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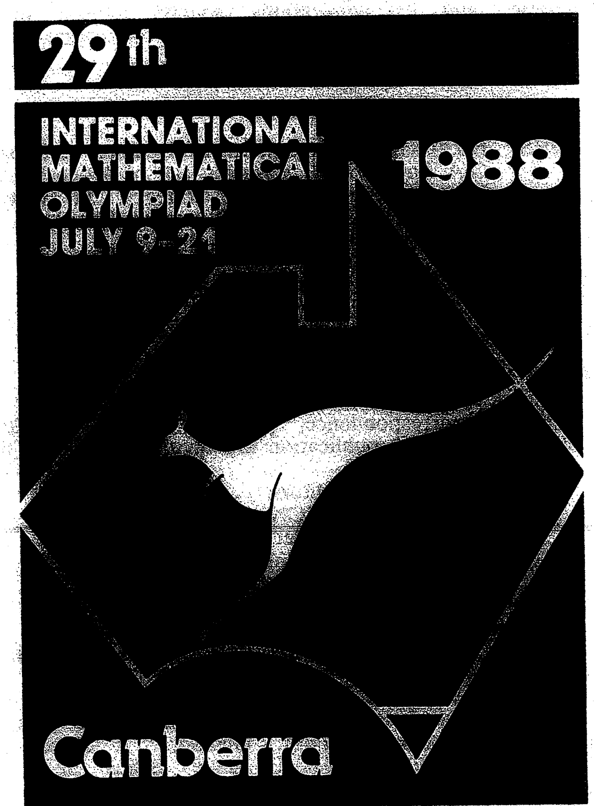

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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of schools, and published by Monash University.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

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This issue opens with Geoff Ball's concise history of the International Mathematical Olympiads and a report by Judith Downes on a budding mathematical video-project for the Australian Mathematical Olympiad. Very Australian are also Frank Hansford-Miller's numbers. Leigh Thompson proposes exercises, some of them endless, for personal computers. In the issue's vital-statistical segment, first G.A. Watterson considers links between AIDS, a virulent disease, and Thomas Bayes (died 1761), a Protestant clergyman; then Peter Kloeden discusses sweet peas, a species of legumina, and Gregor Mendel, a Catholic monk.

CONTENTS

THE FRONT COVER

FUNCTION is pleased to display the official logo, signifying the International Mathematical Olympiad, and welcomes the top mathematicians amongst secondary students throughout the world who are joining with the Australians in the country's celebrations. On this occasion FUNCTION salutes its readers in the true mathematical spirit that transcends geographical and man-made boundaries.

A team of editors will accompany the events in Canberra.

THE INTERNATIONAL MATHEMATICAL OLYMPIADS : HISTORICAL NOTES

Geoff Ball[1], University of Sydney

The first International Mathematical Olympiad was held in Brasov, Rumania, in 1959 with teams from Bulgaria, Czechoslovakia, Hungary, Poland, the Soviet Union and of course Rumania attending. This humble beginning saw a total of only 52 contestants to whom 3 first, 3 second and 5 third prizes were awarded. With the exception of 1980, an IMO has been held every year since, with 14 different countries hosting the event throughout that period. By 1987, at the 28th IMO in Cuba, the number of participating countries had increased to 42 with 237 participants – a truly international event. Of these participants, 22 received first prizes, 42 second prizes and 56 third prizes.

These statistics illustrate two of the developments that have occurred during the past thirty years. First, more countries have become involved in the organization. The rate at which new countries have entered has varied significantly. The first four Olympiads saw no new countries involved while for each of the next 14 years, the average number of countries participating for the first time was between one and two. In 1964, Mongolia became the first non-European country involved, followed by the first Western hemisphere country, Cuba, in 1971. Since 1981, when 6 newcomers including Australia were invited to attend, all continents have participated.

As more countries have become involved, there has been a corresponding variation in the location of the event. Canberra is the 25th different city to have been the venue of an IMO, and it is the first in the Asian-Pacific area. One consequence of this expansion is that the considerable cost of hosting an IMO has been shared more widely. Of course, the other side of this is that with increasing numbers of participants, the costs are increasing proportionally.

The second development illustrated by the statistics is the change that has occurred in the distribution of prizes. Notice that initially fewer than 20% of the contestants received awards. More recently the trend has been towards 50% of participants receiving them in the ratio of approximately 1 : 2 : 3. Special prizes are awarded for particularly elegant or interesting solutions. Each year a number of solutions have been considered for these awards but the number actually given is small. For instance, in 1986 only one special prize was presented.

Such developments and other variations in the regulations of an IMO are proposed by the host country and endorsed by the International Jury which includes the team leaders from each of the participating countries. The Jury has the further tasks of selecting the contest papers from among the questions submitted earlier by the participating countries, of grading their own team's solutions, of presenting these solutions to the coordinators from the host country and finally of deciding on the awards.

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[1]The author is Deputy Leader of the Australian Mathematical Olympiad Team.

Many countries have had a long tradition of mathematics competitions. The Concours Générale in France originated in the 18th century while Hungary has a history of almost 100 years of conducting mathematical contests, following on from the Eötvös which began in 1894. On the other hand, other countries, including Australia, have no such long tradition. Indeed, the first nationwide contest in Australia (the Australian Mathematics Competition) was first held in 1976. The Australian Mathematical Olympiad commenced in 1979.

Australia has participated in each IMO since the 1981 contest in Washington, DC, United States of America. However, the decision to accept the invitation to compete was made only after years of patient planning and development. In the mid-70's seeds were planted which grew into the Australian Mathematical Olympiad Committee as a sub-committee of the Australian Academy of Science. This committee contains at least one representative from each State and has been responsible for the selection and preparation of all Australian teams. The following table summarizes Australia's participation in the IMO since 1981.

	1981	1982	1983	1984	1985	1986	1987
Gold	*	*	*	*	1	*	*
Silver	*	*	1	1	1	*	3
Bronze	1	1	2	2	2	5	*

The honour of attending an IMO is appreciated by all the participants. However, the significance of such a collection of young mathematicians may not always be fully appreciated. Matti Lehtinen, who was Secretary of the 1985 IMO Organization Committee and in charge of the problem selection, made the following observation.

> *On the basis of a superficial inspection of the standard mathematical reference journals one is led to believe that many successful IMO participants indeed engage themselves in mathematics. Take year 1965 for instance. All the eight first prize winners of that year have published referenced work in mathematics as well as at least seven of the twelve second prize winners and eight out of the 17 third prize winners. Among the invited speakers of the International Congresses of Mathematics one finds at least eight IMO prize winners, and one has even received the coveted Fields Medal, the mathematics equivalent of the Nobel Prize.*

REMOTE INEQUALITIES

Judith Downes[1]
North Balwyn

One of the difficulties encountered in training potential Mathematical Olympiad students in Australia is the geographic dispersion of the students. The traditional method of overcoming this "tyranny of distance" is to run correspondence courses, consisting of varying amounts of lecture notes, worked problems and problems to be attempted. Unfortunately, the written word often fails to teach students as well as a verbally given explanation can. The reasons for a particular approach being taken, the mathematical operation used to move from one line to the next, and the enthusiasm of a lecturer for a particular result are just some of the items which can be lost in the move from a lecture to notes.

In Victoria in 1986 we attempted to overcome some of the problems of distance learning by developing a technique for reproducing lectures on video tape. Video tapes were used because of their advantages of cheap duplication, easy postage, and the ability of students to stop and review the tape as desired. It is anticipated that tapes, accompanied by notes, can be developed and posted to students around the country.

The lecture taped was "*Inequalities*" by Dr. John Upton, University of Melbourne. Basic inequalities and techniques of solving inequalities were introduced to the students. Graphical solutions complemented algebraic ones where appropriate, and proofs of both the Cauchy-Schwarz Inequality and the Theorem of Means were worked through.

The lecture notes were prepared on a micro-computer, printed, enlarged and filmed with the addition of a voice-over and a pointer. Variations in the tape occurred when the lecturer moved to work with graphs which had been prepared on a white-board. The notes required considerable reformatting from the original in order to present the lecture in blocks of information which both made sense mathematically and were large enough to be read on a television screen.

The lecture moved at a slightly slower pace than would occur in a face-to-face situation. Care was taken to anticipate areas of student difficulty, and on occasion to give two different explanations of a point which might normally prompt a question from a student audience.

The tape has been viewed by a number of students in the Victorian Olympiad programme. The overall response has been positive. Students seem to have gained as much from the video and note combination as they would have from a lecture.

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[1] The author is Victorian State Director of the Australian Mathematical Olympiad.

I have reproduced below two of the problems solved towards the end of the lecture for readers to attempt:

1) If $x,y,z,a > 0$ and $xyz = a(x+y+z)$, find the least value of $xy + yz + zx$.

2) If $a,b,c > 0$ then show

$$9/(a+b+c) \leq [2/(a+b)] + [2/(b+c)] + [2/(c+a)]$$
$$\leq (1/a) + (1/b) + (1/c).$$

* * * * *

PROBLEM SECTION

Leigh Thompson of Bairnsdale has taken up the challenge represented by the solution to Problem 8.4.2 that was given in FUNCTION, Vol. 9, Part 1. Due to Lewis Carroll, and submitted by S.J. Newton, the problem reads:

A room has a light switch at each corner. It is not possible by examining the switch to tell if it is on or off. The light will, however, be off unless all switches are in the "on" position. A person comes into the room and finds the light off, then presses each switch in turn with no result. He then presses again in order the first, second and third switches still with no result. How should be proceed if he is to turn the light on?

The solution in FUNCTION has the following opening: "Label the switches A, B, C, D. Switch A may be in state a or state $\bar{a}$, one of which is off and one on but we do not know which is which. Similarly for the other switches.

Suppose we have initially $abcd$. The first circuit of the room produces in order:

$$\bar{a}bcd,\ \bar{a}\bar{b}cd,\ \bar{a}\bar{b}\bar{c}d,\ \bar{a}\bar{b}\bar{c}\bar{d}.$$

The next gives:

$$a\bar{b}\bar{c}\bar{d},\ ab\bar{c}\bar{d},\ abc\bar{d}.$$

At this point no success has been reached and eight out of a possible 16 configurations have been tested. The person is now at C, and going on to D will merely re-establish the initial set-up.

Nonetheless, this is not a bad way to proceed. ..."

The solution concludes with the remark: "There seem to be no solutions that involve no state's being repeated at all."

However, Leigh Thompson believes he has such a solution. We present it as an arrangement of the 16 possible states which has the property that (i) two neighbouring states can be obtained from each other by pressing a switch and (ii) the first eight states, in their order, are consistent with Lewis Carroll's story.

Here it is:

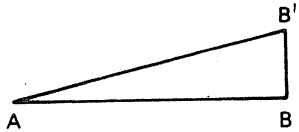
$abcd$, $\bar{a}bcd$, $\bar{a}\bar{b}cd$, $\bar{a}\bar{b}\bar{c}d$, $\bar{a}\bar{b}\bar{c}\bar{d}$, $a\bar{b}\bar{c}\bar{d}$, $ab\bar{c}\bar{d}$, $abc\bar{d}$, $\bar{a}bc\bar{d}$, $\bar{a}\bar{b}c\bar{d}$, $a\bar{b}c\bar{d}$,

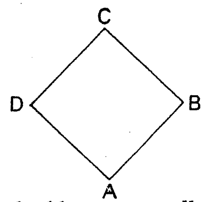
$a\bar{b}cd$, $a\bar{b}\bar{c}d$, $ab\bar{c}d$, $\bar{a}b\bar{c}d$, $\bar{a}b\bar{c}\bar{d}$.

* * * * *

Problem 12.3.1 (Michael A.B. Deakin)

Today (28.3.'88) I visited Calcutta's Birla Museum of Science and Technology and saw a device which consisted of a lozenge-shaped base on each leg of which stood a triangle





all four triangles being congruent. On this base, a double cone rolled with its axis parallel to BD. It appeared to roll "uphill" as it came to equilibrium with the axis vertically above BD. What condition(s) must hold for this motion to occur?

* * * * *

Problem 12.3.2 (Michael A.B. Deakin)

If n is a non-negative integer, then $n!$ is a defined integer, given by $0! = 1! = 1$, $2! = 2$, $3! = 6$, etc. Let $n?$ satisfy $n?! = n!?$, where $n?$ is to take positive integral values (or zero). What values can $n?$ in fact take?

AUSTRALIAN NUMBERS

Frank Hansford-Miller[1]
Murdoch University, Western Australia

When I reached the age of 69 years, more than two years ago now as I write, I was reminded of the Bingo or Lotto caller of this number – "69 – any way up". Yes, there is something special about 69. Indeed, if it is written down on a piece of paper or printed on a wooden counter, it really does not matter which way up it is, it remains 69.

What do we mean exactly by "any way up"? The first figure of 69, the 6, when inverted becomes 9, and the second number of 69, the 9, becomes 6. If, then, the numbers comprising 69 are inverted separately, i.e., individually, 69 becomes 96. So this is not what is meant in this context by "any way up".

To obtain 69 as the result of an operation upon the original 69, we must be more precise. What we are really doing in this operation is rotating the total number, in its entirety, through half a revolution, or 180 degrees; whether this rotation is made clockwise or anti-clockwise is immaterial. If this operation is performed then 69 inverts to 69.

Because Australia is situated geographically in the Southern Hemisphere, the people who live in England and other countries in Europe and North America, and in the Northern Hemisphere generally, frequently refer to Australia as "Down Under" and, jokingly, of being "Upside Down". Qantas, in fact, uses this fact in Britain by printing the word Australia in large print upside down as the bottom line of its airline advertisements.

So, to many in the world, Australia is the "Upside Down" country. It seems appropriate, therefore, to designate numbers such as the above-mentioned 69, which turn upside down, or invert, to themselves, as "Australian Numbers".

I will now give a definition of an Australian Number.

Definition one: an Australian number

An Australian number is a real number in Arabic numerals with the property that when it is inverted in its entirety, commonly called being "turned upside down", which is equivalent to rotation of the entire number either clockwise or anti-clockwise through 180 degrees, it remains the same number.

It is convention to designate negative numbers by a negative sign to the left of the number. Upon inversion such negative signs appear on the right-hand side of the numbers. Thus, clearly, no negative numbers can be Australian numbers. The Corollary follows.

[1] The author is Western Australian Over 70 Marathon Champion.

Corollary

All Australian numbers are non-negative numbers.

Australian numbers group one: single digit components

My family gave me a birthday barbecue party on my 69th birthday and I was expected to make a little speech. I therefore began to develop some further ideas on Australian Numbers. The first question I posed to myself was: "How many Australian Number birthdays had I already experienced in my life?" Having reached the Australian number of 69 years, my second question was: "How many more Australian birthday numbers could I look forward to, God willing, in the future?" I will attempt answers to these questions later in the paper.

First I will discuss the possible types of Australian numbers, beginning with what I consider the simplest type, which I will designate group one, those with single digit components. Clearly the numbers 0 and 1 may be considered members of the group. Both these numbers upon inversion remain the same. (The printers' accretions upon 1 will be ignored.) The numbers 2, 3, 4, 5 and 7 are obviously not Australian. The numbers 6 and 9 invert to 9 and 6, respectively, and taken singly are clearly not Australian. This leaves the number 8. This is a difficult one. What we are looking for in an Australian Number is symmetry about a horizontal line through the centre of the number. Some computer formats do give an 8 this symmetry but an 8 is also frequently printed with the upper loop and the lower loop of differing sizes, the upper loop being smaller than the lower loop. However, the difference seems to me so slight, especially in written script, that, in my judgement, the number 8 should also be listed as an Australian Number.

An Australian Number will be deemed to belong to Group One if it is a Positive Integer and each of its digits inverts separately. The Set of Australian Numbers will be denoted by AUS_1. We first consider the single digit members of AUS_1.

$$\{x \mid 0 \le x < 10, x \text{ belongs to } AUS_1\} = \{0,1,8\}.$$

We can now extend the range of x beyond 10, or 10^1. Consider, first, x within the range 0 to 100, or 0 to 10^2. We obtain:

$$\{x \mid 0 \le x < 100, x \text{ belongs to } AUS_1\} = \{0,1,8,11,88\}.$$

Further extension to the range for x from 0 to 10^3 gives:

$$\{x \mid 0 \le x < 1000, x \text{ belongs to } AUS_1\} = \{0,1,8,11,88,101,111,181,808,818,888\}.$$

And for the range of x from 0 to 10^4, the list of Australian Numbers becomes:

$$\{x \mid 0 \le x < 10000, x \text{ belongs to } AUS_1\} =$$
$$\{0,1,8,11,88,101,111,181,808,818,888,1001,1111,1881,8008,8118,8888\}.$$

The range of x can, of course, be continued indefinitely.

Project 1. List the Australian Numbers in the range of x from 0 to 10^5, inclusive, where x is a Positive Integer.

Project 2. List the Australian Numbers in the range of x from 0 to 10^6, inclusive, where x is a Positive Integer.

These results may be summarized as shown in Table 1.

TABLE I

NUMBER OF AUSTRALIAN GROUP ONE NUMBERS

Range	Number of Australian group one numbers
0 to 10^1	3
0 to 10^2	5
0 to 10^3	13
0 to 10^4	21
0 to 10^5	47
0 to 10^6	73

Another question which now arises is the growing use of the symbol for zero as an accepted initial digit of a number, for example 01, 001, 0001, etc. We are getting familiar with this usage from computer printouts on bank statements, electricity accounts and so on, and from being previously illegitimate usages they are now generally acceptable.

This has consequences for our Australian Numbers. For example, it could bring in as Australian Numbers the following numbers:

010,080

0110, 0880

00100, 00800, 01010, 01110, 01810, 08080, 08180, 08880.

I consider that these are now all perfectly legitimate numbers and henceforth they should be included as true Australian Numbers.

I draw the line, however, against strings of zeros, such as 00, 000, 0000, etc. Surely these are only 0 writ large in a different form. They are not new numbers and as 0 is already included I can see no reason to include these strings of zeros in addition.

Project 3. Rewrite the lists of Australian Numbers in all ranges of x from 0 to 10^6, inclusive, to include numbers beginning with 0.

Project 4. Examine your lists of Australian Numbers Group One to see whether it is possible to discern a pattern and so, by deriving a formula, or otherwise, to deduce the numbers for further ranges of x, say, from 0 to 10^7, inclusive, and from 0 to 10^8, inclusive.

The Characteristics of Australian Numbers Group One

It will have already become obvious that the characteristics of the digits of an Australian Number Group One are that each number must be symmetrical about a horizontal line drawn through its centre. Only with this property will the number invert.

It must next be asked what are the characteristics of Australian Numbers Group One when taken in entirety, so that they invert to the same identical number.

Consideration of the lists of Australian Numbers shown above will indicate that as well as the symmetry of each individual number about a horizontal axis, the number as a whole must also be symmetrical about a vertical line through its centre.

In the case of an Australian Number with an odd number of digits, this vertical axis will run down the centre of the middle number. In the case of a number with an even number of digits then this axis will run vertically between the two middle numbers.

Thus Australian Numbers Group One must be symmetrical about their centres both vertically and horizontally. This is what distinguishes an Australian Number of this type with single digit components from a Non-Australian Number.

Australian Numbers Group Two: Multiple Digit Components

Let us now return to my birthday number of 69. We noted earlier that there was something special about this number but, of course, if we invert the numbers separately, as we do with Australian Numbers Group one, we do not obtain 69, but, instead, we arrive at 96. Clearly a different operation is required for 69 to invert to itself.

As explained in the opening paragraphs, what we have to do with 69 and like numbers is obviously to invert the numbers TOGETHER. In effect, as we have seen, this means rotating the whole number about its centre through 180 degrees.

To distinguish such positive integers from those already considered, we can designate them Australian Numbers Group Two. The multiple digit components of the number must be inverted together as a whole. When this is done the resulting number must be the same number as the one we started with. Let us now consider, first, numbers in the range 0 to 100, as before. Clearly 69 is a member of the Set of Australian Numbers Group Two in this range, and so is 96. There are no others. Duplications of numbers which are already Australian Numbers Group One are excluded.

Australian Numbers Group Two can obviously involve the numbers 6 and 9 only in combination with each other.

Let AUS_2 mean the set of Australian Numbers Group Two, that is, of numbers with multiple digit components, which, when rotated through 180 degrees in entirety, remain unchanged as the same number. Then we can compile the following list of this type of number:

$\{x \mid 0 \le x < 10, \text{ x belongs to } AUS_2\} = \varnothing$, the empty set.

$\{x \mid 0 \le x < 100, \text{ x belongs to } AUS_2\} = \{69,96\}$.

$\{x \mid 0 \le x < 1000, \text{ x belongs to } AUS_2\} = \{69,96,609,619,689,906,916,986\}$.

$\{x \mid 0 \le x < 10{,}000, \text{ x belongs to } AUS_2\} = \{69,96,609,619,689,906,916,$
$986,0690,0960,6009,6119,6699,6889,6969,9006,9116,9696,9886,9966\}$.

$\{x \mid 0 \le x < 100{,}000, \text{ x belongs to } AUS_2\} = \{69,96,609,619,689,906,916,$
$986,0690,0960,6009,6119,6699,6889,6969,9006,9116,9696,9886,9966,$
$06090,06190,06890,09060,09160,09860,60009,60109,60809,61019,$
$61119,61819,66099,66199,66899,68089,68189,68889,69069,69169,$
$69869,90006,90106,90806,91016,91116,91816,96096,96196,96896,$
$98086,98186,98886,99066,99166,99866\}$.

Project 5. List the following Set of Australian Numbers Group Two:

$$\{x \mid 0 \le x < 1{,}000{,}000, \text{ x belongs to } AUS_2\}.$$

As before we can summarize the results (Table II).

TABLE II

NUMBER OF AUSTRALIAN GROUP TWO NUMBERS

Range	Number of Australian group two members
0 to 10^1	0
0 to 10^2	2
0 to 10^3	8
0 to 10^4	20
0 to 10^5	56

Project 6. Examine your lists of Australian Numbers Group Two to see whether it is possible to discern a pattern and so, by deriving a formula, or otherwise, to deduce the numbers for further ranges of x, say, from 0 to 10^6, inclusive, and from 0 to 10^7, inclusive.

Semi-Australian Numbers

We have been careful so far when deriving our Australian Numbers to accept only those resulting from being "turned upside down" in their entirety which are exactly the same as they were to start with. However, numbers consisting of combinations of the digits 0, 1, 6, 8 and 9, but excluding 2, 3, 4, 5 and 7, when "turned upside down" will still remain numbers, though not necessarily the same number and thus not Australian Numbers of the types already considered. We call such numbers Semi-Australian Numbers.

Examples of Semi-Australian Numbers are given below, with the symbol □ meaning "inverts to":

19608 □ 80961 610089 □ 680019 08116961 □ 19691180

Clearly one difference between Australian and Semi-Australian Numbers concerns the numbers 6 and 9. These can now be involved singly, instead of in paired combinations. Another difference is that Semi-Australian Numbers are not symmetrical about a vertical axis through their centres, as are Australian Numbers Group One.

Use of Australian Numbers as a Teaching Aid in Primary Schools

I have produced a separate paper, "*Notes on the Use of Australian Numbers as a Teaching Aid in Schools*", and will mention this only briefly here. Teachers of mathematics to young children have, first, to impart a familiarity with Arabic Numerals to their pupils and, second, to generate a facility in their use. I believe Australian Numbers can be very helpful at this important and difficult early stage as a "fun" way to achieve these objectives.

The finer distinctions between Group One, Group Two and Semi-Australian Numbers should be dispensed with, and, for simplicity, all numbers which "turn upside down" called Australian Numbers.

The children can be issued with, or can cut out their own, discs, on which they write the digits 0 to 9. They can then play at Australian Numbers by finding out which numbers invert. Further sets of numbered discs can be added, so that larger and larger numbers can be considered.

Games can be added to find Australian Birthdays in the class, i.e. Birth Dates which invert to all numbers, as for example: 8-9-1981 □ 1861-6-8.

Finding Australian Number History dates can provide exercises for older pupils. Battle of Hastings, 1066 □ 9901, is a good example, as also is Australian Federation Year, 1901 □ 1061. Extra marks could be given for finding "True" or "Dinky-Di" Australian Number dates, i.e. those that invert to themselves, as for example, 1881 □ 1881 and 1961 □ 1961.

Age and Australian Numbers

This study began with my age of 69 and this association of age with Australian Numbers curiously continues. I will, however, first answer my own earlier questions regarding my birthdays past and, hopefully, in the future.

I make my current tally 0, 1, 8, 010, 11 and 69, giving a total of 6. To come, well, possibly, 080, 88, 96 (my grandmother's life span), 00100, 101, 0110 and 111, now well into the right-hand tail of a distribution of the age of death. The next, 609, unless medical science intervenes, obviously has zero probability. So, 0 to 7 Australian Number Birthdays to come.

This conclusion, however, discounts the longevity of the Biblical patriarch, Methuselah. According to Genesis, chap. 5, v. 27, Methuselah lived 969 years.

Project 7. Determine all Australian Number Birthdays Methuselah, who died aged 969, experienced in his lifetime.

It is curious that Methuselah's age, 969 □ 696, is a Semi-Australian Number. And who is the modern record holder as the longest living person? This is a Pakistani mystic and holy man, Abdul Ma'abood Jilani, who was reported in "*The Australian*" of 2 December 1985 as dying the previous day, aged 169. Well, 169, I nearly wrote "of course", is another Semi-Australian Number, 169 □ 691. Very strange.

Conclusion

It is perhaps premature to suggest that as a result of this sometimes fanciful introductory exploration into the nature of Australian Numbers existing treatises on Number Theory will need to be rewritten. I hope, however, it will prove enjoyable and useful as something new and different as a learning aid to both the very young and the older pupils in our schools.

* * * * *

GENESIS OF ALGEBRA

After 40 days and 40 nights the ark runs aground and the flood subsides. Noah kicks open the ark's door and says: "Behold, the flood has ended! All you animals go forth and multiply!" So the animals wander down the ramp two by two until it's the snakes' turn. They look up at Noah with tearful eyes and say: "But Noah, we cannot multiply, for we are adders." Noah is visibly miffed. Off he goes into the nearest forest and cuts down some trees. He saws and hammers away until he has made a table out of the wood. He drags this back to the ark and puts it in front of the snakes with a huff. "Behold, snakes! I have built a table of logs. Now you adders can multiply!".

* * * * *

LETTER TO THE EDITOR

FUNCTION editor M.A.B. Deakin in a letter of 28 March 1988 from Calcutta:

The following minor item may be of interest to FUNCTION readers.

> "From the Lookout on the Mainmast, one could see twenty Miles in each Direction."
>
> Erica Jong, *Fanny*, p. 475

To make matters worse – these are nautical miles.

> "I forced myself to climb the one-hundred-foot Mast and spy the Seas."
>
> *ibid.*, p. 477.

TASKS FOR YOUR MICROCOMPUTER: e^x, ln x, sin x and cos x

Leigh Thompson
Bairnsdale High School

1. BINOMIAL THEOREM AND APPLICATIONS

The Binomial Theorem states that if a and b are numbers and n is a positive integer, then

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n$$

$$= \sum_{r=0}^{r=n} \binom{n}{r}a^{n-r}b^r.$$

This theorem can be proved by induction as follows.

Assume the Binomial Theorem is true for some value of n, say $n = k$. Then

$$(a+b)^{k+1} = (a+b)(a+b)^k$$

$$= (a+b)\sum_{r=0}^{r=k} \binom{k}{r}a^{k-r}b^r \quad \text{provided the Binomial Theorem is true for } n = k$$

$$= \sum_{r=0}^{r=k} \binom{k}{r}a^{k+1-r}b^r + \sum_{r=0}^{r=k} \binom{k}{r}a^{k-r}b^{r+1}$$

$$= \binom{k}{0}a^{k+1-0}b^0 + \sum_{r=1}^{r=k} \binom{k}{r}a^{k+1-r}b^r + \sum_{r=0}^{r=k-1} \binom{k}{r}a^{k-r}b^{r+1} + \binom{k}{k}a^{k-k}b^{k+1}.$$

Replacing r with $r-1$ in the second summation does not alter the value of this summation :

$$(a+b)^{k+1} = \binom{k}{0}a^{k+1-0}b^0 + \sum_{r=1}^{r=k} \binom{k}{r}a^{k+1-r}b^r + \sum_{r=1}^{r=k} \binom{k}{r-1}a^{k-r+1}b^r + \binom{k}{k}a^{k-k}b^{k+1}$$

$$= \binom{k}{0}a^{k+1-0}b^0 + \sum_{r=1}^{r=k} \left[\binom{k}{r} + \binom{k}{r-1}\right]a^{k+1-r}b^r + \binom{k}{k}a^{k-k}b^{k+1}.$$

But
$$\binom{k}{r} + \binom{k}{r-1} = \frac{k!}{r!(k-r)!} + \frac{k!}{(r-1)!(k-r+1)!}$$

$$= \frac{k!(k-r+1+r)}{r!(k+1-r)!}$$

$$= \frac{(k+1)!}{r!(k+1-r)!}$$

$$= \binom{k+1}{r}, \text{ also } \binom{k}{0} = \binom{k+1}{0} \text{ and } \binom{k}{k} = \binom{k+1}{k+1} ;$$

$$(a+b)^{k+1} = \binom{k+1}{0}a^{k+1-0}b^0 + \sum_{r=1}^{r=k}\binom{k+1}{r}a^{k+1-r}b^r + \binom{k+1}{k+1}a^{k+1-k-1}b^{k+1}$$

$$= \sum_{r=0}^{r=k+1}\binom{k+1}{r}a^{k+1-r}b^r, \text{ which is the Binomial Theorem for } n = k+1;$$

thus if the theorem is true for $n = k$, it is true for $n = k+1$. But it is true when $n = 1$ because $(a+b)^1 = \binom{1}{0}a^{1-0}b^0 + \binom{1}{1}a^{1-1}b^1$

$$= a + b.$$

Hence it is true for $n = 2$ and so for $n = 3$ and so on. Hence it is true for all $n \in \mathbb{N}$.

The Binomial Coefficients form the familiar Pascal Triangle

$$\begin{array}{ccccccccccccccc}
 & & & & & & & \binom{0}{0} & & & & & & & \\
 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & \\
 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & \\
 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & \\
 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} & \\
\binom{7}{0} & & \binom{7}{1} & & \binom{7}{2} & & \binom{7}{3} & & \binom{7}{4} & & \binom{7}{5} & & \binom{7}{6} & & \binom{7}{7}
\end{array}$$

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

1 1 2 3 5 8 13 21

Note that the sums along the dotted diagonals produce the Fibonacci sequence, i.e. 1, 1, 2, 3, 5, 8, 13, 21, ... (each term is the sum of the two preceding terms).

By noting that the sum of two adjacent terms gives the term "in between" on the next row, the following relationship for $n,r \in \mathbb{N}$ can be established:

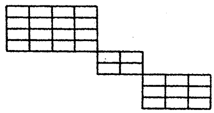
$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r} \text{ as shown earlier, or } \binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1},$$

also $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{r} = \binom{n}{n-r}$.

These Binomial Coefficients can be used to solve problems similar to the following.

Counting only paths that follow the lines and go to the right or downwards, how many paths from P to Q are possible?

P



Q

Answer: $70 \times 6 \times 20 = 8400$

Numbers represent the paths from the top, left corner of each square grid to the corresponding intersection, cf. Pascal's Triangle.

P

•	1	1	1	1					
1	2	3	4	5					
1	3	6	10	15					
1	4	10	20	35					
1	5	15	35	70	1	1			
				1	2	3			
				1	3	6	1	1	1
						1	2	3	4
						1	3	6	10
						1	4	10	20

Q

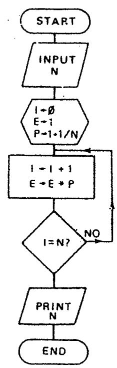
2. e AND e^x

Consider $\lim_{n\to\infty} (1 + \frac{1}{n})^n$. The following table evaluates $(1 + \frac{1}{n})^n$ for various values of n.

n	$(1+\frac{1}{n})^n$
1	2
10	2.59374
100	2.70481
1 000	2.71693
10 000	2.71815
100 000	2.71828
1 000 000	2.71828

The following BASIC program evaluates $(1+\frac{1}{n})^n$ given n.

```
1Ø  INPUT N
2Ø  I=Ø
3Ø  E=1
4Ø  P=1+1/N
5Ø  I=I+1
6Ø  E=E*P
7Ø  IF I<>N THEN 5Ø
8Ø  PRINT E
9Ø  END
```



For large n this program takes much time and roundoff errors become significant.

The Binomial Theorem can be used to evaluate $\lim_{n\to\infty} (1+\frac{1}{n})^n$.

Now $(1+\frac{1}{n})^n = 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}(\frac{1}{n})^2 + \binom{n}{3}(\frac{1}{n})^3 + \ldots$

$= 1 + 1 + \dfrac{n(n-1)}{2!n^2} + \dfrac{n(n-1)(n-2)}{3!n^3} + \ldots$; so it becomes plausible that $\lim_{n\to\infty}(1+\frac{1}{n})^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$

$= 2.71828\ 18284\ 59045 \ldots$. This value is known as e.

More generally, $\lim_{n\to\infty}(1+\frac{x}{n})^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$= e^x$ as one can prove.

For large x this series converges slowly, i.e. many terms $\frac{x^k}{k!}$ have to be evaluated before their sum becomes a satisfactory approximation of e^x. For large negative x, round-off errors are a problem in summing this series with a computer, and a satisfactory approximation of e^x will not be obtained using a simple program to sum the series.

3. log x

We know that $\int_0^r \frac{1}{1+x}\,dx = \log_e(1+r)$ for $r > -1$.[1]

By algebraic division

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

$$\begin{array}{r} 1 - x + x^2 \\ 1+x \overline{)\,1 \qquad\qquad} \\ \underline{1 + x \qquad} \\ -\,x \qquad \\ \underline{-\,x - x^2} \\ x^2 \end{array} \quad \text{etc.}$$

so $\int_0^r \frac{1}{1+x}\,dx = \int_0^r (1-x+x^2-x^3+\ldots)dx$

$$= r - \frac{r^2}{2} + \frac{r^3}{3} - \frac{r^4}{4} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{-(-r)^n}{n}.$$

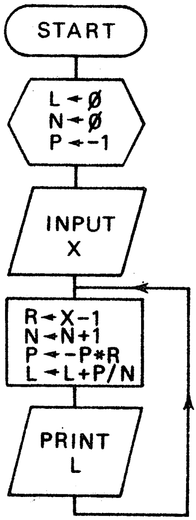
[1] An alternative notation for $\log_e$ is ℓn.

This series has a limit for $-1 < r \le 1$

$$\log_e(1+r) = r - \frac{r^2}{2} + \frac{r^3}{3} - \frac{r^4}{4} + \ldots \qquad \text{for } -1 < r \le 1.$$

The following BASIC program "endlessly" evaluates $\log_e x$ given x where $0 < x \le 2$.

```
10   L=0
20   N=0
30   P=-1
40   INPUT X
50   R=X-1
60   N=N+1
70   P=-P*R
80   L=L+P/N
90   PRINT L
100  GOTO 60
```



The following series can be used to evaluate $\log_e x$ for $x \ge \frac{1}{2}$:

$$\log_e x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x-1}{x}\right)^n.$$

The following changes to the preceding program will evaluate this series:

```
30   P=1
50   R=(X-1)/X
70   P=P*R .
```

4. THE TRIGONOMETRIC FUNCTIONS SIN AND COS

Euler's formula is

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{where} \quad i = \sqrt{-1}$$

$$\text{so} \quad i^2 = -1$$

$$i^3 = -1$$

$$i^4 = 1,$$

and when $\theta = \pi$

$$e^{i\theta} + 1 = 0.$$

Using the series for e^x

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots$$

i.e.
$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots .$$

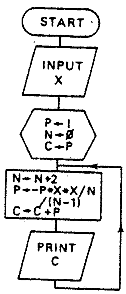
Equating real and imaginary parts

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}.$$

The following BASIC program "endlessly" evaluates $\cos x$ given x.

```
10  INPUT X
20  P=1
30  N=0
40  C=P
50  N=N+2
60  P=-P*X*X/N/(N-1)
70  C=C+P
80  PRINT C
90  GOTO 50
```



The following changes to the preceding program will evaluate sin x given x.

```
20   P=X
30   N=1
40   S=P
70   S=S+P
80   PRINT S.
```

* * * * *

PERDIX

For the last eight years the ICTM (Illinois Council of Teachers of Mathematics and Computer Science) has staged an annual mathematics competition, sponsored in 1988 principally by the C.N.A. Insurance Co., for senior high school students (years, or grades as they call them in the States, 9, 10, 11, and 12). Similar competitions are held in most other States of the United States. High schools are divided into two categories, AA(=large) high schools and A(=small) high schools, for which separate competitions are held.

In Illinois about 200 AA high schools take part in the competition. Illinois is divided into 10 regions and about 20 high schools compete in each regional competition. The winners of the regional competitions then compete in the State Final.

This year the regional competitions for AA high schools were all held on March 12 and one of these took place at Northern Illinois University, hosted by the department of Mathematical Sciences. The mathletes arrived at 9.30 am, the first contests started at 10 am, the final knock-out contest was held at 1.30 pm and the prizes were awarded at 2pm.

The competition aroused a lot of excitement. Mathletes were divided into Juniors (years 9 and 10) and Seniors (years 11 and 12) and while juniors could represent their schools in the senior teams the opposite arrangement, for obvious reasons, is not allowed.

The contests came in a variety of kinds, some of which will be new to Australian readers.

The first contests required written answers and took just under an hour to complete. The subjects were Algebra I and Geometry for Juniors, and Algebra II and Pre-calculus for Seniors. Each school entered two Junior teams of six mathletes, one team to take the Algebra I the other to take the Geometry, and two Senior teams, again each of six mathletes, to compete in the Algebra II and the Pre-calculus written contests. Each competitor's answers are marked and there are, at the end of the competition, both individual prizes, first, second and third, and team prizes, again first, second and third, for each of these written contests and for each of the other contests that make up the competition. The team score for these teams of six in the written contests is the total of the four best individual scores.

For the second hour there were simultaneous separate contests. One was the Calculating Team Contest, the others were the Junior and the Senior Two-person Team Contests.

Each school entered a team of five mathletes, with at least one student from each of the years 9, 10, 11 and 12, for the Calculating Contest. Armed with hand-calculators (battery-operated) the team members solved the calculating problems displayed on an overhead screen. There were 20 problems with a total time allowed of 15 minutes. Team members work separately or together on problems, but only one set of answers, written on the official answer sheet, is accepted from each team.

For the Two-Person Contests each school enters two teams, each of two persons, one a Junior team, the other a Senior team. The Senior and Junior contests are held in different rooms. For each contest 10 questions are shown, one at a time, on a screen. A maximum time of 3 minutes is allowed for each question. The first team to get the correct answer (shown in writing to a proctor standing by the team), within the 3 minutes allowed, gets 7 points, the second team gets 5 points, and all the others getting the answer get 3 points.

In the Eight-Person Contest a time of 20 minutes is available for answering the problem paper. Again a single set of written answers must be submitted by each team.

The Oral Contest provides an interesting variant. For this contest schools are sent written information about the subjects on which the test questions are set. In 1988 there were two subjects, one was transformation geometry, the other was logic and logic circuits. In 1987 the subjects were graph theory (including the idea of an Euler path and the idea of connected graphs), and geometric probability. Competitors have about 5 weeks to study this material and then each school selects a single representative to compete for it in each of the subjects. The test itself consists of questions handed to the mathlete on going into the competition room. Quarter of an hour is allowed for private study of the questions and then 7 minutes is given for an oral exposition, with writing on a blackboard allowed, of answers to the questions. This will be followed by questions from the judges lasting at most 3 minutes.

The points obtained by each school are now totalled. I have indicated a few of the rules for allocating points. The total possible for any school was 1000 points. The top two schools (or more if there are equal scores) now take part in the Knock-out Contest to choose the school to compete in the State Finals. For the Knock-out each school provides a Two-Person team. The rules are the same as before, except that 2 minutes is now the maximum time allowed for each question. The first team to get two correct answers wins.

You might be interested in the questions asked in the Knock-out Contest this year. Here they are (at most 2 minutes allowed for each).

ICMT REGIONAL MATHEMATICS COMPETITION 1988

KNOCK-OUT CONTEST

1. What is the probability, in seven tosses of a fair coin, that five or more heads occur?

2. Compute the sum of the series

$$\frac{4}{5} + \frac{3}{25} + \frac{2}{125} + \frac{2}{625} + \frac{2}{3125} + \cdots$$

3. If $\sin x = \dfrac{1}{\sqrt{(1+u)^2}}$ and $u = \dfrac{1}{3}$ evaluate $\cos^2 x - \sin^2 x$.

4. Let a be the number of integral solutions of

$$7 \le |x-5| \le 10,$$

and let b be the number of integral solutions of $x(x-5) = 22$. Find $a+b$.

5. If the area of an equilateral triangle is 3, what is the length of each side?

6. The radius of a cylinder is decreased by 20% and its height is increased by 30%. What is the change in its volume to the nearest per cent?

7. Find $\sin(2\tan^{-1}3)$.

8. An omelet made with 2 eggs and 30 grams of cheese contains 280 calories. An omelet made with 3 eggs and 10 grams of cheese also contains 280 calories. How many calories in an egg?

9. If $f(x) = x^2 + \dfrac{1}{x^2}$, evaluate

$$(f(x))^3 - 3f(x) - f(x^3).$$

10. If $0 \le \theta \le \pi/2$ and $\cos 2\theta = 4/5$, find $\sin\theta + \cos\theta$.

In fact the winner had been decided after the first three questions: the loser answered the first question correctly first, the winner got the next two questions.

Other types of competition have been used or contemplated. Two that sound interesting are the Relay Competition and the Mad Hatter Competition.

In the Relay Competition a team of eight is entered by each school. The eight competitors are designated definite positions in the relay team. The first question must be answered by competitor number 1, the second by competitor number 2, and so on. The questions are of increasing difficulty. A team scores according to the number of questions answered.

This competition poses interesting tactical questions about the best way to play to win. Should a team place its weakest competitor in place 1? If so perhaps it will never get off the starting blocks. Is it, for example, better to allocate positions in the team so that there is a good chance of answering the first five questions, by putting your best five competitors first, instead of last? Detailed account will have to be taken of the way points are allocated.

In the Mad Hatter Contests the Juniors and Seniors compete separately. For each group the teams from all schools compete at the same time. When the teams are gathered together then the Mad Hatter marathon, as it is called, begins: a sequence of 50 or more questions is displayed on a screen, one at a time, and a maximum of 60 seconds is allowed to find each answer. When a third of the teams (each has a single person whose job it is to signal success) have found the answer, no further claims of success are allowed. Proctors verify, making a random choice from the teams claiming success, whether the claimed correct answers are indeed correct. The first three correct answers found are awarded 7 points each, while any teams chosen giving an incorrect answer get a penalty of 4 points each. Once three teams with correct answers have been found the proctors cease checking further claims and the remaining teams that had claimed correct answers, i.e. those that got their claims in in the first third, get 2 points each. The results of the round are noted and then the next question is displayed on the screen.

In playing the Marathon there is again an interesting strategy problem: should you claim success, even when you have not obtained an answer to a question, in the hope of getting the 2 points for unchecked answers? If so you must do so quickly to be among the first third making claims. Or is this too risky a procedure because you may be found out and lose 4 points?

There are interesting variants to the rules possible for the Marathon. In particular it is important to make the number of points awarded or lost depend on the number of schools competing so as to prevent any one strategy being obviously much better than another.

* * * * *

CANBERRA LOOKS AT MIGRANT AIDS TESTING

The Federal Government yesterday confirmed that it was considering mandatory AIDS tests for potential migrants. The Opposition spokesman on immigration, Mr Cadman, said that carriers of AIDS (acquired immune deficiency syndrome) should not be allowed as immigrants. Mr Michael Cobb (NP, NSW) raised the issue in Parliament, urging the Government not to grant permanent residence in Australia on the basis of a long-lasting relationship with a homosexual partner already here. He called for a mandatory AIDS test for homosexuals asking for permanent residence.

The Health Minister, Dr Blewett, is expected to present the national strategy on AIDS later this year. Mr Holding said the question of AIDS screening for immigrants was being considered. A spokeswoman for Dr Blewett said that only those potential migrants who showed physical signs of AIDS were tested.

Mr Holding said: "The problem is, if we were to screen for AIDS the 132,000 people who currently come to Australia as immigrants, what would we do about the other 3.5 million people who enter Australia each year?"

Excerpts from an article by Ross Peake
in *THE AGE*, Wednesday 25 May 1988.

* * * * *

AIDS AND BAYES

G.A. Watterson
Monash University

AIDS is a major disease, gradually spreading around the world. It is, at the moment, incurable. Clearly the spread of AIDS could be halted if those people having the disease were unable to pass it on to others, perhaps by isolating them from the rest of the community. The editors of a new and interesting magazine, "*Chance: New Directions for Statistics and Computing*", have urged statisticians to explain to the general public how difficult it would be to carry out such an isolation programme. That is the aim of this article.

In order to isolate AIDS sufferers or carriers, it has been suggested in the U.S.A. that the Government should routinely test all immigrants, all prisoners, and all hospital patients, for AIDS. The Secretary for Health and Human Services also announced, in June 1987, a plan to test 45 000 randomly selected people from the general population. Similar proposals have been mentioned in Australia (see above).

But there is one great difficulty with such proposals. If a person was tested and diagnosed as having AIDS, *there is a good chance that the diagnosis would be wrong.* If the person was then isolated, cut off from friends and family, probably forced to give up employment, and if this isolation was by mistake, then not only would a terrible injustice be done but no doubt such a person would also qualify for very substantial financial recompense, via law suits, etc.

Why is there "a good chance that the diagnosis would be wrong"? Well, suppose that the percentage of the population having the AIDS virus is somewhat less than 1%. The figure of 0.6% has been suggested for the U.S.A., and Australia seems to be only a year or two behind the U.S.A. in the spread of AIDS. Expressed as a proportion (or probability), we write

$$\Pr(A) = 0.006 \qquad (1)$$

for the probability that a randomly chosen person would have AIDS (the event "A").

If a person were tested, there might be very high probabilities that the test would indicate the correct diagnosis. For instance, (and this is an estimate made in 1985) suppose that

$$\Pr(+ \mid A) = 0.977 \qquad (2)$$

is the conditional probability that, if the person did have AIDS, then the test would prove positive (i.e. would correctly indicate that he did have AIDS).

Similarly, it is estimated that

$$\Pr(- \mid \text{not } A) = .926 \qquad (3)$$

for the probability that the test would prove negative, conditional on the person not having AIDS. So the test has very good (but not perfect) performance in detecting the presence or absence of the virus. And yet, for a randomly selected person, *if the test proved positive, it is still much more likely than not that the person is not suffering from AIDS.* To show this, we have to use "Bayes' Theorem". This says that the probability of a "false positive" is given by

$$\Pr(\text{not } A \mid +) = \frac{\Pr(\text{not } A \text{ and } +)}{\Pr(+)} \qquad (4)$$

where, for our values given above,

$$\begin{aligned} \Pr(\text{not } A \text{ and } +) &= \Pr(\text{not } A)\Pr(+ \mid \text{not } A) \\ &= (1 - \Pr(A))(1 - \Pr(- \mid \text{not } A)) \\ &= (1 - 0.006)(1 - 0.926), \text{ by (1) and (3)} \\ &= .994 \times .074 \\ &= 0.073556 \qquad (5) \end{aligned}$$

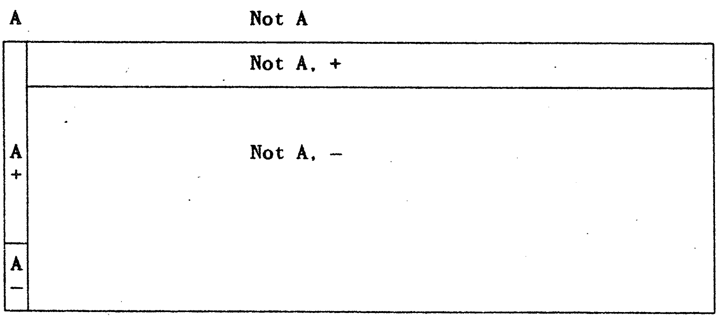
and

$$
\begin{aligned}
\Pr(+) &= \Pr((\text{not A and } +) \text{ or } (\text{A and } +)) \\
&= \Pr(\text{not A and } +) + \Pr(\text{A and } +) \\
&= 0.073556 + \Pr(\text{A}) \Pr(+ \mid \text{A}), \text{ by (5)} \\
&= 0.073556 + 0.006 \times 0.977, \text{ by (1) and (2)} \\
&= 0.079418.
\end{aligned}
$$

Thus, by (4),

$$
\begin{aligned}
\Pr(\text{not A} \mid +) &= \frac{0.073556}{0.079418} \\
&= 0.926. \qquad (6)
\end{aligned}
$$

This shows that in roughly 9 cases out of 10, when the diagnosis was +, it would be wrong! See Figure.



Of both + regions, most is in Not A. (Not to scale.)

Somewhat different answers could be found if we assumed different values of (2) and (3). The values above are quoted from an article in the Journal of the American Medical Association (1985). If the testing were to be done on thousands, or millions, of people, there would be many more mistakes made in the testing (e.g. the tests might be less carefully done, the results of people could get mixed up with those of other people, etc.). So we might expect the high figures proposed in (2) and (3) for correct diagnosis to be unrealistically high. If so, (6) could be too low as an estimate of the false positive rate.

I think our authorities should take great care in using any data from AIDS testing.

YOU CAN ALWAYS BLAME YOUR PARENTS!

Peter Kloeden

Murdoch University

Isn't it interesting how parents are quick to attribute some stunning success of their children to heredity, but even quicker to attribute some less than desirable characteristic to environmental circumstances or, worse still, to inherited shortcomings of the other side of the family? From the similarity of physical appearance and personalities with parents or grandparents, it is obvious that we all inherit some of the characteristics of our forebears. It is also true that some of our characteristics are determined by our physical and social environment. Just how much of which determines a particular characteristic is a moot point and is not without serious socio-political implications, as for instance the debate that rages from time to time about the IQs of the black and white races. But as every farmer knows, some characteristics in animals or crops are most definitely inherited and can be enhanced by selective breeding.

Breeding programmes were carried out on an *ad hoc* and vaguely understood way until early this century, when the studies of a monk from Austrian Silesia, Johann Gregor Mendel, were rediscovered. Mendel spent most of his adult life in the Augustinian monastery of St. Thomas, located in Altbrünn (now a part of Brno, the capital of Moravia, in present-day Czechoslovakia) after studying at the University of Vienna to be a mathematics teacher. One of his duties in the monastery was to supervise the monastery's gardening activities. Over time he realised that the colours of sweet pea flowers in the gardens maintained a fairly steady frequency distribution between the colours red, white and pink, from season to season, and that cross-fertilizing plants of the same or different colours led to various combinations of colours in the next generation of plants with fairly definite frequencies. He carried out systematic experiments, obtaining highly accurate data and was able to make predictions that were convincingly verified by experiment. Eventually Mendel became abbot of his monastery.

Some people now say that Mendel fiddled his data to strengthen his theory, but that's another story. Mendel, who lived in the middle part of the previous century, published his findings in an obscure journal, and they remained unknown until after his death. He and his scientific successors, in particular the English geneticist Galton, believed that the inherited characteristic such as the colour of the sweet pea, was transmitted from generation to generation by means of genes, which were a sort of "atom" of information supposedly of chemical nature. In any individual these genes form larger ensembles called chromosomes, and in every individual plant or animal those chromosomes occur in pairs. (For instance, human beings have 23 pairs of chromosomes.) On reproduction, a chromosome pair of an offspring is formed, a chromosome from each of the parents. It was not until the early 1950s that Crick and Watson, in one of the most fascinating and far-reaching scientific discoveries of all time, worked out the chemical structure of genes and the mechanism by which inherited information is transmitted from generation to generation. An account of this can be read in Watson's book, "*The Double Helix*", named after the geometric structure of the large DNA molecules from which genes and chromosomes are made. However, like Mendel, we do not need to know these scientific details in order to make elementary but useful predictions.

Let us consider the example considered by Mendel himself, that of sweet peas with the colours red, white and pink. We will assume that this colour is determined by a single gene which can take one of two slightly different forms, say "A" and "a". (These are called *alleles*.) In particular, we will assume that the gene pair AA produces a red flower, aa a white flower and Aa a pink flower, a formulation which suggests (and is in fact true) that a pink flower is a hybrid of the red and white flowers. When we cross-fertilize two flowers, the offspring will inherit, with equal probability, one of the two alleles of each of its parents, and its colour will depend on the resulting pair. What this is thus partly depends on the alleles possessed by the parents and partly on chance. Like Mendel we can systematically work out all possible combinations and the probabilities of their outcomes. The simplest cases are when the parents have non-hybrid pairs AA or aa. We have

1. *Both parents* AA (*red*) $\Rightarrow$ *offspring* AA (*red*)

2. *Both parents* aa (*white*) $\Rightarrow$ *offspring* aa (*white*)

3. *One parent* AA (*red*), *the other* aa (*white*) $\Rightarrow$ *offspring* Aa (*pink*)

(The last case occurs since the offspring can only inherit an "A" allele from its red parent and an "a" allele from its white parent.)

When we cross-fertilize a red or white flower with a pink one, we get two equally probable outcomes in each case. Now the probability of getting a particular allele from a given parent is $\frac{1}{2}$ and each parent has two alleles. Consequently the probability of getting a particular pair of alleles from the parents is

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4},$$

since which particular allele an offspring gets from one parent is statistically independent of which one it gets from the other. Note that there are four different possible pairs, depending on which allele comes from which parent, and each of these outcomes is equally probable. Of course some of the pairs may be the same. (It doesn't matter which particular parent an allele comes from, the important thing being the allele itself.) The easiest way to look at what happens here is to form a 2×2 matrix, that is a rectangular array, with the columns depending on which allele is inherited from one parent and the rows depending on which is inherited from the other. It doesn't matter here which parent is represented by the columns or the rows. Similarly the hybrid pairs Aa and aA denote the same pair. Then we have

PINK PARENT

Aa

			A	a
red parent	AA	A	AA	Aa
		A	AA	Aa

AA occurs twice with probability $\frac{1}{4}$ each, so AA occurs with probability $2 \times \frac{1}{4} = \frac{1}{2}$.
Similarly Aa occurs with probability $\frac{1}{2}$.

We summarize:

4. *If we cross an* AA *(red) flower with an* Aa *(pink) flower, the offspring will be* AA *(red) with probability* $\frac{1}{2}$ *and* Aa *(pink) with probability* $\frac{1}{2}$.

An analogous situation holds for a white and a pink flower:

5. *If we cross an* aa *(white) flower with an* Aa *(pink) flower, the offspring will be* aa *(white) with probability* $\frac{1}{2}$ *and* Aa *(pink) with probability* $\frac{1}{2}$.

The remaining case of cross-fertilizing a pink flower with a pink flower is the most interesting, but the most complicated:

PINK 1

		A	a
Pink 2	A	AA	Aa
	a	aA	aa

Here we get AA (red) with probability $\frac{1}{4}$, Aa (pink – don't forget, aA is the same) with probability $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and aa (white) with probability $\frac{1}{4}$. Thus we have

6. *If we cross an* Aa *(pink) flower with an* Aa *(pink) flower, the offspring will be* AA *(red) with probability* $\frac{1}{4}$, Aa *(pink) with probability* $\frac{1}{2}$ *and* aa *(white) with probability* $\frac{1}{4}$.

The situation we have described so far is an elementary application of probability theory. To be fair on Mendel, probability theory was not well understood in his day, and of course he knew nothing about the chemical mechanisms, in fact of any mechanism, underlying inheritance. What makes the above model simple is that each of the three allele pairs AA, Aa and aa corresponds to (or *causes*) an observably distinct phenomenon, namely a red, a pink and a white flower. Things become more complicated when two of the pairs give the same outcome. This occurs when one of the characteristics caused by an allele "B", say, is dominant and the other, due to the allele "b", is recessive. A simplistic (and not quite true) model is with human eye-colour, with the pairs BB and Bb causing the dominant colour, say brown, and the pair bb causing blue eyes. (In this model other colours are excluded.) If we know that the parents are of particular pairs BB, Bb or bb we can easily carry out the same analysis as above for sweet-pea colour., Certainly a blue-eyed person must be bb, but a brown-eyed person can be either BB or Bb. Consequently if all we know is that a parent is brown-eyed then we cannot be certain if the parent is BB or Bb. This could really only be determined by several generations of breeding or by the parent's producing several offspring. For instance, if a brown-eyed parent has offspring with a blue-eyed parent then only a brown-eyed parent of the mixed pair Bb can produce a blue-eyed offspring (bb), whereas a BB parent will always produce a Bb offspring with a blue-eyed partner. The interested reader could probably think of other ways of determining the exact genetical make-up of a brown-eyed individual. Note also how two brown-eyed parents of the mixed Bb pair will have brown-eyed offspring (BB or Bb) with probability $\frac{3}{4}$ and a blue-eyed offspring (bb) with probability $\frac{1}{4}$. This is why the blue-eyed colour here is called a recessive characteristic. We are all aware of such recessive characteristics from our own family backgrounds – I myself have fair hair, while my parents both have dark hair, but one of my grandparents was fair-haired and the other three dark-haired. Of course, the genetical situation in human beings is not as simple as in the above model. Nevertheless, the model does give us some instinctive insight into what is going on.

So far we have only looked at what is happening on the individual level. Of course this is important, but in some instances the genetical characteristics of a population are important, too. Suppose that our jolly monk Mendel (actually he was very shy and timid) sold his sweet pea to make some money to buy books for the monastery. If red flowers fetched a higher price he would naturally want to produce more red flowers. Suppose that there are N flowers in the garden. Since each has two colour genes, there are thus $2N$ alleles "A" or "a" in this population of flowers. Let the fraction of "A" alleles be p (there are thus $2Np$ alleles of type A) and the fraction of "a" alleles be q, so $q = 1-p$. In practice we probably would not know p and q. Rather, our observations would tell us immediately that there are Nf red flowers, Ng pink flowers and Nh white flowers, where the fractions f, g, h add up to 1. Counting up the alleles we have

red (AA)	$A \longrightarrow 2Nf$ times
pink (Aa)	$A \longrightarrow Ng$ times, $a \longrightarrow Ng$ times
white (aa)	$a \longrightarrow 2Nh$ times.

Thus we have the A allele occurring $2Nf + Ng$ times out of $2N$ and the a allele $Ng + 2Nh$ times out of $2N$, where

$$(2Nf + Ng) + (Ng + 2Nh) = 2N(f + g + h) = 2N.$$

From this we can calculate that

$$p = \frac{2Nf+Ng}{2N} = f + \tfrac{1}{2}g$$

and

$$q = \frac{Ng+2Nh}{2N} = \tfrac{1}{2}g + h.$$

This will tell us p and q in terms of f, g and h. Now you may think we could work out f, g and h if we know p and q by these same equations. But that is not possible; at least it is not possible to find a unique triple of values f, g, h because we have one more unknown than we have independent equations. For example, if $p = \frac{1}{4}$ and $q = \frac{3}{4}$ we have

$$f + \tfrac{1}{2}g = \tfrac{1}{4} \text{ and } \tfrac{1}{2}g + h = \tfrac{3}{4}.$$

With $g = 0$ we find that $f = \frac{1}{4}$ and $h = \frac{3}{4}$, whereas with $g = \frac{1}{2}$ we get $f = 0$ and $h = \frac{1}{2}$. In fact there are infinitely many solutions here:

$$f = \tfrac{1}{2}(\tfrac{1}{2} - \alpha),\ g = \alpha,\ h = \tfrac{1}{2}(\tfrac{3}{2} - \alpha),$$

one for each value of α in the interval $0 \le \alpha \le \frac{1}{2}$. (Note that $0 \le f,g,h \le 1$ here, hence α cannot exceed $\frac{1}{2}$.)

In order to pin down a specific solution we must make an assumption on how the members of the population "interact" with one another. In the case of the sweet peas this really means which flowers get to cross-fertilize with which flowers. Of course, if we only grew red (AA) flowers and excluded any possibility of cross-fertilization by pink (Aa) or white (aa) flowers, then the offspring would always be red (AA). This may be possible under laboratory conditions, but would certainly not be true in a garden or for wild flowers. At the opposite extreme is what we now call random mating, that is where cross-fertilization occurs independently of the colours of the parent flowers. This is a fairly realistic assumption here, as it probably is with human eye colour in Australia (but not in Sweden). It is certainly not true for human skin colour. Under the assumption of random mating in a very large population, the probabilities of red (AA), pink (Aa) or white (aa) flowers, that is f, g and h can be straightforwardly calculated using the probabilities (frequencies) of the alleles A and a, namely p and q. Since the events of obtaining an A allele or an a allele from any parent are independent in this case we have $f = p^2$ since the probability of the A allele is p for both A's that make up the AA pair. Similarly we get $h = q^2$ for the aa pair and $g = 2pq$ for the Aa pair (since this can occur in two different ways, depending on which parent provides A and which provides a). Thus we have

$$f = p^2,\ g = 2pq \text{ and } h = q^2.$$

These are known as the *Hardy-Weinberg* laws, so-named in honour of the English pure mathematician Hardy and the German physician Weinberg, who first enunciated them around the turn of the century. While very simple, they have some profound consequences. One of these is that regardless of the initial values of f, g and h after a few generations of random mating, these distributions will equal the values given by the Hardy-Weinberg laws.

Suppose that we have $p = \frac{1}{4}$ and $q = \frac{3}{4}$. Then eventually, the proportions of red, pink and white flowers will be $\frac{1}{16}$, $\frac{3}{8}$ and $\frac{9}{16}$.

Problem (Send your solution to the Editor of Function)

Suppose that eye colour is caused by a single gene with two alleles B and b, where BB and Bb result in brown eyes and bb in blue eyes. Suppose that one population of individuals of whom 64% are blue-eyed is combined with another population of twice the size with 16% blue-eyed. Determine what the proportion of blue-eyed members in the combined population will be after a few generations.

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NO HIGHWAY

In 1742 Goldbach postulated: "Any sum of two squares is a prime".

No highway in the sky,
1951 drama starring
James Stewart and Marlene Dietrich

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PROPHETIC WORDS?

Algebra, in its entirety, contains not a single mental operation that would not have already been used in digital computing.

E. Cassirer (1874-1945)
philosopher, in 1911[!]