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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of secondary schools.

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Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Volume 14 *Part 3*

You would not think, to look at our front cover, that this apparently natural fern-frond was the product of a simple computer algorithm with a partially random input. Yet such is the case. The fern is one of the class of mathematical objects known as fractals, now receiving a lot of attention in the new fashionable study of Chaos. Warren Jennings explains how he drew his fractal fern and how you can make other such patterns for yourself.

We also describe aspects of Dr Andrew Prentice's theory of the origin of the solar system and pose some problems about ships at sea. And of course there are our usual features.

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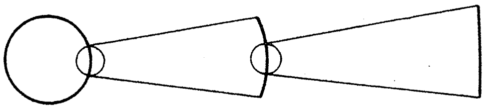
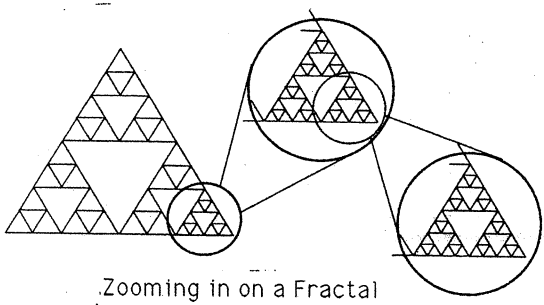
THE FRONT COVER : THE FRACTAL FERN

Warren Jennings, Monash University

With classical mathematics, man has traditionally tried to enforce the straight line upon the world. Look around: in every man-made object the straight line rules. Look at the traditional functions we use: polynomials, trigonometric functions, and exponentials. They all share the property that if you look closely enough at them, they appear to be straight lines.



Zooming in on a Smooth Curve



Zooming in on a Fractal

Fractals are mathematical objects which behave in a different way; when you zoom in on a fractal, you reveal more detail. No matter how far you zoom in, you will just reveal more and more detail. For this reason, fractals are thought to be very good for modelling nature. Consider tree bark. It is rough, and as you look more closely at it, it never becomes smooth, it just reveals more detail.

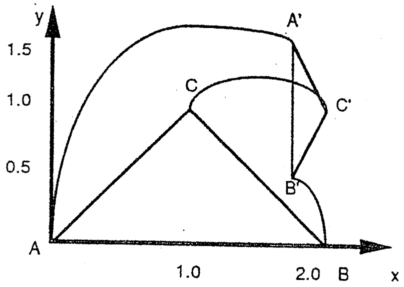
An easy way to generate fractals is using an Iterated Function System (IFS). This is a very compact way of describing complex fractals described by Michael Barnsley. At the heart of the interated function system is the *affine transformation* which is a combination of rotation, stretching and translation. We write an affine transformation as follows:

$$\begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x_{old} \\ y_{old} \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

$$x_{new} = a \cdot x_{old} + b \cdot y_{old} + e$$

$$y_{new} = c \cdot x_{old} + d \cdot y_{old} + f$$

Let's have a look at just what an affine transformation does.



The affine transformation which performs this maps (0,0) to (1·8,1·5) so $e = 1{\cdot}8$, $f = 1{\cdot}5$, (1,1) to (2,1) and (2,0) to (1·8,0·5), so $a = 0{\cdot}9$, $b = 1{\cdot}1$, $c = 0{\cdot}25$ and $d = 0{\cdot}75$.

Given any values of x and y, we can get a new position. To construct an iterated function system, we have a number of such affine transformations, each with an associated probability. From this IFS, we use the *Random Iteration Algorithm* to plot out the picture associated with IFS.. The random iteration algorithm goes as follows:

The Random Iteration Algorithm

1. read in number of affine transformations (NTrans).
2. ptotal = 0
3. For i = 1 to NTrans
   - 3.1 read in a(i),b(i),c(i),d(i),e(i),f(i) and pr
   - 3.2 ptotal = ptotal+pr
   - 3.3 p(i) = ptotal
4. Pick a starting point x_{old}, and y_{old}.
5. Choose at random, one of the affine transformations.
   - 5.1 pick a number from 0 to 1 (rand).
   - 5.2 choose the first affine transformation for which rand < p(i)
6. Calculate x_{new} and y_{new} as shown above using affine transformation number i.
7. Plot a point at x_{new}, y_{new}.
8. Let $x_{old} = x_{new}$ and $y_{old} = y_{new}$.
9. Goto step 4

Some interesting IFS's

Name	a	b	c	d	e	f	p
Fractal Fern	0	0	0	0.16	0	0	0.01
	0.2	-0.26	0.23	0.22	0	1.6	0.07
	-0.15	0.28	0.26	0.24	0	0.44	0.07
	0.85	0.04	-0.04	0.85	0	1.6	0.85
Sierpinski	0.5	0	0	0.5	0	0	0.33
Triangle	0.5	0	0	0.5	1	0	0.33
Gasket	0.5	0	0	0.5	0.5	0.5	0.34
Simple	0	0	0	0.5	0	0	0.05
Fractal	0.1	0	0	0.1	0	0.2	0.15
Tree	0.42	-0.42	0.42	0.42	0	0.2	0.4
	0.42	0.42	-0.42	0.42	0	0.2	0.4

From " A better Way to Compress Images"
Michael Barnsley & Alan SLoan
Byte Magazine - January 1988.

At first it seems quite strange that by choosing from among the affine transformations in the IFS at random, we can end up with a nice picture, regardless of which transformations we choose, when we choose them, and which starting point we select. To explain, take the example of the fractal fern. What happens is that if our current point is part of the fern, then any of the transformations will give us another point on the fern. If our point is not on the fern, then after several repeats these transformations will eventually move us onto the fern. For this reason the fern is called the *attractor* of the IFS – if we keep iterating, our point is attracted to the fern. The list above gives the affine transformations and probabilities for a few intersecting attractors.

It is interesting to note that we will get the same picture regardless of what probabilities we use. All that will change is how fast parts of the picture emerge. The probabilities given above have been found to give a reasonably even spread over the whole picture, but try to experiment with them.

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ALL AT SEA

A.W. Sudbury, Monash University

Here are two problems in which you have to find a moving object which you cannot see, but about which you have some information. The solutions will be published in *Function, Vol.* 14, *Part* 5. I hope that what gets published then is supplied by one of the readers of this article.

The Ship in the Fog

You are at sea and need to deliver much-needed medical supplies to a cruise-ship. You spot the ship and determine its exact position, both distance and angle. However, a dense fog then comes down; in fact a fog so dense that visibility is for all practical purposes zero. You know that the cruise-ship always travels at the same speed v in a straight line (if we ignore the curvature of the earth), but you know nothing about the direction of that line. Your own speed is $v(1+\varepsilon)$, where $\varepsilon > 0$. Describe a path which will ensure that you meet the cruise-ship sometime.

The Abandoned Cruise-ship

An abandoned cruise-ship whose wheel is locked in position is going round and round in a circle, of which you are at the centre. You know the ship has speed v. Again a fog comes down. This time you know the direction of the ship (from you) and which way it was travelling, but not its distance. Can you determine a path which will enable you to rendezvous with the ship if:

(a) you can choose your speed,
(b) your speed is predetermined?

[Note: If you can solve (b), you can solve (a), but not necessarily *vice versa*.]

Now suppose you don't even know which way round the ship is travelling. Can you still find a path that enables you to rendezvous with the ship? This time assume that you have a top speed, and can travel at any speed less than that.

Please send solutions, comments and analyses to the editor.

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Highly Likely

"At the moment the average daily intake is about 1·3 milligrams, but some people may be eating less and some more."

From the *Times* of 14 September 1989, reprinted in *The Mathematical Gazette,* March 1990.

THE TITIUS-BODE LAW OF PLANETARY DISTANCES

A.J.R. Prentice and M.A.B. Deakin,
Monash University

In 1764, the French biologist Charles Bonnet published his book *Contemplation de la nature* (Looking at Nature). The book was rapidly translated into German, Italian, Dutch and English. The German translation was undertaken by Johann Titius, Professor of Natural Philosophy at the University of Wittenberg, and appeared in 1766. Titius did not merely translate what Bonnet wrote; he interpolated passages of his own (but without letting the reader know that they were not to be found in the original French!).

One of these passages gave a formula whereby the distances of the various planets from the sun could be calculated quite readily. We will return to this formula in a minute. It was taken up by the much more famous scientist Johann Bode and inserted into the second (revised) edition of an astronomy text he published in 1772.

Bode did not acknowledge his indebtedness to Titius (at least not at first, although in subsequent writings he did). This fact, and Bode's considerably greater fame, led to Titius' formula being referred to as "Bode's Law", although it is now quite clear that Bode copied the law from Titius.

When Titius and Bode were writing there were six known planets: Mercury, Venus, Earth, Mars, Jupiter and Saturn (those visible to the naked eye). Their distances from the sun are given in Table 1, which lists those distances in *Astronomical Units*. (An astronomical unit is the mean distance from the earth to the sun.)

Table 1

Planet	Distance from Sun in A.U.
Mercury	0·387
Venus	0·723
Earth	1·000
Mars	1·524
Jupiter	5·203
Saturn	9·546

Titius took the number 4; added 3 to it to give 7; added twice 3 to 4 to give 10; added 4×3 to 4 to give 16; added 8×3 to 4 to give 28; added 16×3 to give 52; finally added 32×3 to 4 to give 100. He then divided all these answers by 10. This gives Table 2.

Table 2

Planet	Distance in A.U.	Predicted distance in A.U.
Mercury	0·387	0·4
Venus	0·723	0·7
Earth	1·000	1·0
Mars	1·524	1·6
?	–	2·8
Jupiter	5·203	5·2
Saturn	9·546	10·0

The numbers in the right-hand column are not exactly the same as those in the centre, but they approximate them very closely. Figure 1 shows how close that approximation is.

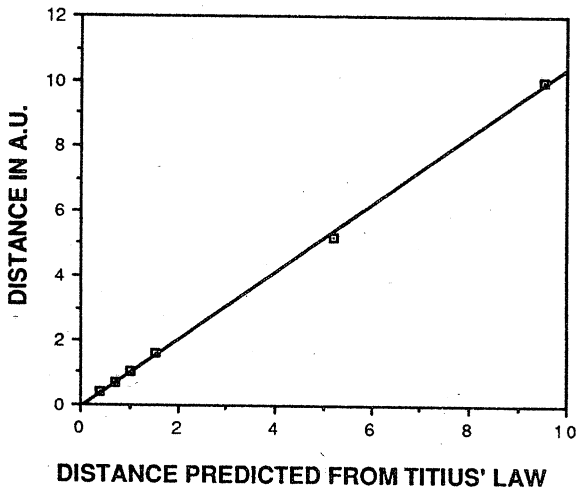

Figure 1

The correlation coefficient between the two columns (that is to say, a measure of how well they agree) is 0·9997. Another way of putting this is to say that $0{\cdot}9997^2$, or 99·94%, of the variance in the observed distances is explained by Titius' Law. The simple mathematical rule explains 99·94% of what we actually observe.

If we try to express the Titius law in algebraic terms we come up with

$$D = \frac{1}{10}\left[\ 3 \times 2^n + 4\ \right], \tag{1}$$

where D is the distance to the planet and n is given by: $-\infty$ for Mercury, 0 for Venus, 1 for Earth, 2 for Mars, 4 for Jupiter, and 5 for Saturn.

Now this leads to a number of questions. First of all, what has happened to $n = 3$? There is a gap in Table 2 at a distance of 2·8 astronomical units from the sun. Titius noticed this, but thought that it would be filled by the (then) undiscovered satellites of Mars. It was Bode who suggested that a new planet would be found there.

In 1801, such a planet was seen by the astronomer Giuseppe Piazzi. It was not immediately clear however whether what Piazzi had seen was really a new planet or merely a comet. Eventually an orbit was predicted by the very great mathematician Karl Friedrich Gauss, and using these calculations Heinrich Olbers was able to sight the planet and thus confirm its true planetary character. Its distance from the sun was 2·767 A.U., very close to the value 2·8 of Table 2.

The new planet was named Ceres and it is now known to be the largest of the asteroids or minor planets. Others were quickly found: Pallas (1802), Juno (1804) and Vesta (1807). These four, if we average their mean distances to the sun, give a figure of 2·641 A.U., in reasonable agreement with the Titius-Bode prediction of 2·8 A.U.

But even before these events, the planets Uranus was discovered. It was found by William Herschel in 1781, but initially misidentified as a comet. Gradually, however, as a result of attempts to calculate its orbit, involving among others Bode himself, it became clear that Uranus was a planet with a mean distance from the sun of 191·8254 A.U., reasonably close to the value 196 A.U. given by Equation (1) if $n = 6$.

In 1846, however, Neptune was discovered. In fact, it had been predicted to exist and was found exactly where it was supposed to be. However, this was luck rather than anything else, as the prediction had assumed that the distance from Neptune to the sun would be given by the Titius-Bode law. Now the true distance from Neptune to the sun is 30·07 A.U., whereas Equation (1) gives (with $n = 7$) 38·8 A.U. and the discrepancy is now rather too large for comfort.

There is also another problem with Equation (1). We have used the values $n = -\infty, 0, 1, \ldots, 7$, but why not $n = -1, -2, -3, \ldots$? These values would seem to predict infinitely many planets between Mercury and Venus. Such planets have not been found.

One approach has been to modify Equation (1) in such a way that this prediction fails. The most successful modification has been to replace Equation (1) by

$$\mathrm{D} = Ar^{n} \tag{2}$$

and seek values for A, r in such a way as to give the best predictions. This was the tactic adopted by Mary Blagg in 1913. She plotted $\log D$ against n, taking $n = -2$ for Mercury, -1 for Venus, 0 for Earth, etc. We do the same but take $n = 1$ for Mercury, 2 for Venus and so on.

The result is Figure 2. The best fit straight line is

$$\log D = -\ 0{\cdot}6929 + 0{\cdot}2388n, \tag{3}$$

where the log is taken to base ten.

This gives the formula

$$D = (0{\cdot}203) \times 1{\cdot}73^n, \qquad (4)$$

which predicts the distances given in Table 3.

Table 3

Planet	Distance in A.U.	Predicted distance in A.U.
Mercury	0·387	0·351
Venus	0·723	0·609
Earth	1·000	1·055
Mars	1·524	1·829
Asteroids	2·641	3·170
Jupiter	5·203	5·494
Saturn	9·546	9·521
Uranus	19·20	16·499
Neptune	30·07	28·594

This time the correlation coefficient is 0·998 and so Equation (4) explains $0{\cdot}998^2$ of the variance. This is 99·7%. Not as spectacularly good as the Figure for Equation (1), but remember we now have nine data points whereas before we had only six. We also have a mathematically more satisfactory formula and a much better fit for Neptune. Figure 2, showing $\log D$ versus n, demonstrates how good the fit is.

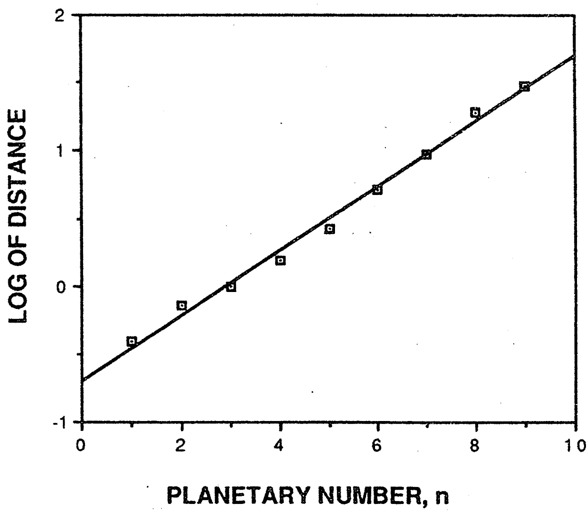

Figure 2

Mary Blagg actually went on to produce even better formulae. We will not, however, follow her in this, but rather take up another of her ideas. This was to investigate similar regularities in the systems of satellites surrounding the major planets: Jupiter, Saturn, Uranus and Neptune.

[Just before we continue with this story, we point out that we will not consider the planet Pluto. There are good reasons to believe that Pluto is not a member of the primary family of planets.]

Now consider the satellites of Jupiter. When Mary Blagg did her analysis, eight of these were known. See Table 4.

Table 4

Satellite	Distance (in Jovian radii)* [1]
Amalthea	3·108
Io	5·905
Europa	9·397
Ganymede	14·99
Callisto	26·37
Himalia	160·8
Elara	164·4
Pasiphae	329·1

The main moons are the so-called Galilean moons (named after Galileo who first studied them systematically): Io, Europa, Ganymede and Callisto. In Figure 3, we show a plot of the logs of their distances against n, where we have taken n to be 4, 5, 6, 7 respectively. [It doesn't really matter *where* we begin counting. Can you see why?]

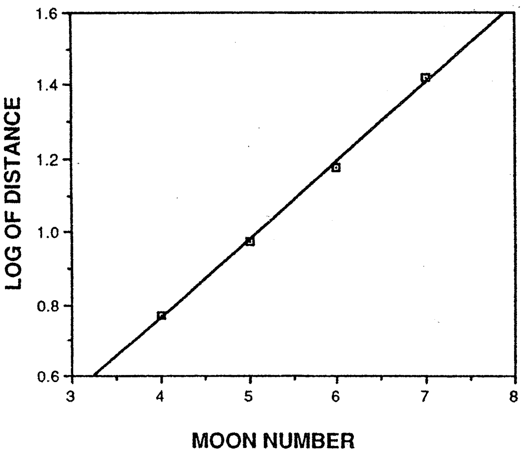

Figure 3

*The word "Jovian" means "pertaining to Jupiter".

Again we see a very accurate straight line fit. It obeys the formula

$$D = 0{\cdot}797 \times 1{\cdot}64^n, \tag{5}$$

and this may now be used with values of n other than those listed above. If we put $n = 2$, we get $D = 3{\cdot}523$, relatively near the distance to Amalthea. The other moons are thought not to belong to the system, but rather to be isolated pieces of "space debris".

Next consider the case of Saturn. Table 5 shows the main moons of Saturn.

Table 5

Satellite	Distance (in Saturnian radii)
Mimas	3·075
Enceladus	3·945
Tethys	4·884
Dione	6·256
Rhea	8·736
Titan	20·25
Hyperion	24·55
Iapetus	59·03
Phoebe	214·7

The two outermost moons, Iapetus and Phoebe, are thought not to be part of the primary system but to have been captured afterwards. We thus leave them out and analyse the others. Figure 4 (overleaf) shows a plot of $\log D$ against n, starting with $n = 1$ in this case. The fit is very good and the line drawn explains 98% of the observed variance. Notice that there is a gap at $n = 6$. There may be a moon yet to be discovered here. The formula for Saturn is

$$D = 2{\cdot}043 \times 1{\cdot}36^n. \tag{6}$$

Thus the ratio between successive orbital radii is $1{\cdot}36$ for Saturn. If we take the actual radius of Mimas' orbit and divide by $1{\cdot}36$ (not quite the same thing as putting $n = -1$ in Equation (6), as Mimas is further out than Equation (6) would predict) we get $D = 2{\cdot}26$, very close to where two small moons (Prometheus and Pandora) and a new ring have recently been discovered. One of us (AJRP) predicted that a ring would be found here, and this has turned out to be correct.

Uranus is the next planet out beyond Saturn and its system contains five main moons as listed in Table 6 overleaf.

Figure 5 (overleaf) shows the graph in this case. The relevant formula is

$$D = 1{\cdot}567 \times 1{\cdot}47^n. \tag{7}$$

In Figure 5 we took n to go from 3 to 7. If we take $n = 1$ in Equation (7) we get $D = 2{\cdot}30$ and if we take $n = 2$ we get $D = 3{\cdot}37$. In 1985 a satellite was found at a distance of 3·28 Uranian radii, and in 1986 a cluster of satellites was found at distances spanning 2·3 radii. If we put $n = 0$ in Equation (7), we get $D = 1{\cdot}567$ and this may correspond to a set of rings between $D = 1{\cdot}6$ and $D = 1{\cdot}95$. These bodies were all predicted by AJRP, prior to their discovery by the space-probes Voyager 1 and Voyager 2.

Table 6

Satellite	Distance (in Uranian radii)
Miranda	4·95
Ariel	7·30
Umbriel	10·15
Titania	16·64
Oberon	22·24

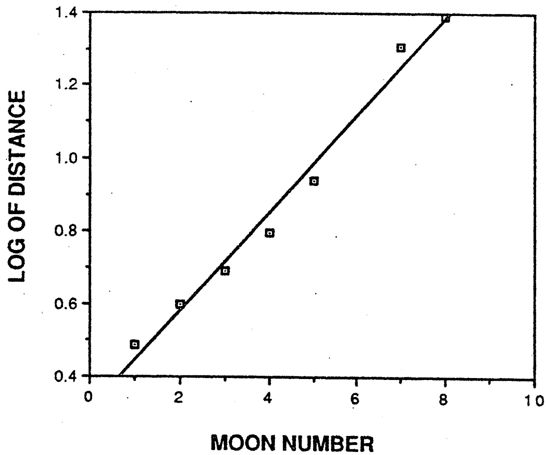

Figure 4

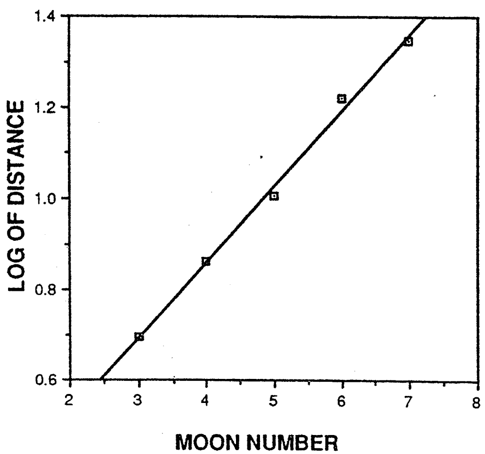

Figure 5

So accurate is the Titius-Bode law that it would seem that it must be an indication that some underlying mechanism is at work. Various researchers have tried to explain it (and others to explain it away). AJRP, however, saw it as a consequence of the modern Laplacian theory, explained in (e.g.) *Function, Vol.* 2, *Part* 2, pp. 14-20. This theory is still regarded by some as controversial, but has an outstanding track record in predicting not only the existence of new bodies in the solar system, but also their composition, temperatures and other physical attributes.

The modern Laplacian theory shows that the Titius-Bode law is a good approximation to the reality, whose full details require much more sophistication and are omitted here. The real test came with the Voyager 2 fly-by of Neptune in late 1989. The British journal *Nature* said, before this encounter: "A theory believed to be successful in predicting the satellites of Uranus is not widely accepted. Much will hang on whether it has been successfully applied to Neptune."

Neptune is indeed a real test case. In the first place, only two moons were known prior to the 1985 fly-by, and both of these are widely conceded to be later captures. Thus, we have no table to display as we did for the earlier planets. Secondly, Neptune is a case where the mere application of Titius-Bode is not enough. We need to apply the full power of the modern Laplacian theory.

However, the modern Laplacian theory predicts that a planet should have been formed at 16 Neptunian radii. If we now take the orbital ratio for the Neptunian system to be 1·47 (as for the similar planet Uranus – see Equation (7)), we find that others should exist at 10·9 Neptunian radii (i.e. 10·9 = 16/1·47), 7·4, 5·0 and 3·4 radii. After this, the Titius-Bode law needs modification, but AJRP predicted moons at 5·0, 3·4, 2·7, 2·4 radii.

On July 7, 1989 a moon was discovered at 4·7 radii – clearly the one expected near 5·0. Take this as a basis, just as we did above with Mimas in the Saturnian system, and rescale the results accordingly. This yields predictions for moons at 3·2, 2·6, 2·35 radii. AJRP predicted these moons just prior to the Voyager 2 discovery of moons at 2·59, 2·5 and 2·1 radii respectively. The error overall is only 6%.

No other theory of the solar system and its origins has had such consistent predictive success as the modern Laplacian theory. Perhaps one day it may achieve acceptance and validate the insight of the obscure Johann Titius, who was too modest to put his name to the law which is his greatest claim to fame.

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LETTERS TO THE EDITOR

More on Trigonometric Functions

I read Brian Weatherson's letter (*Function, Vol.* 14, *Part* 1, p.21) with interest, and offer this simplification of Paragraph 1 of his proof. The statement to be proved is that if A, B, C have rational coordinates, and if $\angle ABC = \theta$, then $\sin 2\theta$ and $\cos 2\theta$ are rational. The proof is as follows.

Since A, B, C have rational coordinates, the lines BA and BC have rational slopes, m_1 and m_2 (say). Then

$$t = \tan\theta = \pm \frac{m_1 - m_2}{1 - m_1 m_2}$$

is also rational.

But it is known that

$$\sin 2\theta = \frac{2t}{1+t^2} \quad \text{and} \quad \cos 2\theta = \frac{1-t^2}{1+t^2}\ .$$

It follows that these quantities are also rational.

John Mack
University of Sydney

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Calculating Ages : How it Works

In *Function, Vol.* 13, *Part* 3, *Vol.* 14, *Part* 1, it was pointed out that if x, y, z are the remainders on dividing a person's age by 3, 5, 7 respectively then the age can be recovered from the formula

$$\text{age} = 70x + 21y + 15z - 105n$$

where n is chosen to make the answer sensible, i.e., between 0 and 104. For example, if the age were 37 then the remainders are 1, 2, 3 and the formula requires calculating

$$(70\times1) + (21\times2) + (15\times2) - 105n$$

$$= 142 - 105n.$$

Taking $n = 1$ we recover the age $142 - 105 = 37$.

Why does this work? The formula is really an application of the "Chinese remainder theorem" of elementary number theory. The theorem is usually written using the idea of "congruence" rather than "remainder".

The idea of congruence is as follows. When we say 37 is congruent to 2 mod 5, which we write

$$37 \equiv 2 \bmod 5$$

we mean that 5 divides $37 - 2$. Note that this is weaker than the idea of remainder; while $37 \equiv 2 \bmod 5$ and 2 is the remainder on dividing 37 by 5, we can also say that $37 \equiv 7 \bmod 5$ (as 5 divides $37 - 7$) whereas, although $37 = 6\times5 + 7$, we do not normally say 7 is the remainder on dividing 37 by 5 as we require remainders to be smaller than the number being divided by.

Written in terms of congruence the age problem becomes the following.

If $\quad \text{age} \equiv x \bmod 3$

$\text{age} \equiv y \bmod 5$

$\text{age} \equiv z \bmod 7$

then

$$\text{age} \equiv 70x + 21y + 15z \bmod 105.$$

This is exactly a special case of the Chinese remainder theorem, which says the following.

If $\quad \text{age} \equiv x_1 \bmod m_1$

$\text{age} \equiv x_2 \bmod m_2$

$\vdots$

$\text{age} \equiv x_k \bmod m_k$

and no two m_i have a common factor, then

$$\text{age} \equiv c_1x_1 + c_2x_2 + \dots + c_kx_k \bmod m_1m_2 \dots m_k$$

where the coefficients c_i satisfy

$$c_i \text{ is the multiple of } \frac{m_1m_2 \dots m_2}{m_i}$$

which is congruent to $1 \bmod m_i$. In the special example $k = 3$, $m_1 = 3$, $m_2 = 5$, $m_3 = 7$, $x_1 = x$, $x_2 = y$, $x_3 = z$, c_1 is a multiple of 5×7 which is congruent to $1 \bmod 3$ (note: $70 = 2 \times (5\times7)$ and $70 - 1$ is divisible by 3) and so on.

If one were to use different moduli one could generate different formulae. For example, using 4, 5, 7 we would observe that

$$3 \times (5\times7) = 105 \equiv 1 \bmod 4$$
$$2 \times (4\times7) = 56 \equiv 1 \bmod 5$$
$$6 \times (4\times5) = 120 \equiv 1 \bmod 7$$

and so we have the result:

If $\text{age} \equiv x \bmod 4$

$\text{age} \equiv y \bmod 5$

$\text{age} \equiv z \bmod 7$

then

$$\text{age} \equiv 105x + 56y + 120z \bmod 140.$$

If the moduli are such that two have a common factor, then the theorem is a little more complicated — however, one can still generate similar formulae.

Rod Worley,
Monash University

[The unusual and perhaps puzzling name "Chinese Remainder Theorem" is, in fact, particularly appropriate. Its first appearance would seem to be in the text Master Sun's Mathematical Manual *published in China before 1000 A.D. Master Sun's problem reads: "There are an unknown number of things. Three by three, two remain; five by five, three remain; seven by seven, two remain. How many things?". We leave this to the reader to solve. For more detail on the history, see* Chinese Mathematics: A Concise History *by Li Yan and Du Shiran, translated by J.N. Crossley and A.W.-C. Lun, and published by Clarendon Press. Eds.]*

* * * * *

Euler Nods

Your correspondent's friend Dr Fwls is no fool (*Function, Vol.* 14, *Part* 2), or at least he is in very good company. In the English translation of his book *The Elements of Algebra* (vol. I, tr. Rev. J. Hewlett, 3rd ed., London 1822), the distinguished mathematician Euler writes:

"Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$; and $\sqrt{4}$ or 2 for the value of the product of $\sqrt{-1}$ by $\sqrt{-4}$. Thus we see that two imaginary numbers, multiplied together, produce a real, or possible one. But, on the contrary, a possible number, multiplied by an impossible number, gives always an imaginary product: thus, $\sqrt{-3}$ by $\sqrt{+5}$, gives $\sqrt{-15}$. It is the same with regard to division; for $\sqrt{a}$ divided by $\sqrt{b}$ making $\sqrt{a/b}$, it is evident that $\sqrt{-a}$ divided by $\sqrt{-1}$ will make $\sqrt{+4}$, or 2; that $\sqrt{+3}$ divided by $\sqrt{-3}$ will give $\sqrt{-1}$; and that 1 divided by $\sqrt{-1}$ gives $\sqrt{+1/-1}$, or $\sqrt{-1}$; because 1 is equal to $\sqrt{+1}$."

Thus, according to Euler (i multiplied by i) = +1 and not −1, as we reckon. Of course this causes difficulties since $\sqrt{+1}$ is also ± 1, whence $i = \pm 1$.

Fortunately lesser mathematicians have not made this error.

Harold Erulene
Leningrad, formerly St. Petersburg,
Russia

P.S. I found this quoted in your Monash colleague's book: *The Emergence of Number* (2nd ed., World Scientific, Singapore, 1987) by J.N. Crossley, p.93.

[J.N. Crossley was in fact the medium by which Euler's alter ego *communicated with us. The resolution of the apparent paradoxes is a matter of some subtlety. Eds.]*

* * * * *

HISTORY OF MATHEMATICS SECTION

EDITOR: M.A.B. DEAKIN

Space, Time, Motion and Calculus (With an Excursus on Buddhism)

In the 5th century BC there lived a philosopher whom we know as Zeno of Elea. He was born about 495 BC and died about 430 BC, 65 years later. What he is now remembered for is his questioning of the nature of space and of time: whether they are continuous or discrete.

It would seem that he came to the conclusion that any account we might give of this matter contained absurdities or contradictions and that therefore space and time were illusions. We cannot be entirely clear on this, because we have no direct record of what Zeno actually said, only of what others said that he said. Scholars still dispute how accurately he has been reported and what his purpose was.

However, he is credited with four paradoxes known respectively as the Dichotomy, the Achilles, the Arrow and the Stadium. These four arguments have come down to us via Aristotle, a later (384-322 BC) philosopher who was concerned to refute them. Nonetheless, he refers to them as "four arguments of Zeno's about movement which are hard to refute".

The Dichotomy, or division into two, says that "movement is impossible because the moving body must arrive at the halfway point before it arrives at the end". To elaborate: suppose the body gets to the halfway point, then it still has a certain distance to travel; it must now get to the halfway point of *that* distance, and so on for ever. The journey involves an infinite sequence of steps.

Alternatively, we may present it in reverse for even starker effect. To reach the halfway point, it would previously have had to reach the quarterway point, and before that the eighthway point and so on. Motion could never begin!

The Achilles paradox uses similar ideas, but is a bit more complicated. It is probably the best known of the four. Achilles, a Greek athlete, sets out to race a slower opponent, generally taken to be a tortoise. Suppose for definiteness he gives the tortoise a kilometre start and suppose that he runs 100 times faster than does the tortoise. Achilles covers his kilometre and is then ten metres behind the tortoise. By the time Achilles has run those ten metres, the tortoise has lumbered on by ten more centimetres, and as Achilles runs those ten centimetres, the tortoise manages to stay ahead, if only by a millimetre, and so on. Achilles never quite catches up.

These two paradoxes suppose space and time to be continuous. Now suppose them to be discrete. The Arrow paradox proceeds by supposing as Aristotle put it, that time consists "of moments". Then during such a moment, an arrow in flight must be somewhere. At the next moment, it must be somewhere else. It must have moved discontinuously between the two moments. What could have caused this instantaneous change in position? In fact at any "moment", the arrow is at rest!

The Stadium paradox has been well explained by C.P. Rourke in an article in *Manifold,* a U.K. student journal, now no longer published, but with which *Function* once had an exchange agreement. We will follow Rourke here in considering trains on adjacent tracks, rather than horse-drawn carriages in a stadium. Look at Figure 1. This shows two trains passing each other in

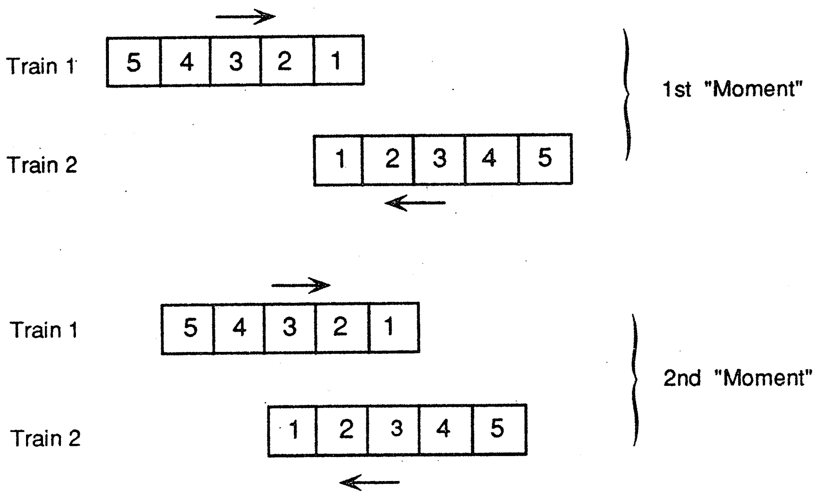


Figure 1

opposite directions. Each, like the arrow in the previous paradox, is motionless during any moment of time. During the first moment, Carriage no. 1 of Train no. 1 is opposite Carriage no. 1 of Train no. 2 (upper picture).

In the next moment, if we adjust matters carefully, each train will be one carriage further on. But now Carriage no. 1 of Train no. 1 is opposite Carriage no. 3 of Train no. 2 and *vice versa.* But Carriage no. 1 of Train no. 1 was thus *never* opposite Carriage no. 2 of Train no. 2.

It is not entirely clear quite why Zeno produced his paradoxes. Almost certainly they were not intended to influence mathematical thought in the way that they have. Possibly Zeno intended to say that space and time are illusions. This is the thrust of another account of Zeno's thought by the Greek philosopher Plato (in a book called *Parmenides* — named after another Greek philosopher to whose thought Zeno is supposed to have been sympathetic).

This quasi-mystical view finds an echo in Buddhist thought. Buddhism was founded by Gautama (the Buddha) in the 6th century BC. Originally it was a monastic discipline stressing the attainment of Nirvana (happiness) by means of self-denial. In the centuries after Gautama's death a school of thought arose that claimed that the self-sacrifice of particularly saintly men (*Boddhisattvas*) could gain Nirvana for others as well as for themselves. This school is known as *Mahayana* ("Greater Vehicle") Buddhism, and its adherents refer to the more traditional Buddhists as *Hinayana* ("Lesser Vehicle"). Those more traditional disciples of Gautama, however, refer to their own beliefs as *Therevada*, meaning "Doctrine of the Elders".

Therevada Buddhism believes not so much in reincarnation as in a succession of states of consciousness which, incidentally, may include consciousness of birth-events or death-events. In the first century AD, the Therevada philosopher Nagarjuna asked how long these states of consciousness last. He held that the question could not be answered, for if they lasted some finite time, we would have no way to measure that time, and they might as well be eternal. Thus they did not persist at all and were merely illusory. This was the origin of what Buddhists call "The Doctrine of the Void".

This analysis relates most clearly to Zeno's Arrow paradox and it is just possible that Nagarjuna's thought derives in some measure from Zeno's. Zeno's paradoxes, as we have seen, came down to us through the writing of Aristotle. Aristotle was at one time the tutor to the Greek prince Alexander the Great, who later conquered Northern India and ruled it as part of an extended Greek empire. It could be that some of the influence from that period continued to survive some four centuries later when Nagarjuna lived. Equally, he may have thought of the idea himself.

For the most part, Western thought sees space and time as both being continuous. The paradoxes we must come to terms with, therefore, are the Dichotomy and the Achilles. This aspect of the resolution began with the Greek philosopher Eudoxus, whose work is reported in Euclid's *Elements*. It has been much refined since and even today work continues to extend and refine it. Whether our idealised mathematical space-time concepts accurately describe either the real universe, or our own perception of that universe, is a moot point and I won't dwell on it here. Let us look rather at two ways in which these Zenonian paradoxes have entered mathematical thought.

First take the Dichotomy and consider the total distance travelled if we add up all the successive fractions of the total. We get

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \qquad (1)$$

and we would like to say that, if we added up all the infinitely many fractions, we would get a total of 1.

But clearly we can't add up infinitely many numbers; it would take forever. So how can we assign this total?

This problem was considered by the great French mathematician Augustin- Louis Cauchy in his 1821 textbook *Cours d'analyse*. Cauchy's analysis is a little more general than Expression (1) suggests, but we can present its tenor and substance in this simpler case.

If we stop, as we must, after adding *finitely* many terms of Expression (1), we get a number rather less than 1. In fact, if we add n terms (for some finite number n), then the total will be $1 - 1/2^n$. The discrepancy between this number and 1 is never zero, but we can make it arbitrarily small. Thus if I ask for a discrepancy of less than one part in a million, this can be achieved by taking n such that

$$2^n > 1\,000\,000$$

i.e., taking logarithms to base 10,

$$n \log 2 > 6$$

which gives us $n \geq 20$.

Thus, although we can never get *to* a total of 1, we can approach it arbitrarily closely simply by adding sufficiently many terms. Nor is there any number, other than 1, which has this property. Any number less than 1 is ultimately surpassed (by the same argument as used above); and as we approach, but never reach, 1, we do not approach arbitrarily closely to any number greater than 1.

Thus we assign the value 1 to the infinite sum in Expression 1, because the *finite* sums we construct from Expression 1 have these properties.

A second mathematical example relates to Achilles and the tortoise. Achilles does in fact pass the tortoise, but this paradox relates to the divisibility not only of the distances covered, but also of time. Achilles and the tortoise are each supposed to be travelling at constant speed, so that if we measure the distance covered in any measured interval of time, then the quotient gives the speed. If we take a smaller interval of time, the distance will be correspondingly smaller, but the quotient will be unaltered.

Adding all the distance increments in the manner just described produces the point at which Achilles will pass the tortoise. This can be verified independently from (e.g.) a space-time graph or (equivalently) a pair of simultaneous equations.

Suppose, however, we take a different tack and assume a situation in which the speed is not constant. For definiteness, consider the situation in which a heavy body falls from rest. The distance (s) that it travels is proportional to the square of the time (t) during which it has been falling. Thus we have

$$s = kt^2. \qquad (2)$$

Figure 2 shows the space-time graph for this law (first discovered by Galileo). Now here, if we take some value T, of t, and let α be some increment in time, the distance travelled in the time-interval α will then be

$$k(T+\alpha)^2 - kT^2 \qquad (3)$$

and so the average speed is

$$\frac{k}{\alpha}\left\{(T+\alpha)^2 - T^2\right\}. \qquad (4)$$

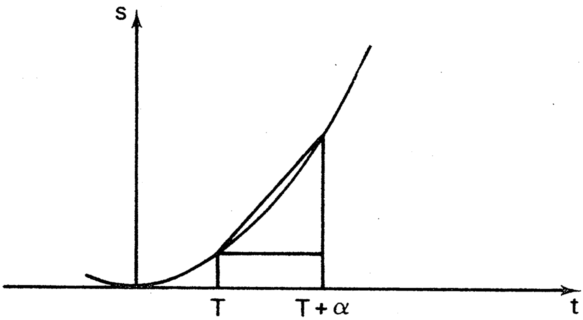


Figure 2

But this speed depends not only on k, but also on T and, in fact, also on α. Newton and Leibnitz in their development of the calculus had tried to use values of α that were, in essence, zero, but had run into problems, as Bishop George Berkeley pointed out, with Equation (4), because we can't divide by zero. (For an account of the controversy this generated, see *Function*, *Vol.* 11, *Part* 2, p.42.)

Cauchy's concept of limit also serves to rigorise the calculus — that is to say, it provides a contradiction-free account of it. Expression (4) may be simplified to give

$$k(2T+\alpha) \qquad (5)$$

as long as $\alpha \neq 0$.

But now let α get smaller and smaller. We can make Expression (5) get arbitrarily close to $2kT$ merely by choosing a sufficiently small α. We *explicitly avoid* putting α exactly equal to zero, but the limit of Expression (5) ceases to involve α, as by making it small enough, we can make its effect correspondingly small.

These are just two examples of how Zeno's querying of time and space have led (*inter alia*) to fruitful mathematics.

* * * * *

COMPUTER SECTION

EDITOR: R.T. WORLEY

Primes and Computers II

In the last issue I discussed building a table of primes using Eratosthenes' Sieve, and promised some graphs to illustrate the answers to some interesting questions, based on a list of the first million primes. In addition to these graphs (see Appendix 1), you may care to look at the graph plotting the distribution of the first 1500 primes in the article by Arnulf Riedl in *Function* 11, *Vol.* 1. It should be pointed out that the curves in Figures 1 and 2 are in fact not quite smooth. They only appear that way due to the scaling. The graph in Riedl's article, which has a different scale, illustrates this.

For this issue I shall consider the problem of determining if a given number is prime. The methods that can be used depend on the size of the number given, and also whether you want to be absolutely certain if the number is prime. The obvious method of testing if the number is divisible by any smaller number is quite time-consuming and unsuitable for all but very small numbers, so when using a computer other methods are used.

If the number given is small, then one can just look it up in the table of primes (assuming your table of primes goes far enough that the number would be in the table if it were prime). There is also the following method.

Primes have some properties that few other numbers have. We can test if our given number has these properties. If it doesn't, then our number certainly is not prime. However, if it does have the properties it probably is prime, but there is a slight chance it may be one of the few other numbers that have the properties. Depending on the properties we check, the chance that the number is not a prime but still has the properties is extremely small.

For example, take the property that if the number n is prime then it has the property that n divides $2^n - 2$. If we are given a number n and are asked if it is prime, we could calculate $2^n - 2$ and see if n divides it. For example $2^{15} - 2 = 32768 - 2 = 32766$ is not divisible by 15 so we can be sure 15 is not prime (actually we know this without all that calculation because we can see that $15 = 3\times5$). However, $2^{17} - 2 = 131072 - 2 = 131070 = 17\times7710$ is divisible by 17, so we suspect 17 is prime (actually we know it really is prime, but we cannot decide this on the basis of the results of our test). Likewise, $2^{341} - 2$ is divisible by 341 and on the basis of our test we suspect (wrongly) that 341 is prime. In fact, 341 is the smallest number that the test falsely leads us to suspect is prime. However, we can modify our test so we will be able to determine correctly if any given number less than 10^{10} is prime.

Firstly we calculate with a fairly powerful computer a list of all those numbers less than 10^{10} that satisfy our test. Next we calculate a list of all primes less than 10^{10}. We then compare the lists and put in a list all those numbers which our test falsely leads us to suspect is prime. (There are 14884 such 'false primes'.) Now if we are given a number n less than 10^{10} we can see if n divides $2^n - 2$. If it doesn't, then we are sure n is not prime. However, if it does, then we see if the number is in our list of false primes. If it's in our list we know it is not a prime, otherwise it is a prime. Since the list of all primes less than 10^{10} would require a lot of space to store, this method may be a better method, once we had the list of 14884 false primes.

We can improve on this test a little. If n is prime then n also divides $3^n - 3$, $5^n - 5$, and $7^n - 7$. If we check these as well we will have a smaller list of false primes to look up when our number passes the tests. There are slightly more sophisticated tests, rather similar to these, such that they give only one false prime less than $25{\times}10^9$. By using more tests one can determine that only one number in 10^{60} is a false prime. In this case we do not bother calculating a list of the false primes, as it would require too much computing power, and are happy to say that a number that passes the test is almost certainly prime.

By using more sophisticated tests, which need more computing power, we can use enough tests to be certain that our given number n is not a false prime, so if it passes the tests it will be prime. By this means it is possible to prove numbers of 210 digits prime in less than 10 minutes on a very fast computer.

Although it may seem necessary to calculate $2^n - 2$ in order to see if n divides it, this is not the case. This is fortunate, because the normal arithmetic routines provided with computers cannot calculate $2^n - 2$ if n is larger than 32. The following BASIC program will print a list of those n which do divide $2^n - 2$ and are less than 1000. The numbers printed out which are not in the list of primes printed by the program in the previous article are the false primes. Note that the program uses a 'fast binary powering' routine. To calculate, for example, 2^13 it uses the fact that 13 = 8 + 4 + 1 and calculates 2^1 * 2^4 * 2^8. It also takes the remainders on division by n at each step, rather than waiting till the end. It can be shown that this gives the correct result.

```
100 N=1000: REM small upper limit on prime/false prime
110 P#=2: REM this is the 2 in 2^n-2
120 FOR I=3 TO N: REM I is the 'n' to be tested
130 GOSUB 8000: REM calculate in J remainder on dividing 2^I by I
140 IF J#=P# THEN PRINT I: REM I divides 2^I-2, so print I
150 NEXT I
160 STOP
7999 REM this is the fast 'binary powering routine'
8000 K#=P#: REM set K#=2^1
8010 J#=1#
8020 L=I
8030 IF L=0 THEN RETURN
8040 L1 = L\2: REM equivalent to L1 = INT(L/2)
8050 IF L-2*L1<1 THEN GOTO 8080
8059 REM multiply J# by current power of 2, and take remainder
8060 J#=(J#*K#)
8070 J# = J# - INT(J#/I) * I
8079 REM square current power of 2, and take remainder
8080 K#=K#*K#
8090 K# = K# - INT(K#/I) * I
8100 IF L=0 THEN RETURN
8110 L=L1
8120 GOTO 8040
```

Appendix 1

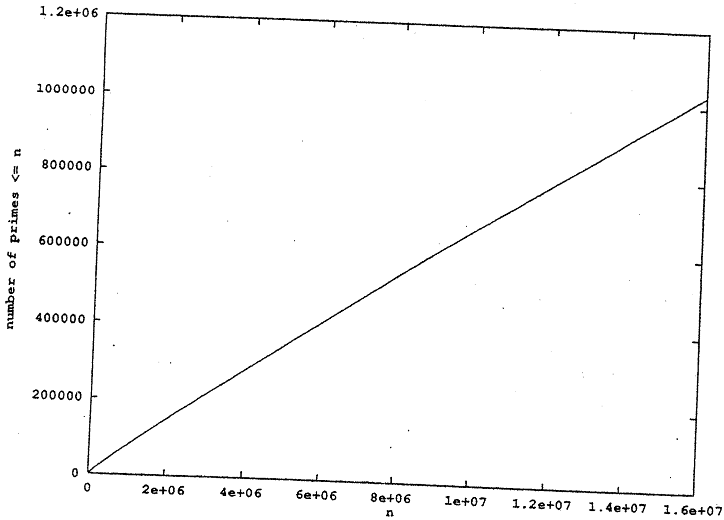


Fig. 1. Graph giving number of primes less than or equal to n

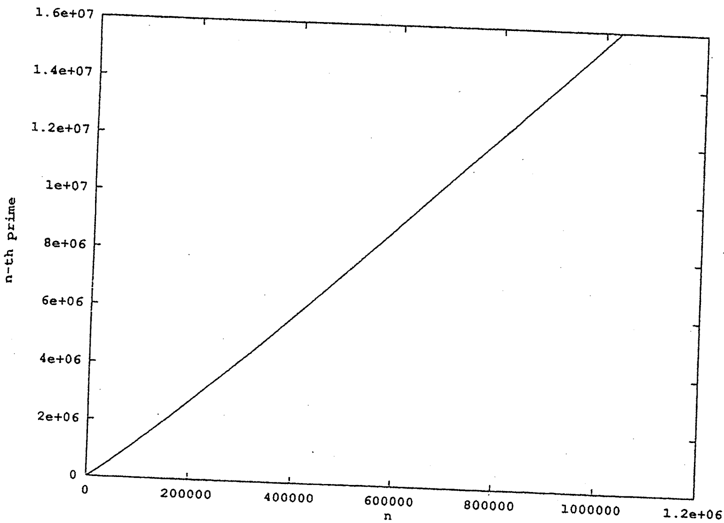


Fig. 2. Graph giving value of n-th prime (reflection of Fig. 1)

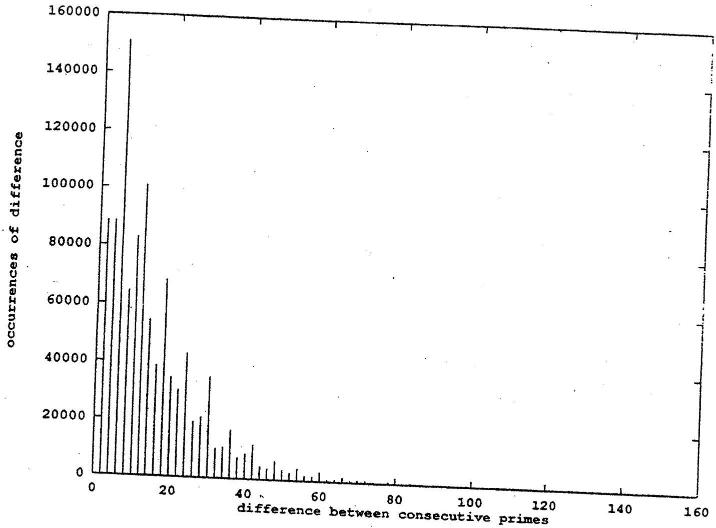


Fig. 3. Graph giving number of occurrences of differences between consecutive primes (for primes less than 16000000)

last digit	occurs
1	257724
3	257943
7	257725
9	257742

Table giving number of occurrences of primes less than 16000000 with specified last digit

Appendix 2

Using the binomial theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 \dots + \binom{n}{n-1}ab^{n-1} + b^n$$

and the observation that if n is prime then the binomial coefficient $\binom{n}{k}$ is divisible by n for $1 \le k \le n-1$ (because n will be a factor in the numerator but not the denominator), we can show that n divides $2^n - 2$ if n is prime. For, taking $a = b = 1$ in the binomial theorem we find that

$$2^n = (1+1)^n = 1^n + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 \dots + \binom{n}{n-1}1 \cdot 1^{n-1} + 1^n,$$

that is,

$$2^n = 1 + \binom{n}{1} + \binom{n}{2} \dots + \binom{n}{n-1} + 1,$$

or

$$2^n - 2 = \binom{n}{1} + \binom{n}{2} \dots + \binom{n}{n-1}.$$

Since every term on the right side of this equation is divisible by n when n is prime, the result n divides $2^n - 2$ follows. Furthermore, taking $a = 2$ and $b = 1$ it is now possible to show n divides $3^n - 3$, and so on.

* * * * *

John Brown's Number

John Brown is a Californian mathematician and his number is

$$391581 \times 2^{216193} - 1.$$

This is, at present, the largest known prime number. There are actually infinitely many primes, a fact which was known to Euclid, so there is no such thing as a "largest prime". However, much energy has been expended in calculating larger and larger prime numbers. John Brown's number has 65087 digits.

Most previous records have been held by numbers of the form $2^n - 1$; such primes are called Mersenne primes and their primality is (relatively) easy to test. John Brown's number is unusual in not being of this form.

How long this new record will last is an interesting point. Most likely someone is already working on breaking it.

PROBLEMS SECTION

EDITOR: H. LAUSCH

This time my job has been made very easy by some of our readers: *Function* received new problems as well as plenty of solutions to problems that had appeared in earlier issues. Many thanks to all contributors.

Solutions

Problems 12.4.5, 12.4.6 and 12.4.7 were solved by Mark Kisin, first-year student at Monash University and winner of a silver medal at the 30th International Mathematical Olympiad, Braunschweig, FRG, 1989. Let us see how Mark tackled them.

Problem 12.4.5. Let $a \in R$ and $f : R \to R$ such that

$$f(x+y) = f(x)f(a-y) + f(y)f(a-x) \quad \text{for all} \quad x, y \in R,$$

$$f(0) = \frac{1}{2}.$$

Prove that f is a constant.

Solution. Put $y = a - x$. Then

$$f(x+y) = f(a) = f(x)^2 + f(a-x)^2. \tag{1}$$

Put $x = y = 0$. Then $f(0) = \frac{1}{2} = 2f(0)f(a)$, hence $f(a) = \frac{1}{2}$. Equation (1) implies that

$$f(x)^2 + f(a-x)^2 = \frac{1}{2}. \tag{2}$$

Ed.: Now follow two spectacular steps in Mark's argument. He brings into play two classical <u>*inequalities*</u> *in order to establish the* <u>*equality*</u> *of $f(x)$ with $\frac{1}{2}$ for all values of x! The inequalities are (a) the "triangle inequality": if a and b are real numbers, then $|a+b| \leq |a|+|b|$; and (b) the "arithmetical mean-geometrical mean (AM-GM) inequality": if a and b are non-negative real numbers, then $\sqrt{ab} \leq \frac{1}{2}(a+b)$.*

$$\begin{aligned}
|f(x+y)| &= |f(x)f(a-y) + f(y)f(a-x)| \\
&\leq |f(x)f(a-y)| + |f(y)f(a-y)| && \text{[triangle inequality]} \\
&\leq \tfrac{1}{2}(f(x)^2 + f(a-y)^2) + \tfrac{1}{2}(f(y)^2 + f(a-x)^2) && \text{[AM-GM inequality applied twice]} \\
&= \tfrac{1}{2}(f(x)^2 + f(a-x)^2 + f(y)^2 + f(a-y)^2) \\
&= \tfrac{1}{2} && \text{by (2).}
\end{aligned}$$

Since $x + y$ can take on every real number as its value, we face $|f(x)| \le \frac{1}{2}$ for all values of x. Consequently, $f(x)^2 + f(a-x)^2 \le \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. From (2) we know that $f(x)^2 + f(a-x)^2 = \frac{1}{2}$. Thus, in order for equality to hold, $f(x) = f(a-x) = \frac{1}{2}$ for all values of x. Therefore f is constant.

Problem 12.4.6. Is it true that 832^n and 2^n have an equal number of digits?

Solution. Let k be the number of digits in 832^n, then $10^{k-1} \le 832^n < 10^k$. Since $832 > 10$, we have $k > n$ so that $2^n | 10^k$. As $2 | 832$, it follows that $2^n | 832^n$, hence also $2^n | 10^k - 832^n$; in particular, $2^n \le 10^k - 832^n$, i.e. $832^n + 2^n \le 10^k$. It remains to show that equality cannot hold. We note that $832^n + 2^n \equiv 2^n + 2^n = 2^{n+1} \mod 10$. Of course, 10 is not a factor of any power of 2 so that equality cannot hold.

Problem 12.4.7. Prove that for any $a, b \in R$, $a < b$, there exists $n \in N$ and $c_i \in \{-1,1\}$, $i = 1,\ldots,n$ such that

$$a < c_1 + \frac{c_2}{2} + \ldots + \frac{c_n}{n} < b.$$

Solution. We first observe that, however small $b - a$ might be, there is always a positive integer m such that $\frac{1}{m} < b - a$ [simply choose an integer m that is larger than $\frac{1}{b-a}$]. Then put $c_1 = c_2 = \ldots = c_m = 1$. If $a < 1 + \frac{1}{2} + \ldots + \frac{1}{m} < b$, then we take $n = m$ and are done. Then we observe that if K is any positive number and we take a sufficiently large positive integer N, then $\frac{1}{m+1} + \frac{1}{m+2} + \ldots \frac{1}{N} > K$. We take advantage of this observation:

(a) If $1 + \frac{1}{2} + \ldots + \frac{1}{m} < a$, then there exists a positive integer $n > m$ such that $1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \le a < 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} + \frac{1}{n}$, but as $n > m$, we have $\frac{1}{n} < \frac{1}{m} < b - a$ by the way we chose m. It follows that $1 + \frac{1}{2} + \ldots + \frac{1}{n-1} + \frac{1}{n} < a + (b-a) = b$ so that we choose $c_{m+1} = \ldots = c_n = 1$ to obtain the required inequality.

(b) If $b < 1 + \frac{1}{2} + \ldots + \frac{1}{m}$, then there exists a positive integer $n > m$ such that

$$1 + \frac{1}{2} + \ldots + \frac{1}{m} - \left(\frac{1}{m+1} + \ldots + \frac{1}{n-1}\right) \ge b$$
$$> 1 + \frac{1}{2} + \ldots + \frac{1}{m} -$$
$$- \left(\frac{1}{m+1} + \ldots + \frac{1}{n-1} + \frac{1}{n}\right),$$

and since again $\frac{1}{n} < \frac{1}{m} < b - a$, we have $-\frac{1}{n} > -\frac{1}{m} > a - b$. It follows that $1 + \frac{1}{2} + \dots + \frac{1}{m} - \left[\frac{1}{m+1} + \dots + \frac{1}{n-1} + \frac{1}{n}\right] > b + (a-b) = a$ so that this time we choose $c_{m+1} = \dots = c_n = -1$ to obtain the required inequality. Noting that we have covered all possible cases finishes our proof.

John Barton, North Carlton, Victoria, solved Problems 14.1.5, 14.1.7 and 14.1.8:

Problem 14.1.5. Let a_0 be an arbitrary integer and suppose that $a_1,a_2,a_3,\dots$ are numbers for which the equation

$$a_n = n + (-1)^n \cdot a_{n-1}$$

holds.

(a) Show that the sequence $a_0,a_1,a_2,\dots$ contains at least 1990 numbers that are equal.

(b) Determine the least value of N such that the finite sequence $a_0,\dots,a_N$ contains 1990 numbers that are equal.

Solution. Calculating the first several terms of the sequence suggests that $a_{3m+1} = a_0$ and $a_{4m+1} = 1 - a_0$ for each integer m, and we verify this:

$$a_{4m-1} = 4m - 1 - a_{4m-2} \quad (m > 2)$$

$$a_{4m-2} = 4m - 2 + a_{4m-3}$$

so that

$$a_{4m-1} = 1 - a_{4m-3}. \qquad (1)$$

Similarly

$$a_{4m-3} = 4m - 3 - a_{4m-4}$$

$$a_{4m-4} = 4m - 4 + a_{4m-5}$$

so that

$$a_{4m-3} = 1 - a_{4m-5}. \qquad (2)$$

From (1) and (2), $a_{4m-1} = a_{4m-5}$.

Since

$$a_3 = 3 - a_2 = 3 - (2+a_1)$$
$$= 3 - 2 - (1-a_0)$$
$$= a_0,$$

we obtain $a_{4m-1} = a_0$ for $m = 1,2,3,\dots$.

From (2) $$a_{4m-3} = 1 - a_{4m-5} = 1 - a_{4m-1} = 1 - a_0,$$

so that

$$a_{4m+1} = 1 - a_0 \quad \text{for} \quad m = 0,1,2,\dots .$$

Beginning with a_0, the first repetition occurs for $a_3 = a_0$, so that, to have 1990 equal numbers we need the set $\{a_{4m-1} | m = 1,2,\dots,l\}$ to have 1989 elements, equivalently the set $\{4,8,12,16,\dots,4l\}$ to have 1989 elements, whence $4l = 4 \cdot 1989$ so that $N = 4l - 1 = 7955$.

Editor's comment: The occurrence of the number 1990 was, as can be seen, a red herring and not really vital for this problem.

Problem 14.1.7. Let n be a positive integer. What is the value of the sum

$$\sum_0^n \frac{(-1)^{\left[\frac{3^k}{2}\right]}}{3^k} \, ?$$

Solution. We shall assume that the sum is to be over the integers $0,1,2,\dots,n$. The integers $\left[\frac{3^k}{2}\right]$ are alternately even and odd as k runs through the natural numbers. For, supposing that $3^j = 4l + 1$ for some integers j and l, then

$3^{j+1} = 3(4l+1) = 4l' - 1$ where $l' = 3l + 1$, and

$3^{j+2} = 3(4l'-1) = 4l'' + 1$ where $l'' = 3l' - 1$. Hence

$$\left[\frac{3^j}{2}\right] = 2l, \quad \left[\frac{3^{j+1}}{2}\right] = 2l' - 1, \quad \left[\frac{3^{j+2}}{2}\right] = 2l''.$$

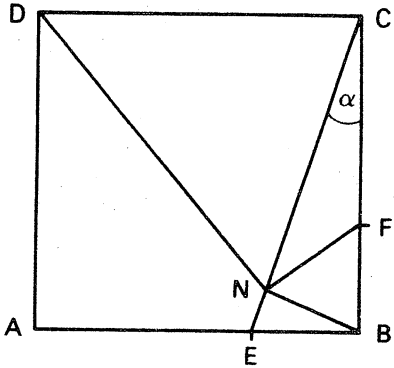
Hence the sum is

$$\sum_{k=0}^{n} \left(-\frac{1}{3}\right)^k = \frac{3}{4}\left\{1 - (-3)^{-n-1}\right\}.$$

Problem 14.1.8. Let $ABCD$ be a square. Choose any point E on AB and then let F be the point on BC which is determined by the condition $BE = BF$. Let N be the foot of the altitude of the right-angled triangle EBC.

Show that $\angle DNF$ is a right angle.

Solution.



With $BC = 1$, $BCE = \alpha$, $BN = \sin\alpha$, $BE = \tan\alpha$, origin B, first axis BA, second axis BC, we have the following components for the indicated vectors:

$$\overrightarrow{NF} : (-\sin\alpha\cos\alpha,\ \tan\alpha - \sin^2\alpha)$$

$$\overrightarrow{ND} : (1 - \sin\cdot\alpha\cdot\cos\cdot\alpha,\ \cos^2\alpha)$$

Now $\overrightarrow{NF} \cdot \overrightarrow{ND} = -\sin\alpha\cos(1 - \sin\alpha\cos\alpha) + \cos^2\alpha(\tan\alpha - \sin^2\alpha) = 0$. Since neither $\overrightarrow{NF}$ nor $\overrightarrow{ND}$ is the zero vector, these vectors are perpendicular.

Problems

a. Old problems

Apologies to our readers for misprints in two problems of *Volume* 14, *Part* 1. Corrected as they now stand they read:

Problem 14.1.2. Determine all real numbers x which satisfy the equation

$$\sqrt{x^2-[x^2]} = 3 - x.$$

Problem 14.1.4. Prove that the following inequalities hold for every positive integer n :

$$\frac{3}{2} \cdot \left(\sqrt[3]{(n+1)^2} - 1\right) < 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \dots + \frac{1}{\sqrt[3]{n}} < \frac{3}{2} \cdot \sqrt[3]{n^2}.$$

b. New problems

Function is delighted to put before it's readers two problems that have been received from Ethiopia. To *Function* subscriber K.R.S. Sastry of Addis Ababa many thanks for their communication.

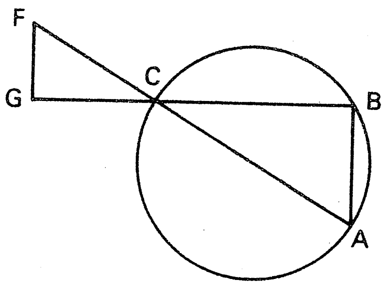
Problem 14.3.1. $ABCD$ is a rhombus and E, the intersection point of its diagonals AC and BD. Let r_1, r_2, r_3 and r denote the radii of the incircles of the triangles ABE, ABC, ABD and of the rhombus $ABCD$ respectively. Prove that

$$\frac{1}{r} + \frac{1}{r_1} = \frac{1}{r_2} + \frac{1}{r_3} .$$

Problem 14.3.2. ABC is a triangle right-angled at A, and D is the foot of the altitude from A. Let X and Y be the incentres of triangles ABD and ADC, respectively. Determine the angles of triangle AXY in terms of triangle ABC.

Note (Ed.): K.R.S. Sastry refers to Problem 5 of the 29th International Mathematical Olympiad held in 1988 in Canberra. For the benefit of our more recent subscribers we quote this problem: "ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incentres of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that $S \geq 2T$."

Problem 14.3.3. In his writings of 1638 and 1644 the Danish astronomer Christian Longomontanus, an assistant of Tycho Brahe, employed the following construction: Let AF and BG be line segments that intersect in the point C, and let k be the circle having AC as a diameter. Suppose that B lies on the periphery of k and FG is perpendicular to BG. Furthermore, let $CF = 70$ and $AC = 86$ and $AB = 43$.



Longomontanus claimed that BG and the semicircular arc ABC were of equal length. Was Longomontanus right?

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