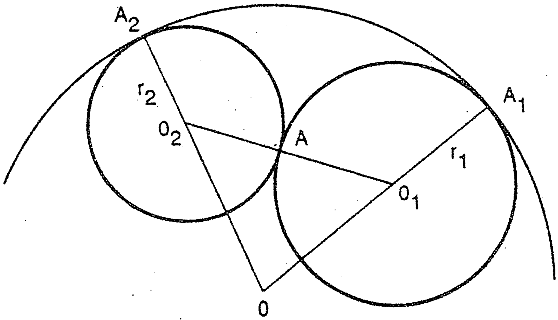
A triangle with sides a, b and c is given. Denote by s the semiperimeter, that is $s = (a+b+c)/2$. Construct a triangle with sides $s - a$, $s - b$ and $s - c$. This process is repeated until a triangle can no longer be constructed with the side lengths given.

For which original triangles can this process be repeated indefinitely?

Question 2 (proposed by Canada):

In a circle C with centre O and radius r, let C_1, C_2 be two circles with centres O_1, O_2 and radii r_1, r_2 respectively, so that each circle C_i is internally tangent to C at A_i and so that C_1, C_2 are externally tangent to each other at A.

Prove that the three lines OA, O_1A_2 and O_2A_1 are concurrent.



Question 3 (proposed by the Republic of Korea):

Let n be an integer such that $n > 3$. Suppose that we choose three numbers from the set $\{2, 3, ..., n\}$. Using each of these three numbers only once and using addition, multiplication and parenthesis, let us form all possible combinations.

(a) Show that if we choose all three numbers greater than $n/2$, then the values of these combinations are all distinct.

(b) Let p be a prime number such that $p \leq \sqrt{n}$. Show that the number of ways of choosing three numbers so that the smallest one is p and the values of the combinations are not all distinct is precisely the number of positive divisors of $p - 1$.

Question 4 (proposed by Mexico):

Determine all pairs (h, s) of positive integers with the following property: If one draws h horizontal lines and another s lines which satisfy

(i) they are not horizontal,
(ii) no two of them are parallel,
(iii) no three of the $h + s$ lines are concurrent,

then the number of regions formed by these $h + s$ lines is 1992.

Question 5 (proposed by New Zealand):

Find a sequence of maximal length consisting of non-zero integers in which the sum of any seven consecutive terms is positive and that of any eleven consecutive terms is negative.

The results of the Australian Division were:

Gold Certificate:	Anthony Henderson (year 11), Sydney Grammar School, NSW;
Silver Certificates:	Benjamin Burton (12), John Paul College, Queensland; Frank Calegari (11), Melbourne Church of England Grammar School, Victoria;
Bronze Certificates:	Adrian Banner (12), Sydney Grammar School, NSW; Lawrence Ip (12), Melbourne Church of England Grammar School, Victoria; Rupert McCallum (11), North Sydney Boys' High School, NSW; Brett Pearce (12), St. Michael's Grammar, Victoria;
Honourable Mentions:	Geoffrey Brent (12), Canberra Grammar School, ACT; William Hawkins (10), Canberra Grammar School, ACT; Michael Russell (12), Collegiate School of St. Peter, SA.

Congratulations to all!

The Australian team for the International Mathematical Olympiad (IMO) of 1992

This year's IMO, the thirty-third in history, is to be held at Moscow from July 10 to July 21. As Head of State of the host country, Russian President Boris Yeltsin is official Chairman of the IMO for 1992. Eighteen students, including all winners of the Asian Pacific Mathematics Olympiad, congregated at Sydney for twelve days to undergo a densely packed training programme. At this training school our IMO team for this year was selected after two more examinations. It is the youngest ever Australian IMO team, of average age 16 years and six months. These young people, whom Function *wishes every success, are:*

Adrian Banner,
Benjamin Burton,
Frank Calegari,
Anthony Henderson,
Lawrence Ip,
Rupert McCallum.
Geoffrey Brent (reserve).

An Olympian's progress

Terence Tao, born on July 17, 1975, celebrated his eleventh, twelfth and thirteenth birthdays while attending International Mathematical Olympiads as member of an Australian team. In 1986 he won a bronze, in 1987 a silver and in our bicentennial year, when the IMO was held in Canberra, a gold medal. In 1989 Terence would no longer have qualified for participating at an IMO as he had enrolled at a university.

Now being hardly 17, he is the youngest Australian to be given a Fulbright award, which is a very prestigious scholarship in the United States of America. He has graduated from the University of Adelaide with first class honours in Mathematics. Terence proposes to pursue PhD studies in pure mathematics from this year until 1996. He is also co-author of a book on problem solving in mathematics.