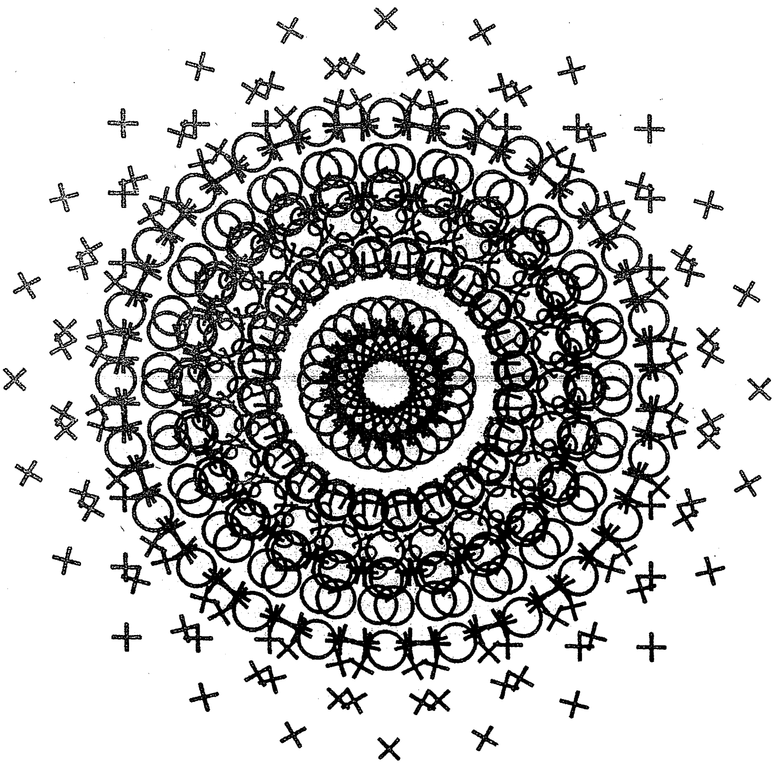
Function

Volume 13 Part 5 October 1989



A SCHOOL MATHEMATICS MAGAZINE

Reg. by Aust. Post Publ. No. VBH0171

FUNCTION is a mathematics magazine addressed principally to students in the upper forms of schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

* * * * *

EDITORS: G.B. Preston (chairman), M.A.B.Deakin, H. Lausch, G.A.Watterson, R.T.Worley (all of Monash University); K.McR. Evans (Scotch College); J.B.Henry (Victoria College, Rusden); P.E.Kloeden (Murdoch University); J.M.Mack (University of Sydney); D. Easdown (Curtin University); Marta Sved (University of Adelaide).

BUSINESS MANAGER: Mary Beal (03) 565-4445

TEXT PRODUCTION: Anne-Marie Vandenberg

ART WORK: Jean Sheldon

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
FUNCTION,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

The magazine is published five times a year, appearing in February, April, June, August, October. Price for five issues (including postage): $12.00*; single issues $2.40. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

* $6.00 for *bona fide* secondary or tertiary students.

* * * * *

Registered for posting as a periodical - "Category B"
ISSN 0313 - 6825

* * * * *

Published by Monash University Mathematics Department.

Our leading article, by D.F. Charles, of Pascoe Vale Girls' High School, is about that perennial problem of trying to find examples to illustrate a situation, in this instance the turning points of a cubic curve, without introducing too much complication extraneous to the main idea being illustrated.

Our choice for the "Ten years ago" entry is the article by Kathleen Ollerenshaw with its message, of permanent importance, on what mathematics is really about.

Neil Barnett's article points to some of the mathematical thinking that is a necessary support for current weights and measures legislation, while Geoff Watterson has some interesting comments about pilots' and first officers' pay before the recent pilots' strike.

We hope you like the article by S. Trompler, taken from our sister magazine, the Belgian *Maths-Jeunes*, giving the story of the introduction of the metre. Do you think the metric system, with its large numbers, is an improvement on the system it replaced? Or is it a disaster?

* * * * *

CONTENTS

FRONT COVER

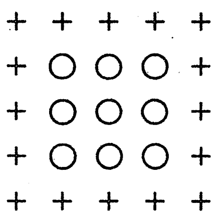
by

Jean Sheldon

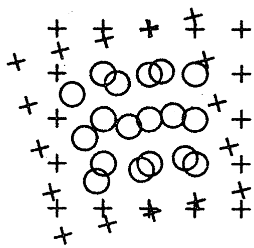
The front cover diagram was generated, using an Apple Mac II computer, starting with a 5 by 5 square consisting of a border of plus signs, with a 3 by 3 interior of circles (see diagram, step 1), rotated repeatedly through an angle of 15° (see diagram, steps 2 and 3), giving a total of 24 repetitions, each superimposed on the rest.

The software package used on the Mac II was the Adobe Illustrator '88.

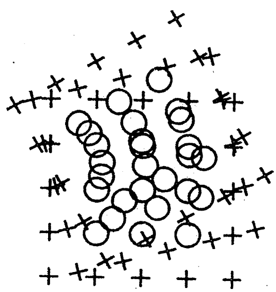
The starting point of the construction, the 5 by 5 square, figures as the generating idea for solving a mathematical problem in Kathleen Ollerenshaw's article (p. 138).



Step 1



Step 2



Step 3

PLEASANT CUBICS

D.F. Charles, Pascoe Vale Girls' High School

As a teacher one often needs to make up cubic functions of the form

$$f(x) = x^3 + ax^2 + bx + c$$

for student work sheets, tests, exams and the like. The students are to sketch the curve of the function by finding the x-intercepts, the y-intercept, and the coordinates and nature of the turning points.

The teacher may sit down and choose 3 integers, then construct a cubic equation with these integers as solutions. For instance, choose -1, -3, 2 so that

$$f(x) = (x + 1)(x - 2)(x + 3).$$

At least one of the solutions must be an integer so that the students can factorize the function by the factor theorem. In the example above

$$f(x) = x^3 + 2x^2 - 5x - 6,$$

the student may try $x = 2$ and find that $f(x) = 0$ so that $(x-2)$ is a factor. She then divides $f(x)$ by $(x-2)$, obtaining a quadratic as the quotient, then proceeds to factorize the quadratic. This all runs smoothly. Next the student tries to find the x-coordinates of the turning points by differentiating the function, setting the derivative to zero, then solving. In our example

$$f'(x) = 3x^2 + 4x - 5;$$

hence, to find the turning points we must solve

$$3x^2 + 4x - 5 = 0.$$

In most cases the student must complete the square or use the formula to find solutions. This process can be needlessly time-consuming and "messy", with irrational solutions.

If the ability to solve a quadratic by these means is not the concept being tested, if time is of the essence, and if not many marks are allocated to the problem then the teacher might try to find a cubic function whose derivative factorizes by inspection, that is, a cubic function whose graph has all important points rational.

These functions may often be hard to find, so the teacher might:

(i) choose an example from the book which he knows will work; or

(ii) choose a cubic with 2 identical rational solutions so that x-values of all important points are rational. (One turning point then always coincides with one of the intercepts.)

It would be interesting to be able to construct quickly a cubic function with 3 different solutions whose derivative factorizes by inspection.

Unfortunately a turning point in a cubic is not exactly in the middle of the interval between two adjacent x-intercepts. Not only that, if you widen or narrow one of these intervals, the turning point in the other interval will move. For example, an attempt to lengthen the interval of length b, in the top diagram of Figure 1, may result in a change of the kind shown: the length a of the other interval also changes, as does the position of the turning point TP.

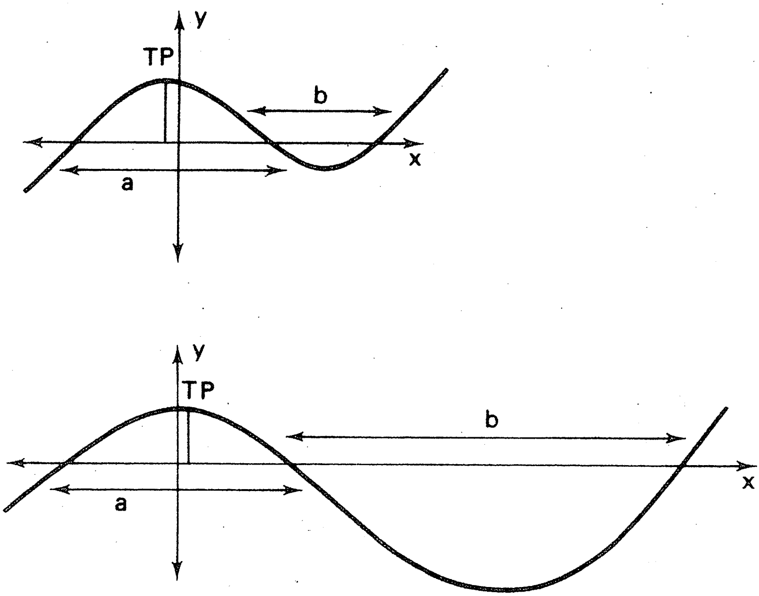


Figure 1

Knowing this we construct a cubic, one of whose solutions is 0; i.e. we take

$x_{123} = O, A, B$, where $A, B \in N$, the set of integers;

$$\therefore \quad f(x) = x(x-A)(X-B)$$

$$= x^3 - (A+B)x^2 + ABx.$$

Differentiating f we obtain

$$f'(x) = 3x^2 - 2(A+B)x + AB.$$

At the turning points $f'(x) = 0$, so

$$x_{12} = \frac{2(A+B) \pm \sqrt{4(A+B)^2-12AB}}{6} .$$

For rational solutions $\sqrt{(A+B)^2-3AB}$ must be an integer.

$\therefore \quad \boxed{A^2 + B^2 - AB = C^2}$ where $C \in N$.

Since this equation looks like Pythagoras' Theorem, we could try to generate integer solutions in a similar way to which Pythagorean triads are generated.

Note that if we find integer solutions to A and B then the required ratios a and b will be given by:

$a{:}b = |B| - |A| : |A|$ if the signs of A and B are the same and $|B| > |A|$,

and by:

$a{:}b = |A| : |B|$ if the signs of A and B are different.

Two such solution generators I have found are:

(i) $\boxed{A : B = -4n : (n+1)(n-3)}$

and

(ii) $\boxed{A : B = (n^2-1)^2 : n^4 - 4n^3 + 2n^2 - 4n + 1}$.

where n is an integer.

The second of these seems a bit clumsy looking and I am not sure whether they eventually generate the same solutions or whether either or both generate all primitive solutions, since although A and B become large with increasing n (especially in the second one), sometimes a common factor cancels out, reproducing one of the earlier ratios, or creating a new one. I would appreciate hearing from any reader who can find or knows of any simpler or more general solution generators.

Let's try generator (i) with $n = 2$ ($n = 1$ gives a cubic with only 2 intercepts).

$$A : B = -4 \times 2 : (2+1)(2-3)$$

$$= -8 : -3 = 8 : 3.$$

This gives a ratio of distances a and b of

$a : b = 3 : 5$.

This is the ratio with smallest integers. To test it we construct a cubic function whose intervals between x-intercepts are in the ratio $3 : 5$.

Try $x_{123} = 1, \xrightarrow{3} 4, \xrightarrow{5} 9$

$$f(x) = (x-1)(x-4)(x-9)$$

$$= x^3 - 14x^2 + 49x - 36$$

$$f'(x) = 3x^2 - 28x + 49$$

$$= (3x-7)(x-7);\ \text{giving rational } x\text{-values.}$$

With $n = 1$ generator (ii), like generator (i), gives a trivial solution and with $n = 2$ reproduces the above ratio. With $n = 3$ it gives a ratio of $a : b = 16 : 5$.

Trying $x_{123} = 10, \xrightarrow{5} 5, \xrightarrow{16} 11$ gives:

$$f(x) = x^3 + 4x^2 - 115x - 550$$

$$f'(x) = 3x^2 + 8x - 115$$

$$= (3x + 23)(x-5).$$

Testing other values of n gives other rational x-values.

In practice the teacher need only remember the ratio $3 : 5$ as this is sufficient, with various transformations, to generate any number of different cubic functions, all of which are "pleasant".

* * * * *

Ten years ago

METHODS OF PROOF

Dame Kathleen Ollerenshaw, Institute of Mathematics and its Applications†

There are several recognised methods of mathematical proof, but proof is never absolute and there are varying degrees of rigour. A mathematical proof is only valid within the limits of the definitions laid down. We can change the rules or move the goal posts in mathematics as surely as we do in other evolving activities and start an entirely new ball game, often very fruitfully. Geometrical truths, though eternal in a Euclidean world, did not suffice for the geometry of outer space. Proof may traditionally be by the "direct method" (as with Pythagoras's theorem); by one of the indirect methods; such as *reductio ad absurdum*; or by the method of exhaustion of all possibilities. An example of the latter is the standard proof that there can be (and are) only five regular Platonic Solids – the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. Sir Hermann Bondi and I used this method when solving and finding the correct answer to the classical *Nine Prisoners Problem* this time last year.†† The recently accomplished computer-crushing solution to the famous *Four Colour Map Problem* is an example of proof by exhaustion of defined possibilities.#

As an example of proof by translation, in this instance by projective geometry, here is one of the most beautiful results of all mathematics found by Pascal (1623 – 1662) at the age of 16.

Pascal's theorem states that:

If a hexagon is such that its six points of intersection lie on any conic, then the three points of intersection of opposite sides lie on one straight line.

The points can be taken in any order (Fig.1). The theorem can be proved by using "cross-ratios" (which I shall not explain here) and for the simplest of the conics – the circle. By projection

---

† A professional organisation of mathematicians, based in the U.K.. This article is excerpted from Dame Kathleen's presidential address, entitled *The Magic of Mathematics*, and is reproduced, with permission, from the Institute's *Bulletin* (Vol. 15, No.1, Jan. 1979).

†† 1977.

See *Function*, Vol.1, No.1.

this establishes its truth for all conics as cross-ratios remain invariant under projection and all conics are projections of one another.

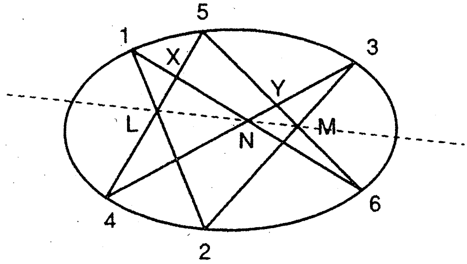


Figure 1

The result is astonishing, beautiful and of great generality, and the proof is elegance itself. To add to the magic, 144 years after the death of Pascal in 1662, another Frenchman, C.W. Brianchon, when still a student, discovered by means of the "principle of duality" a related theorem.

Brianchon's theorem states that:

If a hexagon formed by six straight lines is such that they are tangents to a conic, then the three lines joining opposite vertices intersect at a point (Fig.2).

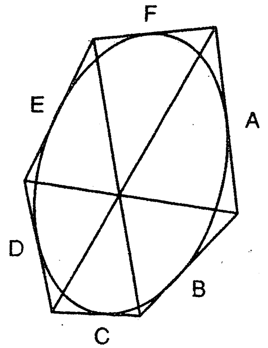


Figure 2

The figures for these two propositions do not look alike, which emphasises the power of the particular method of proof employed, in which one result is deduced from the other.

I had thought to show you several classical proofs of special beauty. I shall ration myself to just two, the first in probability which I came across for the first time only recently. This is known as *Buffon's Needle Problem*. In a Sultan's palace the floors were tiled to give thin parallel lines, narrowly spaced at a distance *d* apart. The ladies of the harem would keep dropping their embroidery needles. The Sultan decided to lay bets on the probability of a needle when dropped crossing a line. If the length of the needle is ℓ, $\ell < d$, what odds was he to lay to be sure over time of being the winner? Buffon's solution involves the multiplication of two probabilities and a difficult integration, the answer being $2\ell/\pi d$. Buffon died in 1788. More than a hundred years later another Frenchman, Rabier, provided a marvellously simple method of arriving at the same solution. I use Rabier's description as quoted in Coolidges's "Mathematics of the Great Amateurs". "The probability of the needle crossing a line is the expectation of a man who is to receive one crown if a crossing takes place. This expectation is the sum of the expectations of the various elements of the needle, and these are unaltered if the needle is bent into a circle. The probability of a crossing is now the ratio of the diameter of this circle, namely ℓ/π, to d, the distance between the lines. But if the bent needle crosses a line once, it will cross it twice, so the expectation is $2\ell/\pi d$" (Fig.3). I had a real thrill out of this and I have revelled in it

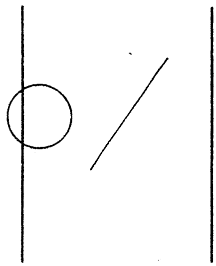


Figure 3

– a difficult problem until Rabier thought of bending the needle into a circle, rather like Christopher Columbus and the egg.

The second example is simpler. In the February 1978 *Bulletin*† Professor Patterson of Aberdeen, at the end of an article about "Mathematical Challenge," the competition initiated in Scotland for sixth-formers, put forward a problem. I quote: "Consider the game of noughts and crosses. In how many ways can a line of three noughts or three crosses be achieved? The answer is eight. There is a three-dimensional version (trade-named Plato). This has 27 holes and two players, each with a set of coloured marbles, who place these in turn in the holes. The winner is the player who achieves, when all the holes are filled, the larger number of rows-of-three in any direction. Question: How many different ways are there of achieving a row of three?" The makers of the game give an incorrect answer 48, whereas by counting carefully and remembering the diagonals we can check that there are 49. The problem Professor Patterson posed was to extend this to n dimensions and to give the number of rows-of-three in an

---

† The *Bulletin of the Institute of Mathematics and its Applications*.

n-dimensional hypercube. Several people sent the correct answer, most by inelegant methods or in inelegant form, but a proof sent by Dr David Singmaster, which it later transpired had already been given in 1941 in the *Scientific American*, has a delightful simplicity.

Consider first the two-dimensional noughts-and-crosses matrix (Fig. 4). There are nine positions represented here by the nine little circles or "holes". The three horizontal rows of holes,

```
+ + + + +
+ o o o +
+ o o o +
+ o o o +
+ + + + +
```

three vertical rows and two diagonals, give the eight possible rows of three. Now surround this matrix of holes with a boundary of crosses (shown here as plus signs). There must plainly be 16 crosses, that is $5^2 - 3^2 = 25 - 9$. Then, through any particular cross, say the cross in the first column and second row, one and only one line can be drawn which passes through three holes. Moreover, this line will pass through one and only one other cross. This is true for each of the crosses. It follows that the number of rows-of-three for the holes of the original matrix is $\frac{1}{2}(5^2 - 3^2) = 8$ as we already know. In exactly the same way in the three-dimensional game Plato, the 27 holes can be surrounded by a boundary of $(5^3 - 3^3) = 98$ crosses, and precisely the same argument establishes that the number of rows-of-three which can be achieved with the 27 holes is $\frac{1}{2}(5^3 - 3^3) = 49$. Moreover, the argument can be extended to give $\frac{1}{2}\{(k + 2)^n - k^n)\}$ for the number of rows of k in an n-dimensional k-hypercube. When $k = 4$ which is another well-known form of the game, there are 76 rows of four where $n = 3$.

* * * * *

WEIGHTS AND MEASURES AND THE OVERFILL PROBLEM

Neil S. Barnett, Footscray Institute of Technology

In order to protect consumers, every country, at least in the Western world, has its legislation on "weights and measures". This legislation is to ensure that if a producer stamps on his product that it weighs 300 grams (for example), then the purchaser has redress at law if the product weight varies appreciably from this. Meeting weight (or volume) specifications exactly is rarely possible; neither is it a reasonable expectation, given the variety and quantity of products that our consumer society demands. Nonetheless, some measure of fairness needs to be defined and enforced by law if the purchaser is not to be exploited.

It is a fact of life that all things vary and so, even with the best of intentions, if a producer doesn't have clear information about the extent of the variation of his products and processes, he may well be, unwittingly, selling his customers short. Weights and measures legislation exists to protect against unscrupulous operators and also to impress on producers the need to monitor the variation of production, so that they don't inadvertently fail to meet their commitment to customers.

The unscrupulous operator, if he is indeed aware of specific weights and measures legislation, will balance the risk of being caught breaching the legislation, and the related penalty, with the profit he can make in the meantime. The genuine producer, however, will consider the variation that he has to contend with, consider the legislation and stamp his product accordingly. If he is wise, he will look closely at the variation he has, assess what it is costing him and seek to reduce it. In so doing, he will continue to meet legislation requirements, but at a lower cost, and as a result his company will be more profitable and his product more competitively priced.

There is always a risk that a producer, whether unscrupulous or genuine, will breach weights and measures legislation. The honest producer would like this risk to be small. The unscrupulous producer must balance the extra profit he gets from breaking the law with the size of the penalty if he happens to be caught and must also take into account the cost of bad publicity, should he be caught.

Suppose that a company is producing jars of jam, stamped 200 g. The company knows that if it sets its filling machine to 210 g rarely, if ever, will the actual weight fall below 200 g; however, many jars will contain 215 g or more. This variation can be due to the operation of the filling machine and also to the weights of jars. The extra jam it "gives away" is a cost to the company. If it could improve the performance of the filling machine so that it could be set at 205 g and almost never produce jars actually less than 200 g or greater than 210 g, then the customer would have the same assurance of quantity, but at less cost to the company. This saving the producer can either pass on to the consumer as a reduction in price (and hence be more competitive) or retain as additional profit. One cannot over-emphasise the importance of competitiveness (not only in price) of fast-turnover consumer products.

Recently I was asked to examine the problem of overfill in relation to weights and measures legislation for a company producing various foods marketed in cans, jars and packets. They were fully aware of the relevant weights and measures legislation and were keen both to serve their customers well and to avoid "giving away" product because of excessive variation in their filling process. My initial action was to examine closely the implications of the weights and measures legislation in force in Victoria and to perform some simple calculations similar to the following.

Weights and measures legislation is uniform throughout Australia. The established practice of any weights and measures inspector is to obtain and sample at random twelve articles of the same product at any one time. To conform with the necessary requirements:

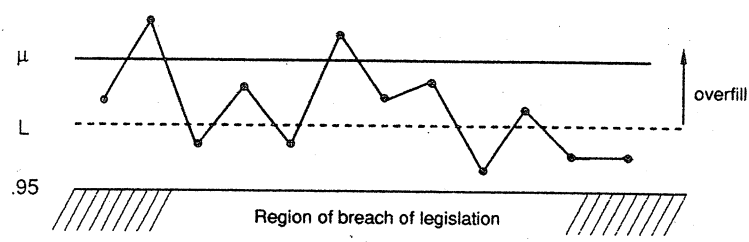
(i) no single item of these twelve shall be less than 95% of the stated label weight;

and

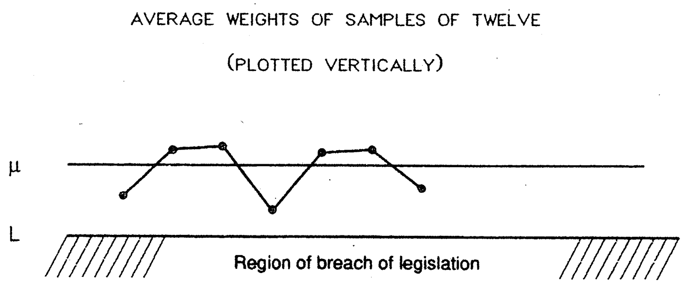
(ii) the average weight of these twelve items must be at least equal to the stated label weight.

The legislation is a deal more detailed than this, but the above requirements (i) and (ii) are all that need concern us here. A couple of diagrams help illustrate these two requirements, but it should be remembered that unless every container is weighed, and topped up if found to be underweight, there is no 100% guarantee that weights and measures legislation will not be breached. The aim is to make the risk, in terms of a probability, small.

INDIVIDUAL WEIGHTS (PLOTTED VERTICALLY)



μ is the setting of the filling machine,
L is the label weight on the filled container.



Provided there is a degree of stability in the operation of the filling machine, for any given setting, the probability of a sample of twelve breaching the legislation can be calculated. By the "stability" of the filling operation we mean that the process variability can be estimated, usually by calculating the standard deviation of the weights of a substantial sample of filled containers. If this stability is not present, then the risk of breach of legislation (as measured by a probability) is an unknown factor. We assume that the weights found are normally distributed, with mean μ, when the filler is set at μ, and standard deviation σ. (It is assumed that weight variation of final product is a consequence mainly of variation of the filler and not the containers – this can easily be checked if there is some doubt.)

Having made a decision on μ, the setting of the filler, then the probability† of breaching legislation with individual weights X is $1 - [Pr(X > \cdot 95L)]^{12}$ and of breaching with the average weight $\overline{X}$ is $Pr(\overline{X} < L)$. If we denote these two occurrences by S and A, respectively, then the probability of breaching legislation is given by $Pr(S \cup A)$, which is $Pr(S) + Pr(A) - Pr(S \cap A)$.

With a setting of μ on the filler, the average overfill per container is $\mu - L$. Suppose that in an effort to fill a 440 g container the filler is set at 442 g and that a reliable estimate of the filling standard deviation under stable conditions is 3. Then

$$Pr(S) = 1 - [Pr(X > \cdot 95 \times 440)]^{12},$$

and by using an appropriate computer program is found to be approximately 12×10^{-9}. The risk on the mean, however, is much larger: $Pr(\overline{X} < 440)$ turns out to be approximately 0·01. Thus the total risk, allowing for the two approximations, is not more than

† We denote the probability of the occurrence of A by $Pr(A)$.

$$7{\cdot}69 \times 10^{-15} + 0{\cdot}01$$

$$= 0{\cdot}01, \text{ to two decimal places,}$$

i.e. of the order of 1 chance in 100.

Suppose now the filler setting is dropped back to 441 g in order to reduce overfill. $P(S)$ doesn't change appreciably, being approximately $1{\cdot}094 \times 10^{-13}$. The mean risk, however, becomes approximately

$$= 0{\cdot}125.$$

As before, this latter is the dominant term, and the risk of breaching weights and measures legislation is approximately 0·125, i.e. about 12 chances in 100, considerably greater than before; but now the average overfill per container is reduced by 50%.

Just what constitutes an acceptable risk must be a company policy decision. Suppose that, with a current σ value of 3, a risk of the order of 1 in a 100 is deemed acceptable. Where the filling process is stable, attempts can be made to reduce σ whilst maintaining the level of risk, the gain being a reduction in average overfill per can. Suppose a company goal is set to reduce σ to 2.5. Then in order to retain the current level of risk we can calculate that the filler setting may be taken at $\mu = 441{\cdot}67$. The average overfill per can is then reduced by 0·33 g. This may seem a fairly insignificant amount. However, suppose the product costs 1¢ per g to manufacture and that the company produces 500,000 containers of this size annually. The average annual saving would amount to $1650. The company may well use the same filler for perhaps fifteen other sizes and products at a comparable overfill and production rate. Clearly, then, by "sharpening" the filling variability tens of thousands of dollars might be saved annually on overfill alone.

Finally, it is worth pointing out that, when considering what constitutes an acceptable risk, it is the risk of breaching legislation when the product is tested by weights and measures inspectors that has been estimated. A chance of one in a hundred of breaching the law thus translates into an average rate of one offence in every one hundred check weighings by weights and measures inspectors. One should thus take into account how often inspections may reasonably be expected.

* * * * *

A PILOT'S LOT

G.A. Watterson, Monash University

After the Australian airline pilots resigned their jobs, the airline companies placed advertisements in the newspapers telling us about the incomes and flying hours (for the year ending June, 1989) of 10 Captains and 10 First Officers, randomly chosen. The data are reproduced below.

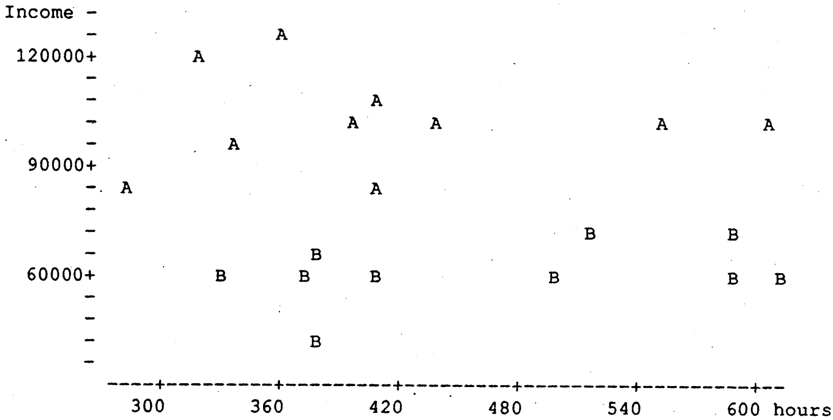
	Year's Income $	Year's flying hours	Average weekly stick hours
Captain A	103,055	439.4	9.6
Captain B	84,123	280.3	6.1
Captain C	104,861	607.5	13.2
Captain D	103,268	551.0	12.0
Captain E	97,610	333.8	7.2
Captain F	126,528	362.4	7.9
Captain G	85,498	409.6	8.9
Captain H	99,385	395.3	8.6
Captain I	120,999	316.2	6.9
Captain J	106,896	409.0	8.9

	Year's Income $	Year's flying hours	Average weekly stick hours
First Officer A	61,270	373.5	8.1
First Officer B	72,121	588.2	12.7
First Officer C	60,796	409.2	8.9
First Officer D	62,115	585.3	12.7
First Officer E	65,798	380.8	8.3
First Officer F	59,726	610.2	13.3
First Officer G	74,740	514.4	11.2
First Officer H	42,425	377.7	8.2
First Officer I	60,814	329.0	7.2
First Officer J	61,844	496.5	10.8

How might we react to such data?

My first calculation was designed to find out how many weeks a pilot worked in a year. It is easy to check that if we divide their year's flying hours by 46 we get the quoted average weekly "stick" hours (with two exceptions, which might be just minor rounding off errors). So I conclude that all pilots worked 46 weeks per year, leaving 6 weeks for holidays.

I then asked my computer to plot the pilots' incomes *versus* their year's flying hours. It produced this (not very accurately plotted) graph.



A = Captains   B = First officers

What is immediately clear is that Captains earn rather more money than First Officers. It is also clear that Captains' incomes cannot be easily related to their flying hours. I asked the computer to fit a straight line, as best it could, through the *A* symbols. The result was the line with equation:

$$\text{Captain's Income} = 100{,}944 + 5.5 \times \text{hours}.$$

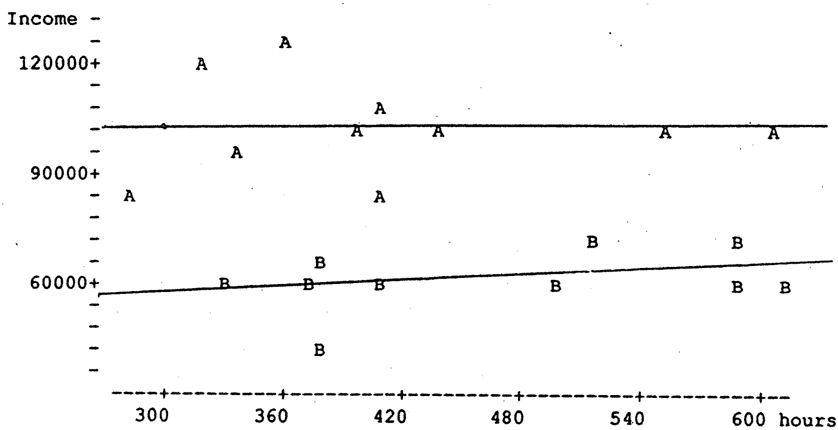
Notice that this mathematical model would predict that a captain with *no* flying hours would earn $100,944, while one with 600 hours would earn $104,244. The difference between the two is hardly worth bothering about – and in fact is not "statistically significant". The data is consistent with the theory that Captains' incomes vary *randomly* around $103,222 (the average of the 10 Captains' incomes), irrespective of hours flown. (The variation might be partly explained by age, years of service, qualifications, etc..)

The incomes of First Officers have a best-fitting straight line:

$$\text{First Officer's Income} = 46{,}888 + 32.8 \times \text{hours}.$$

While this does suggest that their incomes are more dependent on flying hours, yet even here, the data are consistent with the model that First Officers' incomes vary *randomly* around their mean income, $62,165, irrespective of flying hours.

The following Figure shows the data, together with the two lines superimposed.



A = Captains   B = First officers

What with six weeks' holidays, and pay little influenced by actual flying hours, a "pilot's lot was definitely a happy one".

* * * * *

Comment on the pilot problem by a pilot's wife on an ABC talk-back programme:

"The pilots all think that much the best way to solve the problem would be, instead of advertising abroad for new pilots, to advertise abroad for a new prime minister."

* * * * *

ROUNDING UP, DOWN AND OFF

Michael A.B. Deakin, Monash University

Some years ago, an employee of a U.S. bank made a lot of money for himself. He noted that most bank calculations do not work out exactly. For example, if an account holds (say) \$980.16 for a year at 20% p.a., compounded annually, there is due to be paid into it at the end of the year a sum of \$196.032. Of course, the amount actually credited is \$106.03. What our enterprising employee did was to instruct the computer to direct the "spare" 0.2 of a cent into his own account.

Now 0.2 of a cent is hardly a thing on which to build a fortune. Or so we might think. But the employee had the bank's computer do this with *every one* of its calculations. (He also programmed it to erase, when it had finished, all records of this transfer of funds, except in the growing size of his own account.) So for every calculation undertaken, the chap scored, on average, half a cent. By the time he was found out, he had made several million dollars.

The effect of such roundings is to alter the amount that we find for compound interest on investments. For example, if we invest a *principal*, P, of \$1000 for a *number*, N (say 30) of years at a rate R % per annum, compounded annually (here take $R = 20$, say), we have at the end an amount $A(N)$ given by

$$A(N) = P\left(1 + \frac{R}{100}\right)^N. \tag{1}$$

In our case, this works out to be \$237,376.31.

But if at the end of each year, we round down to the nearest cent, the amount actually credited turns out to be \$237,373.94, some \$2.37 less. This is hardly a matter of great concern, but some years ago I had some fun looking for a formula which might estimate these small corrections.

Similar things happen if at each stage we round up to the nearest cent. In this particular case, we get \$237,378.41, an additional \$2.10, again a drop in the ocean compared with nearly a quarter of a million dollars, but nonetheless not entirely zero.

Even if we round off to the nearest cent, using a 5/4 convention (that is to say, 1.49...¢ becomes 1¢, but 1.50...¢ becomes 2¢), we do not get the precise answer (except in rare cases). In the example given above, this procedure yields \$237,376.63, close to the value given by Equation (1) but nonetheless 32¢ off.

Now, again, nobody is going to worry unduly about this 32¢ – unless like our enterprising bank officer, you get lots and lots of such amounts.

But it did occur to me that effects like these might be more important with credit card accounts and other situations where interest is computed daily. I began by considering a debt of \$500 on a credit card charging 20%

p.a. (MasterCard have recently gone from 19.2% to 21%, and I chose 20% as a "round" number near these; it was a fortunate choice, as will appear later.) I followed the debt for a month under a regime of daily compounding, using rounding down, rounding up and 5/4 round-off. The results are shown in Table 1.

		COMPOUND INTEREST ROUNDING			
	DAYS	**FORMULA**	**ROUND DOWN**	**ROUND UP**	**5/4 ROUND**
PRINCIPAL	1	\$500.27	\$500.27	\$500.28	\$500.27
\$500.00	2	\$500.55	\$500.54	\$500.56	\$500.54
	3	\$500.82	\$500.81	\$500.84	\$500.81
	4	\$501.10	\$501.08	\$501.12	\$501.08
RATE p.a.	5	\$501.37	\$501.35	\$501.40	\$501.35
20.00%	6	\$501.65	\$501.62	\$501.68	\$501.62
	7	\$501.92	\$501.89	\$501.96	\$501.89
	8	\$502.20	\$502.16	\$502.24	\$502.17
NO. OF DAYS	9	\$502.47	\$502.43	\$502.52	\$502.45
31	10	\$502.75	\$502.70	\$502.80	\$502.73
	11	\$503.02	\$502.97	\$503.08	\$503.01
Factor	12	\$503.30	\$503.24	\$503.36	\$503.29
1.0005479452	13	\$503.57	\$503.51	\$503.64	\$503.57
	14	\$503.85	\$503.78	\$503.92	\$503.85
	15	\$504.13	\$504.05	\$504.20	\$504.13
	16	\$504.40	\$504.32	\$504.48	\$504.41
	17	\$504.68	\$504.59	\$504.76	\$504.69
	18	\$504.95	\$504.86	\$505.04	\$504.97
	19	\$505.23	\$505.13	\$505.32	\$505.25
	20	\$505.51	\$505.40	\$505.60	\$505.53
	21	\$505.79	\$505.67	\$505.88	\$505.81
	22	\$506.06	\$505.94	\$506.16	\$506.09
	23	\$506.34	\$506.21	\$506.44	\$506.37
	24	\$506.62	\$506.48	\$506.72	\$506.65
	25	\$506.89	\$506.75	\$507.00	\$506.93
	26	\$507.17	\$507.02	\$507.28	\$507.21
	27	\$507.45	\$507.29	\$507.56	\$507.49
	28	\$507.73	\$507.56	\$507.84	\$507.77
	29	\$508.01	\$507.83	\$508.12	\$508.05
	30	\$508.28	\$508.10	\$508.40	\$508.33
	31	\$508.56	\$508.37	\$508.68	\$508.61

Table 1

20% p.a. is (20/365)% *per diem* and so each of the entries in the "formula" column is multiplied by a factor of $1 + (20/365)/100$, i.e. 1.000 547 945 2055.., which the computer has stored to ten decimal places and displayed in the left-hand column.

The different rounding conventions lead to small differences in the debt at the end of the month. These are not particularly significant and I did not expect that they would be. To put them in perspective, calculate the *effective* rate of interest each represents. E.g. if we round down, what rate, compounded daily, will give us \$508.37 instead of \$508.56 at the end of

		COMPOUND INTEREST ROUNDING			
	DAYS	FORMULA	ROUND DOWN	ROUND UP	5/4 ROUND
PRINCIPAL	1	\$50.03	\$50.02	\$50.03	\$50.03
\$50.00	2	\$50.05	\$50.04	\$50.06	\$50.06
	3	\$50.08	\$50.06	\$50.09	\$50.09
	4	\$50.11	\$50.08	\$50.12	\$50.12
RATE p.a.	5	\$50.14	\$50.10	\$50.15	\$50.15
20.00%	6	\$50.16	\$50.12	\$50.18	\$50.18
	7	\$50.19	\$50.14	\$50.21	\$50.21
	8	\$50.22	\$50.16	\$50.24	\$50.24
NO. OF DAYS	9	\$50.25	\$50.18	\$50.27	\$50.27
31	10	\$50.27	\$50.20	\$50.30	\$50.30
	11	\$50.30	\$50.22	\$50.33	\$50.33
Factor	12	\$50.33	\$50.24	\$50.36	\$50.36
1.0005479452	13	\$50.36	\$50.26	\$50.39	\$50.39
	14	\$50.38	\$50.28	\$50.42	\$50.42
	15	\$50.41	\$50.30	\$50.45	\$50.45
	16	\$50.44	\$50.32	\$50.48	\$50.48
	17	\$50.47	\$50.34	\$50.51	\$50.51
	18	\$50.50	\$50.36	\$50.54	\$50.54
	19	\$50.52	\$50.38	\$50.57	\$50.57
	20	\$50.55	\$50.40	\$50.60	\$50.60
	21	\$50.58	\$50.42	\$50.63	\$50.63
	22	\$50.61	\$50.44	\$50.66	\$50.66
	23	\$50.63	\$50.46	\$50.69	\$50.69
	24	\$50.66	\$50.48	\$50.72	\$50.72
	25	\$50.69	\$50.50	\$50.75	\$50.75
	26	\$50.72	\$50.52	\$50.78	\$50.78
	27	\$50.75	\$50.54	\$50.81	\$50.81
	28	\$50.77	\$50.56	\$50.84	\$50.84
	29	\$50.80	\$50.58	\$50.87	\$50.87
	30	\$50.83	\$50.60	\$50.90	\$50.90
	31	\$50.86	\$50.62	\$50.93	\$50.93

Table 2

the month? I.e. we solve

$$\$508.37 = \$500\left(1 + \frac{R}{36500}\right)^{31} \qquad (2)$$

for R. The result is 19.55%. Similarly, rounding up corresponds to a rate of 20.27% or 5/4 rounding off to a rate of 20.11%.

It surprised me a little to see this last figure so high; in the earlier case, the $1000 invested for 30 years, the effective rates were 20% in each case, to a very high accuracy and the 5/4 round-off estimation had strayed from the true figure only in a minor, random way.

		COMPOUND INTEREST ROUNDING			
	DAYS	FORMULA	ROUND DOWN	ROUND UP	5/4 ROUND
PRINCIPAL	1	$5.00	$5.00	$5.01	$5.00
$5.00	2	$5.01	$5.00	$5.02	$5.00
	3	$5.01	$5.00	$5.03	$5.00
	4	$5.01	$5.00	$5.04	$5.00
RATE p.a.	5	$5.01	$5.00	$5.05	$5.00
20.00%	6	$5.02	$5.00	$5.06	$5.00
	7	$5.02	$5.00	$5.07	$5.00
	8	$5.02	$5.00	$5.08	$5.00
NO. OF DAYS	9	$5.02	$5.00	$5.09	$5.00
31	10	$5.03	$5.00	$5.10	$5.00
	11	$5.03	$5.00	$5.11	$5.00
Factor	12	$5.03	$5.00	$5.12	$5.00
1.0005479452	13	$5.04	$5.00	$5.13	$5.00
	14	$5.04	$5.00	$5.14	$5.00
	15	$5.04	$5.00	$5.15	$5.00
	16	$5.04	$5.00	$5.16	$5.00
	17	$5.05	$5.00	$5.17	$5.00
	18	$5.05	$5.00	$5.18	$5.00
	19	$5.05	$5.00	$5.19	$5.00
	20	$5.06	$5.00	$5.20	$5.00
	21	$5.06	$5.00	$5.21	$5.00
	22	$5.06	$5.00	$5.22	$5.00
	23	$5.06	$5.00	$5.23	$5.00
	24	$5.07	$5.00	$5.24	$5.00
	25	$5.07	$5.00	$5.25	$5.00
	26	$5.07	$5.00	$5.26	$5.00
	27	$5.07	$5.00	$5.27	$5.00
	28	$5.08	$5.00	$5.28	$5.00
	29	$5.08	$5.00	$5.29	$5.00
	30	$5.08	$5.00	$5.30	$5.00
	31	$5.09	$5.00	$5.31	$5.00

Table 3

What I really had in mind, however, was the investigation of how these effective rates varied with the principal. To this end, I had the computer produce an analogue of Table 1, but for a debt of $50 instead of $500. This produced Table 2, and this does show the effect I was looking for. The effective rate of interest produced by rounding down is now only 14.5%, while that produced by rounding up is now 21.7%. But now look at the 5/4 round-off column. It is exactly the same as the rounding-up column! Surely this can't be simply the result of chance.

Indeed, it isn't. We see why this correspondence occurs if we look at an even simpler case. Table 3 shows the situation with a principal of $5. Now here, if we do the calculation by the formula we find a small amount of interest accruing each day, so that after a month has passed it has amounted to about $8\frac{1}{2}$¢. But if we round down each time, we reset the amount back to $5 and it never has a chance to increase. The effective rate of interest is now zero.

		COMPOUND INTEREST ROUNDING			
	DAYS	**FORMULA**	**ROUND DOWN**	**ROUND UP**	**5/4 ROUND**
PRINCIPAL	1	$9.12	$9.12	$9.13	$9.12
$9.12	2	$9.13	$9.12	$9.14	$9.12
	3	$9.13	$9.12	$9.15	$9.12
	4	$9.14	$9.12	$9.16	$9.12
RATE p.a.	5	$9.15	$9.12	$9.17	$9.12
20.00%	6	$9.15	$9.12	$9.18	$9.12
	7	$9.16	$9.12	$9.19	$9.12
	8	$9.16	$9.12	$9.20	$9.12
NO. OF DAYS	9	$9.17	$9.12	$9.21	$9.12
31	10	$9.17	$9.12	$9.22	$9.12
	11	$9.18	$9.12	$9.23	$9.12
Factor	12	$9.18	$9.12	$9.24	$9.12
1.0005479452	13	$9.19	$9.12	$9.25	$9.12
	14	$9.19	$9.12	$9.26	$9.12
	15	$9.20	$9.12	$9.27	$9.12
	16	$9.20	$9.12	$9.28	$9.12
	17	$9.21	$9.12	$9.29	$9.12
	18	$9.21	$9.12	$9.30	$9.12
	19	$9.22	$9.12	$9.31	$9.12
	20	$9.22	$9.12	$9.32	$9.12
	21	$9.23	$9.12	$9.33	$9.12
	22	$9.23	$9.12	$9.34	$9.12
	23	$9.24	$9.12	$9.35	$9.12
	24	$9.24	$9.12	$9.36	$9.12
	25	$9.25	$9.12	$9.37	$9.12
	26	$9.25	$9.12	$9.38	$9.12
	27	$9.26	$9.12	$9.39	$9.12
	28	$9.26	$9.12	$9.40	$9.12
	29	$9.27	$9.12	$9.41	$9.12
	30	$9.27	$9.12	$9.42	$9.12
	31	$9.28	$9.12	$9.43	$9.12

Table 4

Contrast this with the effect of rounding up. At the end of Day 1, the amount should be $5.0027..., hardly any advance on $5. The accrued interest is just over a quarter of a cent, but we told the computer to round up, so round up it does and it keeps rounding up like this each day for the whole month, so that at the end of the month the interest comes to 31¢. The effective interest rate is a whopping 70.90%!

The 5/4 round-off works just like the round-down. The computer sees the 0.27 of a cent, rightly judges it to be nearer to nothing than to one cent, rounds down to zero and this resets the amount at $5. It does this for each of the succeeding days as well. So no interest ever accrues.

			COMPOUND INTEREST ROUNDING		
	DAYS	**FORMULA**	**ROUND DOWN**	**ROUND UP**	**5/4 ROUND**
PRINCIPAL	1	$9.14	$9.13	$9.14	$9.14
$9.13	2	$9.14	$9.13	$9.15	$9.15
	3	$9.15	$9.13	$9.16	$9.16
	4	$9.15	$9.13	$9.17	$9.17
RATE p.a.	5	$9.16	$9.13	$9.18	$9.18
20.00%	6	$9.16	$9.13	$9.19	$9.19
	7	$9.17	$9.13	$9.20	$9.20
	8	$9.17	$9.13	$9.21	$9.21
NO. OF DAYS	9	$9.18	$9.13	$9.22	$9.22
31	10	$9.18	$9.13	$9.23	$9.23
	11	$9.19	$9.13	$9.24	$9.24
Factor	12	$9.19	$9.13	$9.25	$9.25
1.0005479452	13	$9.20	$9.13	$9.26	$9.26
	14	$9.20	$9.13	$9.27	$9.27
	15	$9.21	$9.13	$9.28	$9.28
	16	$9.21	$9.13	$9.29	$9.29
	17	$9.22	$9.13	$9.30	$9.30
	18	$9.22	$9.13	$9.31	$9.31
	19	$9.23	$9.13	$9.32	$9.32
	20	$9.23	$9.13	$9.33	$9.33
	21	$9.24	$9.13	$9.34	$9.34
	22	$9.24	$9.13	$9.35	$9.35
	23	$9.25	$9.13	$9.36	$9.36
	24	$9.25	$9.13	$9.37	$9.37
	25	$9.26	$9.13	$9.38	$9.38
	26	$9.26	$9.13	$9.39	$9.39
	27	$9.27	$9.13	$9.40	$9.40
	28	$9.27	$9.13	$9.41	$9.41
	29	$9.28	$9.13	$9.42	$9.42
	30	$9.28	$9.13	$9.43	$9.43
	31	$9.29	$9.13	$9.44	$9.44

Table 5

Clearly this is the pattern if the principal is small. For values of P less than $5 the relative discrepancies become even larger. If $P = \$0.01$, we find an effective interest rate for round-up of over 40,000%. But how small is small? When does this effect stop?

To answer this, note that if the interest on the first day is less than half a cent, 5/4 round-off and round-down will reset the amount to zero. The cut-off therefore occurs when

$$P\left(1 + \frac{20}{36500}\right) = P + 0.005. \tag{3}$$

Equation (3) has the exact solution $P = \$9.125$. (The choice of 20 as the value for R is what leads to this relatively simple result.) So the pattern already seen in Table 3 should also appear if $P = \$9.12$, but not if $P = \$9.13$.

		COMPOUND INTEREST ROUNDING			
	DAYS	FORMULA	ROUND DOWN	ROUND UP	5/4 ROUND
PRINCIPAL	1	$18.26	$18.25	$18.26	$18.26
$18.25	2	$18.27	$18.25	$18.28	$18.27
	3	$18.28	$18.25	$18.30	$18.28
	4	$18.29	$18.25	$18.32	$18.29
RATE p.a.	5	$18.30	$18.25	$18.34	$18.30
20.00%	6	$18.31	$18.25	$18.36	$18.31
	7	$18.32	$18.25	$18.38	$18.32
	8	$18.33	$18.25	$18.40	$18.33
NO. OF DAYS	9	$18.34	$18.25	$18.42	$18.34
31	10	$18.35	$18.25	$18.44	$18.35
	11	$18.36	$18.25	$18.46	$18.36
Factor	12	$18.37	$18.25	$18.48	$18.37
1.0005479452	13	$18.38	$18.25	$18.50	$18.38
	14	$18.39	$18.25	$18.52	$18.39
	15	$18.40	$18.25	$18.54	$18.40
	16	$18.41	$18.25	$18.56	$18.41
	17	$18.42	$18.25	$18.58	$18.42
	18	$18.43	$18.25	$18.60	$18.43
	19	$18.44	$18.25	$18.62	$18.44
	20	$18.45	$18.25	$18.64	$18.45
	21	$18.46	$18.25	$18.66	$18.46
	22	$18.47	$18.25	$18.68	$18.47
	23	$18.48	$18.25	$18.70	$18.48
	24	$18.49	$18.25	$18.72	$18.49
	25	$18.50	$18.25	$18.74	$18.50
	26	$18.51	$18.25	$18.76	$18.51
	27	$18.52	$18.25	$18.78	$18.52
	28	$18.53	$18.25	$18.80	$18.53
	29	$18.54	$18.25	$18.82	$18.54
	30	$18.55	$18.25	$18.84	$18.55
	31	$18.56	$18.25	$18.86	$18.56

Table 6

A glance at Tables 4, 5 shows that this is indeed the case. A further glance at Table 5 shows us something else. The 5/4 round-off column now follows the round-up, whereas before it followed the round-down.

This is because, now we have passed the critical threshold of \$9.125, the small discrepancy is larger than half a cent, and so the 5/4 rule means that we round up.

This pattern should continue until we reach a principal of 2 × \$9.125, i.e. \$18.25. The interest for one day at 20% p.a. on \$18.25 is precisely one cent, and so now the 5.4 rule rounds down and the round-down rule will actually give some interest.

		COMPOUND INTEREST ROUNDING			
	DAYS	FORMULA	ROUND DOWN	ROUND UP	5/4 ROUND
PRINCIPAL	1	\$18.27	\$18.27	\$18.28	\$18.27
\$18.26	2	\$18.28	\$18.28	\$18.30	\$18.28
	3	\$18.29	\$18.29	\$18.32	\$18.29
	4	\$18.30	\$18.30	\$18.34	\$18.30
RATE p.a.	5	\$18.31	\$18.31	\$18.36	\$18.31
20.00%	6	\$18.32	\$18.32	\$18.38	\$18.32
	7	\$18.33	\$18.33	\$18.40	\$18.33
	8	\$18.34	\$18.34	\$18.42	\$18.34
NO. OF DAYS	9	\$18.35	\$18.35	\$18.44	\$18.35
31	10	\$18.36	\$18.36	\$18.46	\$18.36
	11	\$18.37	\$18.37	\$18.48	\$18.37
Factor	12	\$18.38	\$18.38	\$18.50	\$18.38
1.0005479452	13	\$18.39	\$18.39	\$18.52	\$18.39
	14	\$18.40	\$18.40	\$18.54	\$18.40
	15	\$18.41	\$18.41	\$18.56	\$18.41
	16	\$18.42	\$18.42	\$18.58	\$18.42
	17	\$18.43	\$18.43	\$18.60	\$18.43
	18	\$18.44	\$18.44	\$18.62	\$18.44
	19	\$18.45	\$18.45	\$18.64	\$18.45
	20	\$18.46	\$18.46	\$18.66	\$18.46
	21	\$18.47	\$18.47	\$18.68	\$18.47
	22	\$18.48	\$18.48	\$18.70	\$18.48
	23	\$18.49	\$18.49	\$18.72	\$18.49
	24	\$18.50	\$18.50	\$18.74	\$18.50
	25	\$18.51	\$18.51	\$18.76	\$18.51
	26	\$18.52	\$18.52	\$18.78	\$18.52
	27	\$18.53	\$18.53	\$18.80	\$18.53
	28	\$18.54	\$18.54	\$18.82	\$18.54
	29	\$18.55	\$18.55	\$18.84	\$18.55
	30	\$18.56	\$18.56	\$18.86	\$18.56
	31	\$18.57	\$18.57	\$18.88	\$18.57

Table 7

Table 6 is the computer's attempt to verify this prediction, and we see immediately that in the "Round-down" column, something has gone badly wrong. This shows that you can't always trust computers. The problem is the factor 1 + 20/36500. This is 1.000 547 945 2055..., but the computer carried only the first ten decimal places. Thus, it "thought" the interest was 0.999 998 996 250 and, seeing a number less than one, even if only by a cat's whisker, it did as it was told and rounded it down to zero.

I overcame this problem, by kick-starting the calculation (Table 7): I added the one cent interest for the first day to the 18.26, before starting the computer. The correct reading for Day n in Table 6 may be discovered by looking up the entry for Day $(n-1)$ in Table 7. All the features we predicted for Table 6 show up as they ought to in Table 7.

		COMPOUND INTEREST ROUNDING				
	DAYS	FORMULA	ROUND DOWN	ROUND UP	5/4 ROUND	
PRINCIPAL	1	$1000.55	$1000.54	$1000.55	$1000.55	
$1000.00	2	$1001.10	$1001.08	$1001.10	$1001.10	
	3	$1001.64	$1001.62	$1001.65	$1001.65	
	4	$1002.19	$1002.16	$1002.20	$1002.20	THRESHOLD
RATE p.a.	5	$1002.74	$1002.70	$1002.75	$1002.75	PASSED
20.00%	6	$1003.29	$1003.24	$1003.30	$1003.30	HERE
	7	$1003.84	$1003.78	$1003.85	$1003.85	←
	8	$1004.39	$1004.33	$1004.41	$1004.40	
NO. OF DAYS	9	$1004.94	$1004.88	$1004.97	$1004.95	
31	10	$1005.49	$1005.43	$1005.53	$1005.50	
	11	$1006.04	$1005.98	$1006.09	$1006.05	
Factor	12	$1006.60	$1006.53	$1006.65	$1006.60	
1.0005479452	13	$1007.15	$1007.08	$1007.21	$1007.15	
	14	$1007.70	$1007.63	$1007.77	$1007.70	
	15	$1008.25	$1008.18	$1008.33	$1008.25	
	16	$1008.80	$1008.73	$1008.89	$1008.80	
	17	$1009.36	$1009.28	$1009.45	$1009.35	
	18	$1009.91	$1009.83	$1010.01	$1009.90	
	19	$1010.46	$1010.38	$1010.57	$1010.45	
	20	$1011.02	$1010.93	$1011.13	$1011.00	
	21	$1011.57	$1011.48	$1011.69	$1011.55	THRESHOLD
	22	$1012.12	$1012.03	$1012.25	$1012.10	PASSED
	23	$1012.68	$1012.58	$1012.81	$1012.65	HERE
	24	$1013.23	$1013.13	$1013.37	$1013.20	←
	25	$1013.79	$1013.68	$1013.93	$1013.76	
	26	$1014.34	$1014.23	$1014.49	$1014.32	
	27	$1014.90	$1014.78	$1015.05	$1014.88	
	28	$1015.46	$1015.33	$1015.61	$1015.44	
	29	$1016.01	$1015.88	$1016.17	$1016.00	
	30	$1016.57	$1016.43	$1016.73	$1016.56	
	31	$1017.13	$1016.98	$1017.29	$1017.12	

Table 8

Table 8

Thus a pattern becomes established: after each \$9.125, the 5/4 column reverses direction, rounding up where it used to round down or *vice versa*. The computer, doing a little rounding of its own, will misbehave for any multiple of \$18.25, although not as drastically as in Table 6. This now explains the feature that we first noted with Table 2. The amounts here all lie in the range between 5 × \$9.125 (= \$45.625) and 6 × \$9.125 (= \$54.75) and as this is a "round-up" region, that is what we see.

Now look again at Table 1. Up to Day 7, the 5/4 rounding produces the same result as rounding down. But 55 × \$9.125 = \$501.875 and this threshold is crossed between Day 6 (\$501.62) and Day 7 (\$501.89). So now the 5/4 process begins to round up, and it continues to do so till the end of the table, because the next threshold (\$511.00) is not reached before the end of the month.

Finally, look at Table 8. Here I took the case P = \$1000 and I had the computer point out where, now, two thresholds had been passed.

As P gets bigger and bigger, the likelihood of passing a threshold increases. The number of thresholds to be found within a run of 31 days goes up accordingly. This means that the patterns become less obvious and we get a trend to the random behaviour noted in the first example – the one with annual compounding.

You might like to explore the behaviour more. There is a lot more to discover. When P gets bigger and bigger, how many thresholds do we expect to see? What if we pass two thresholds in a day? How big would P have to be for this to occur? What about three? And so on.

Finally, I should mention that these patterns occur because we are taking powers of a factor only just a little bigger than 1. That is to say, we are looking at expressions of the form $(1+\epsilon)^N$ where ϵ is a small number. The smallness of ϵ is, however, offset by the fact that we are using quite high values of N, going up to 31 in our case.

The study of such expressions in these circumstances is precisely what led to the study of exponential functions. In particular, if we take $\epsilon = 1/N$ and let N get larger and larger, we approach the number e, the base of the natural logarithms. Thus a study of compound interest leads naturally to that elusive number.

In fact, one dollar, invested at 100% p.a., but compounding all the time, produces e dollars at the end of the year. I did this with daily compounding. One dollar, invested at 100%, compounded daily, gives $\left(1+\frac{1}{365}\right)^{365}$ dollars at the end of the year. This is \$2.71456... . Compare this with the value of e.

* * * * *

A heritage of the French revolution: the long and tortuous history of the metre†

by S. Trompler

A universal measure of length taken from nature is, of all the good effects which have come to us from the French Revolution, that which cost us the least, and if this great change initially ran into opposition this was solely because of an attitude of inertia and laziness which always begins by ridiculing new and useful ideas.

Jean Baptiste Delambre

The need and the desire for a universal measure of length is very old. The more people move from one place to another, the greater the inconvenience of having different units of measure proves to be.

At the end of the 18th century units of length and weight varied not only from one country to another but equally within each country from one region to another.

CAESAR already sought to impose on Gaul the Roman laws of weights and measures and arranged for the setting up of standard lengths and weights in certain sacred places and punished severely the makers of false measures; CHARLEMAGNE also established standard weights and measures, kept at the Palais Royal, and declared that identical measures should be used throughout his empire. But rebellious lords effectively annulled these decisions, each one making it a point of honour to have his own units of measure.

Many different attempts have been made at establishing a uniform system, but all foundered. We shall speak only of the most recent.

Two natural units of length co-existed in the second half of the 17th century: the length of a pendulum with a period of one second and a fraction of the length of a meridian.

Scholars were particularly attracted to the pendulum: its length is easy to establish at any time and in any place. It would suffice to determine the connexion between this length and other units used, in order to control and define them.

The length of the terrestrial meridian, made in ancient times by ERATOSTHENES, was made again and improved in 1670 by Abbot PICARD (he measured the meridian arc between Paris and Amiens). Abbot MOUTON was the

---

† Translated from *Math-Jeunes*, No. 45, 2.1, 1989, pp.3-6. [*Math-Jeunes* is a Belgian journal, which began publication 10 years ago, and with which *Function* has reciprocal translation rights.]

first to propose a decimal division: he took a meridian length as starting point and subdivided it until he found a manageable length; he called the length of one minute of arc on the meridian a *milliare*. This unit is the nautical mile, known previously, but the idea of applying it to all measures of length and from there extending it to surfaces and volumes was new.

The length of a pendulum which has a period of one second served as a control.

But alas, in 1673 RICHER discovered that a pendulum with a period of one second was shorter at Cayenne than it was at Paris! *"No universal pendulum, no universal measure of length offered by Nature to man"*, exclaimed MOUTON.

LA CONDAMINE took up the torch, and it is he who is the true originator of the metric system. Since the pendulum measure varies from place to place on the earth's surface, it is necessary to fix a particular place.

In 1747, he made his measurement at Quito, on the equator, and forged there a bronze rod of the same length as the pendulum, and set it in marble on which were engraved these words: *"Model of a simple equinoxial pendulum having, at ground level at Quito, a period of one second, prototype of a universal unit of length; may it also be universal"* (this rod is still in place at Quito).

But he died in 1774 without his proposal for universality being accepted and adopted. Next, CONDORCET took up the challenge, but he preferred latitude 45° because *"Its situation in the middle of countries where science flourishes allows one to check the standard length easily and also as often as one wishes"*. The 45° parallel passes near to Bordeaux and in 1775 the (government) Minister TURGOT ordered a start to be made.

Unfortunately a very accurate pendulum is needed and that of the observatory was not working properly. By the time another had been constructed TURGOT was in disgrace: once more the project was abandoned.

Was this the end of the attempt? No. TALLEYRAND elaborated a new project for which he sought the collaboration of England. The reason for this was that the Royal Society of London, established before the Paris Academy of Sciences, would almost certainly not recognize any research carried out without its involvement. Without England accepting the new measure the hope for universality would fail. Moreover, the two organizations had just collaborated in drawing up new maps of their countries and England was thinking of reforming its own system of measure. This was therefore the right time.

RIGGS MILLER presented TALLEYRAND's project, a little modified, to the Chamber of Communes in 1790, taking as base the length of the London pendulum. The French agreed. All was well set, but ... a squabble between England and Spain, an ally of France, was threatening plans. England let things drag and, after an election, RIGGS MILLER was no longer a member of the Chamber of Communes. The project was buried once again.

However, Spain and the United States began to favour the enterprise. So much the worse for England, France would act without her and, in 1791, the Academy of Science proposed a new measure based upon the terrestrial meridian. The principal promoters were CONDORCET, LAGRANGE, LAPLACE, MONGE, COULOMB, DELAMBRE, LAVOISIER. What a collection of sages! How could one fail to succeed with men so eminent!

In 1792 DELAMBRE and MECHAIN set out to measure the length of the meridian from Dunkirk to Barcelona. DELAMBRE writes: "*MECHAIN measured the length from Barcelona to Rodez as being 170,000 toises*† . *I counted the length from Rodez to Dunkirk as being 380,000 toises*".

But that was not all. The local inhabitants place no confidence in their efforts: "*They recognised us, remembered that we had wished to place a marker on the top of Montjai tower, they led us to the top, they dragged us across fields in teeming rain ...*".

Nevertheless the work advances. They are on the right path.

Alas, on 8 August, 1793, academies are suppressed. Happily, the Temporary Commission of Weights and Measures continues functioning. But not for long: LAVOISIER is arrested on 28 November. DELAMBRE recounts: "*The Commission have asked the Committee for Public Safety to allow LAVOISIER, accompanied each day by a gendarme, to continue the work he has begun. Such favours have been granted rather widely to much less celebrated men and for reasons more or less trivial.*"

The Committee for Public Safety reply to their petition: "*considering how much being delegated an official task or given a mission contributes to improving one's happiness, the government has decided that such tasks will be given only to those worthy of trust because of their staunch support of the Republic and their hatred for kings; with the agreement of the members of the Committee of Instruction specially concerned with weights and measures, we decree that BORDA, LAVOISIER, LAPLACE, COULOMB, BRISSON and DELAMBRE cease, starting from today, to be members of the Commission ...*".

Happily, revolutionary opinion evolves, those in command change and, without awaiting the termination of these decrees (they will be annulled only in 1798), the law of 7 April 1795 institutes a metric decimal system in France, determines the nomenclature for the new units of length (mètre), of area (are), of volume (litre), and of weight (gramme); these units will be subject to later change, and various standard units, starting in 1799, will be constructed with increasing care for exactness. The definition of mètre will vary several times and it has been fixed since 1983 as the length travelled in a vacuum by light in $\frac{1}{299792458}$ of a second.

Having come to the end of our epic on the metre, we can perhaps end with the words of the unfortunate LAVOISIER: "*Nothing greater, more simple, or more coherent in all its parts has been devised by man.*"

REFERENCES

[1] J.B. DELAMBRE, The foundation of the decimal metric system, *Mémoires de l'Institut de France*, Paris, 1806.

[2] A. FAVRE, The origins of the metric system, *P.U.F.*, 1931.

[3] H. MOREAU, The metric system, *Nature*, 1948.

† An ancient French measure of length 6 feet. (The foot was a measure of length in both England and France in the 18th century.)

INDEX, VOLUME 13