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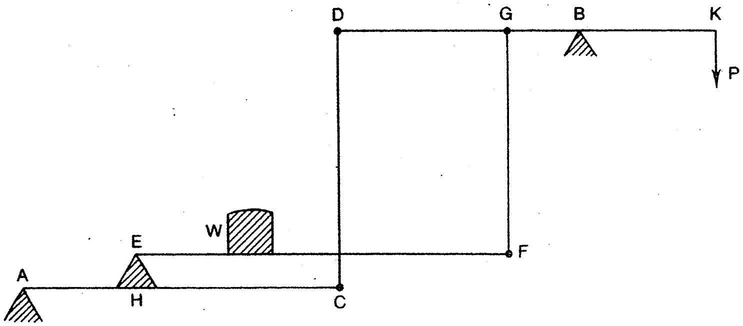

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Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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FRONT COVER

The picture on the front cover is taken from Barnard's *Elementary Statics* (MacMillan 1949), in which it illustrates an article entitled "Weighing Machine". The balance is called "Quintenz's Balance" and we reproduce below Barnard's article.

A simple form of weighing machine or weigh-bridge consists of a series of jointed bars as shown on the front cover.

A, B are fixed fulcrums about which the bars can rotate. EF is the platform on which the body to be weighed is placed. C, D, F, G are joints.

E rests on a fulcrum fixed to AC.

Now, if the bars move, C and D will move the same distance, say x downwards.

E and H will move a distance $\dfrac{AH}{AC}x$.

F and G will move a distance $\dfrac{BG}{BD}x$.

Therefore if $\dfrac{AH}{AC} = \dfrac{BG}{BD}$, E and F will move the same distance, and therefore the whole platform moves the same distance.

If we call the distance the platform moves x' ,

G moves down a distance x' ,

K moves up a distance $\dfrac{KB}{BG}x'$,

and therefore if a weight P at K is required to balance W ,

$$P\,\frac{KB}{BG}\,x' = Wx' ,$$

$$W = \frac{KB}{BG}\,P .$$

As the whole platform moves the same distance, it does not matter on what portion of it the load is placed.

Often this balance is made with $KB = 10\ BG$, and is then called a Decimal Balance.

The work done by the parts of the machine is not taken into account in the above theory.

COOKING MEAT AND POTATOES

Michael A.B. Deakin, Monash University

We consider an old-fashioned (i.e. convective) oven and we wish to use it to bake the family roast. How long will it take?

There is an old rule-of-thumb that says "20 minutes + 20 minutes per lb.", which we would have to metricate as "20 minutes + 45 minutes per kilogram."

This rule has been criticised. Stephen Kline in his book *Similitude and Approximation Theory* analyses the problem in very considerable detail, and one deduces from a footnote on p.98 of this work that his analysis threatened the domestic harmony of the Kline household in the early days of their marriage. A formula like Kline's was produced by P.J.Blennerhasset in *Function's* N.S.W. counterpart *Parabola* (Vol.17, No.3, pp.2-5). Kline points out that the problem is one of great importance, not only for the cooking of food, but for many industrial processes as well.

Let us see what cooking involves. Suppose first that we are boiling a potato, and let us approximate and suppose the potato is spherical. As we put the potato into the boiling water, we raise the surface temperature of the potato to that of boiling water. The interior of the potato is still at room temperature at this point. But as time goes by, the internal temperature rises. If we take the potato out too soon, an uncooked core will remain. The potato, as a whole, is cooked when this core has a radius of zero.

Food cooks as it undergoes an irreversible chemical change - when it is raised to a certain temperature, called the *cooking temperature*. This, in the case of the potato, is less than that of boiling water, but above room temperature. Heat diffuses into the potato, gradually raising the internal temperature, until after a long time its temperature throughout is that of boiling water. Before this occurs, when the coolest part of the potato (its centre) reaches the cooking temperature, the potato is done and is ready to eat.

This is the same problem that confronts the family roast. Initially at room temperature, it is placed in a hot environment and the heat must diffuse into its interior until its entire bulk has been heated to the cooking temperature or above.

To simplify matters, approximate the roast by a sphere. For many roasts, this is not a bad approximation. Essentially, we only need a boneless and relatively compact roast if our analysis is to apply.

(Some years ago, as legend has it, a large firm of poultry breeders asked a mathematician to analyse the heat distribution in their incubators. The unfortunate consultant began his report "Consider a spherical chicken ...". This did not endear him to his employers, but it probably gave a very good approximation.)

Let us list the quantities involved and suppose (provisionally only) that we measure them in S.I. units (listed also).

There is the *cooking time*, t (seconds), there are three temperatures : the *initial temperature* T_i, the *oven temperature* T_o, and the *cooking temperature* T_c (all measured in degrees Kelvin), there is the *volume* V of the roast (measured in cubic metres!) and finally a constant k, called the *thermal diffusivity* (measured in m^2s^{-1}), that describes the rate of heat-flow in the meat.

Now these units are ill-adapted to our problem, and in practice we would use more realistic ones. Here, however, it turns out not to matter. Our formula is to be deduced from the consideration that it must hold good in *any* units. (The systematic study of this question is called *Dimensional Analysis*; see *Function Vol.*10, *Part* 1, pp.14-22.)

Take first the various temperatures. The actual heat flow is caused by the difference $T_o - T_i$. Cooking is complete when this is reduced to $T_o - T_c$ everywhere. Now consider the ratio of these two temperature differences

$$\frac{T_o - T_i}{T_o - T_c} \quad ;$$

this will have no units; it will be a pure number.

The other three variables involved can also be combined to make a pure number. The ratio

$$\frac{kt}{V^{2/3}}$$

is independent of our units of measurement.

Thus a formula that is valid for all systems of measurement is

$$\frac{kt}{V^{2/3}} = f\left(\frac{T_o - T_i}{T_o - T_c}\right) \qquad (1)$$

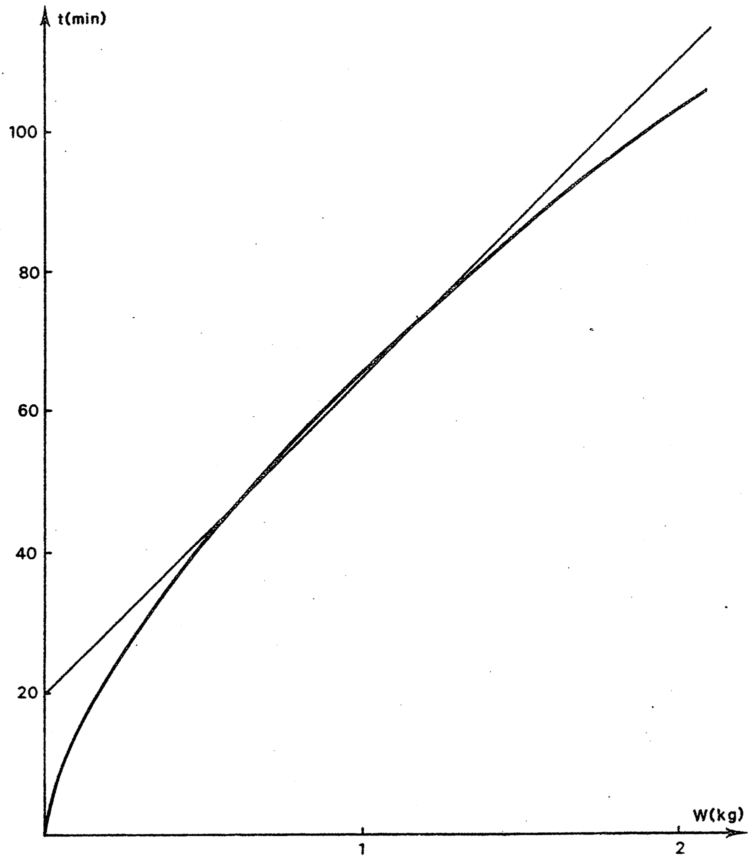


Figure 1.

where f is some unknown function.

Now meat is measured by weight rather than by volume, and we have

$$W = \rho V \tag{2}$$

where ρ is the density and W is the weight.

Combining Equations (1), (2) gives, after some rearrangement,

$$t = W^{2/3} \left\{ \frac{1}{k \rho^{2/3}} \; f \left[\frac{T_o - T_i}{T_o - T_c} \right] \right\} \tag{3}$$

or

$$t = KW^{2/3} \tag{4}$$

where K abbreviates the bracketed expression in Equation (3).

For this same problem, Blennerhasset deduced a more explicit (but approximate) formula

$$t = W^{2/3} \left\{ \left[\frac{3}{4 \pi \rho} \right]^{2/3} \frac{1}{k \pi^2} \; ln \left[\frac{T_o - T_i}{T_o - T_c} \right] \right\} \tag{5}$$

(where ln stands for the natural logarithm) using much more advanced methods, and quite difficult mathematics.

Kline considered a more complicated problem and took into account some complexities ignored here. His formula, however, is compatible also with Equation (3) and hence Equation (4).

Thus the rule-of-thumb enunciated at the beginning of this article is incorrect. Both Blennerhasset and Kline point this out and this would seem to have been the cause of the latter's row with Mrs Kline. (They seem to have ended up compromising by using a meat thermometer.)

Nonetheless I thought generations of housewives could hardly be wrong, and they are not. Figure 1 plots the rule-of-thumb against Equation (4) with $K = 65$ in the appropriate units (minutes $Kg^{-3/2}$). The fit is very accurate over a quite realistic range.

So the rule-of-thumb can be accepted - even by pig-headed mathematicians.

RENE DESCARTES†

John Stillwell, Monash University

René Descartes was born in La Haye (now called La Haye-Descartes) in the French province of Touraine in 1596, and died in Stockholm in 1650. His father Joachim was a councillor in the high court of Rennes in Brittany, while his mother Jeanne was the daughter of a lieutenant general from Poitiers, and the owner of property which was eventually sold to assure Descartes of financial independence. His mother died in 1597, and Descartes was raised by his maternal grandmother and a nurse. He does not seem to have been close to his father, brother or sister, seldom mentioning them to others, and writing to them only on matters of business.

Joachim Descartes was away from home for half the year because of his court duties, but saw enough of René to observe his exceptional curiosity, and called him his "little philosopher". In 1606, he enrolled him in the Jesuit College of La Flèche, which had recently been founded by Henry IV in Anjou. The young Descartes was given special privileges at school, in recognition of his intellectual promise and delicate health. He was one of the few boys to have his own room, was permitted books forbidden to other students, and was allowed to stay in bed until late in the morning. Spending several morning hours in bed thinking and writing became his lifelong habit and, when he finally had to break it in the Swedish winter, the consequences were fatal.

The most dramatic event of his schooldays was the assassination of Henry IV in 1610. Since Henry IV was not only the founder of the school, but also the most popular king in French history, his death was a profound shock. La Flèche became the venue for an elaborate funeral ceremony, the climax of which was the burial of the king's heart. Descartes was one of 24 students chosen to participate in the ceremony.

He left La Flèche in 1614 and, after legal studies at Poitiers which seem to have left no impression on him, went to Holland as an unpaid volunteer in the army of Prince Maurice of Nassau in 1618. This was not an unusual decision for a young Frenchman of means at the time, since the Dutch were fighting France's enemy, Spain, and Descartes seems to have joined the army to see the world, not because of any taste for barracks life or combat. As it happened, there was a lull in the war at the time, and Descartes had two years of virtual leisure to reflect on science and philosophy.

When in Breda, on 10 November 1618, he saw a mathematical problem posted on a wall. Since his Dutch was not yet fluent, he

† Extract from a book on the history of mathematics being written by John Stillwell.

asked a bystander to translate for him. This was how Descartes met Isaac Beeckman, who became his first instructor in mathematics, and a lifelong friend. The following November 10, Descartes was in Bavaria. He spent a day of intense thought in a heated room ("stove" he called it) and that night had a dream he later considered to be a revelation of the path he should follow in developing his philosophy. Whether the dream also revealed the path to analytic geometry, as some have conjectured, will probably never be known. Descartes' own description of the dream has been lost, and we have only a summary by his first biographer, Baillet ([1691], p.85), which is not helpful. In any case, it seems a little ludicrous to award Descartes priority over Fermat on the basis of a dream. Could a counterclaim of priority be lodged if a teenage dream of Fermat came to light?

In 1628, Descartes moved to Holland, where he spent most of the rest of his life. He lived a simple but leisurely life and finally settled down to working out the ideas conceived 9 years before. The relative isolation suited him, as he was hostile to other scientific giants of his time - such as Galileo, Fermat and Pascal - and preferred to communicate with scholars who could understand him without challenging his superiority. One such was Marin Mersenne, who had been a senior student at La Flèche in Descartes' time, and was his main scientific contact in France. Others were Princess Elizabeth of Bohemia and Queen Christina of Sweden, with both of whom Descartes had an extensive correspondence.

A positive side to Descartes' intolerance of intellectual rivals was an apparently genuine interest in the affairs of his neighbours in Holland. He encouraged local youths who showed talent in mathematics, and was known in the region as someone to turn to in times of trouble (see Vrooman [1970, pp. 194-196]. The one serious love of his life was a servant girl named Helen, who bore him a daughter, Francine, in 1635. Admittedly, his interest in this case did not extend to marrying Helen, but the death of Francine from scarlet fever in 1640 caused him the greatest sorrow of his life.

In 1649 Descartes agreed to journey to Stockholm to become tutor to Queen Christina. This was the culmination of his correspondence with her, and of negotiations through Descartes' friend Chanut, the French ambassador. The Queen, who was noted for her physical as well as mental vigour, slept no more than 5 hours a night and rose at 4 a.m. Descartes had to arrive at 5 a.m. to give her lessons in philosophy. The programme commenced on 14 January 1650, during the coldest winter for over 60 years. One can imagine the shock to Descartes' system of such early rising followed by a journey from the ambassador's residence to the palace. However, it was actually Chanut who succumbed to the cold first. On January 18 he came down with pneumonia, and Descartes apparently caught it from him. Chanut recovered, but Descartes did not, and he died on 11 February 1650.

(Continued on page 88)

THE PROBABILITY OF WINNING THE VFL GRAND FINAL

S.N.Ethier, University of Utah

At the conclusion of the home-and-away rounds, the top five teams in the Victorian Football League meet in a series of finals matches culminating in the Grand Final. The VFL Finals Series consists of four rounds of matches played on four successive weekends. Let us refer to the team at the top of the league ladder as Team 1, the team in second position on the ladder as Team 2, and so on. In the first round, Teams 4 and 5 meet in the Elimination Final (the loser of which is eliminated from the competition), Teams 2 and 3 meet in the Qualifying Final, and Team 1 receives a bye. In the second round, the winner of the Elimination Final meets the loser of the Qualifying Final in the First Semi-Final (the loser of which is eliminated), and the winner of the Qualifying Final meets Team 1 in the Second Semi-Final. In the third round, the winner of the First Semi-Final meets the loser of the Second Semi-Final in the Preliminary Final (the loser of which is eliminated), and the winner of the Second Semi-Final receives a bye. Finally, in the fourth round, the winners of the Preliminary Final and the Second Semi-Final meet in the Grand Final. See Figure 1.

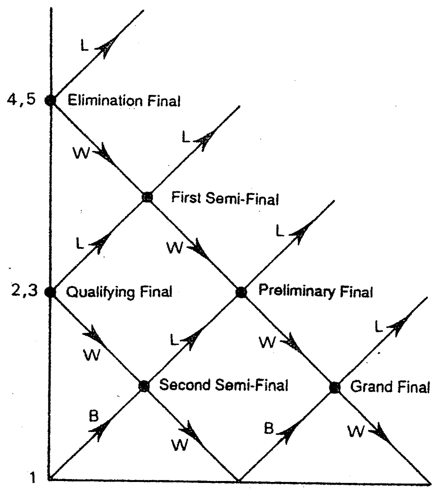


Figure 1. The five-team VFL Finals Series.

To the best of our knowledge, Professor W.J. Ewens of Monash University was the first to raise (and answer) the following question. Assuming that the top five teams are evenly matched and the results of the six finals matches are determined independently, what is the probability that Team k ($1 \leq k \leq 5$) wins the Grand Final? The purpose of the present article is to try to answer this question not just for the five-team Finals Series described above, but for an n-team Finals Series.

The five-team format has been used since 1972. From 1931 to 1971 an analogous four-team Finals Series was used. See Figure 2. Both systems were devised by K.G.McIntyre.

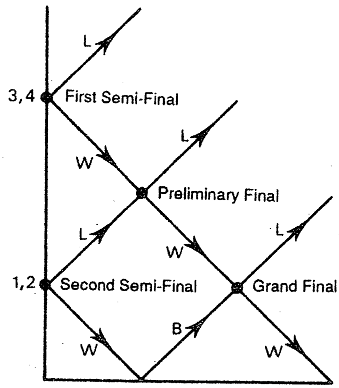


Figure 2. The four-team VFL Finals Series.

Let us inductively define an n-team Finals Series that generalizes the special cases $n = 5$ and $n = 4$. A two-team Finals Series consists of a single match, the Grand Final. An n-team Finals Series consists of $n - 1$ rounds. In the first round, matches are played between Teams n and $n - 1$, Teams $n - 2$ and $n - 3$, and so on, and Team 1 receives a bye if n is odd. The winner of the match between Teams k and $k - 1$ becomes Team $k - 1$ in the next round, and the loser becomes Team k in the next round unless $k = n$, in which case the loser is eliminated. Rounds 2 through $n - 1$ of the n-team Finals Series are just Rounds 1 through $n - 2$ of an $(n - 1)$-team Finals Series between the newly designated Teams 1 to $n - 1$.

We invite the reader to verify that an n-team Finals Series consists of $(n^2 - 1)/4$ matches is n is odd, $n^2/4$ matches if n is even.

All probabilities in this article will be computed under the assumptions that the teams are evenly matched and the results of the various matches are determined independently. Of course, these assumptions (especially the first one) are rather dubious, but they will allow us to determine the extent to which the higher-ranked teams are inherently advantaged by the system. Alternatively, we can regard the assumptions as an idealization corresponding to the hypothetical case of league parity.

Let us begin with the case $n = 5$. Referring to Figure 1, we note that Team 5 (or Team 4) can win the Grand Final only by winning four consecutive games, and this happens with probability

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16} .$$

Team 3 (or Team 2) can win the Grand Final with any one of three sequences of wins, losses, and byes, *LWWW*, *WLWW*, or *WWBW*, and therefore it wins with the probability

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{16} + \frac{1}{16} + \frac{1}{8} = \frac{1}{4}$$

Finally, Team 1 can win the Grand Final with either of two sequences, *BLWW* or *BWBW* , and therefore it wins with probability

$$1 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8} .$$

Notice that the five probabilities obtained here (one for each of the five teams) sum up to 1, as they should.

A similar argument gives the probabilities in the case $n = 4$. See Table 1.

	$n = 4$			$n = 5$	
k	probability	observed frequency 1931-1971	k	probability	observed frequency 1972-1986
1	3/8	23/41	1	6/16	7/15
2	3/8	14/41	2	4/16	4/15
3	1/8	0	3	4/16	4/15
4	1/8	4/41	4	1/16	0
			5	1/16	0

Table 1. The probability that Team k wins the Grand Final in an n-team Finals Series.

The above solution for the case $n = 5$ is perhaps the simplest in that case, but it does not easily generalize to the case of arbitrary n. Let us describe a second solution for the case $n = 5$, which does generalize.

We suppose that each time a team receives a bye, a coin is tossed, and the result of the coin toss (heads or tails) is recorded instead of the bye. (However, the team still receives its bye, and so the result of the coin toss has absolutely no effect on the outcome of the Finals Series.) Figure 1 is then replaced by Figure 3. Again Team 5 (or Team 4) can win the Grand Final only with four consecutive wins, *WWWW*. Team 3 (or Team 2) can win with any one of the four sequences of wins, losses, heads, and tails, *LWWW*, *WLWW*, *WWHW*, or *WWTW*. Team 1 can win with any one of the six sequences, *HLWW*, *HWHW*, *HWTW*, *TWHW*, *TWTW*, or *TLWW*. The advantage of this approach is that each sequence of wins, losses, heads, and tails of length four has probability 1/16, and so we immediately get the probabilities in Table 1 for $n = 5$ by counting the number of sequences and dividing by 16.

We turn now to the case of arbitrary $n \geq 2$. It will be convenient to introduce a coordinate system in Figures 1-3. Think of the horizontal line in each figure as the x-axis, the vertical line as the y-axis. In the case $n = 5$ (Figure 1), Teams 5 and 5 begin at (0,4), Teams 2 and 3 at (0,2), and Team 1 at (0,0); the Grand Final winner finishes at (4,0). In the case $n = 4$ (Figure 2), Teams 3 and 4 begin at (0,3), Teams 1 and 2 begin at (0,1), and the Grand Final winner finishes at (3,0). In the case of arbitrary n, Team k begins at $(0, s(k,n))$, where

$$s(k,n) = \begin{cases} k - 1 & \text{if } k \text{ and } n \text{ are both odd or both even,} \\ k & \text{otherwise} \end{cases},$$

and the Grand Final winner finishes at $(n-1,0)$. Each sequence of wins, losses, heads, and tails of length $n-1$ has probability $(1/2)^{n-1}$. Therefore, to determine the probability that Team k wins the Grand Final, we need only count the number of admissible paths from $(0,s(k,n))$ to $(n-1,0)$ and divide by 2^{n-1}.

Each such path corresponds to a sequence of +1's and −1's of length $n-1$, specified by the sequence of changes in the y-coordinate of the path as the x-coordinate increases one unit at a time from 0 to $n-1$. (For example, the path *TWHW* for Team 1 when $n = 5$ would correspond to the sequence −1, +1, +1, −1; see Figure 3.) Moreover, since the net decrease in the y-coordinate of such a path is $s(k,n)$, the corresponding sequence of +1's and −1's must have $s(k,n)$ more −1's than +1's, hence $(n-1-s(k,n))/2$ +1's and $(n-1+s(k,n))/2$ −1's. Noting that the correspondence is one-to-one, we need only determine the number of such sequences.

More generally, how many permutations (or orderings) of m letters, of which j are A's and the remainder are B's, are possible? If the letters were all distinct, say $A_1, A_2, \ldots, A_j, B_1, B_2, \ldots, B_{m-j}$, the answer would be $m! = m(m-1) \ldots 3.2.1$. However, in the list of these $m!$ permutations, each permutation of the j A's and $m-j$ B's (with subscripts removed) appears $j!.(m-j)!$ times. So the answer is

$$\frac{m!}{j!(m-j)!},$$

which is the binomial coefficient

$$\binom{m}{j}$$

From the three preceding paragraphs, we can finally conclude that

$$Pr\{\text{Team } k \text{ (of } n \text{ teams) wins Grand Final}\}$$

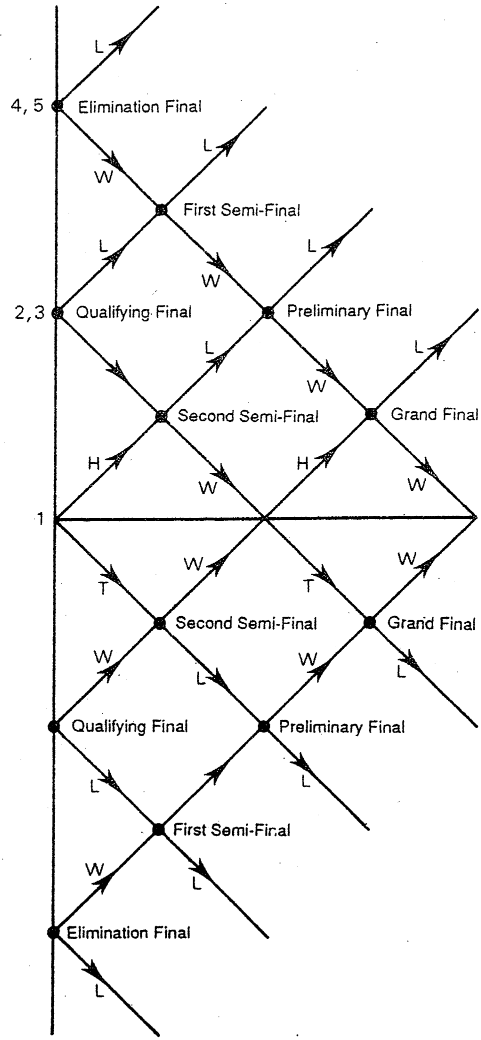
$$= \binom{n-1}{(n-1-s(k,n))/2} \Big/ 2^{n-1} . \qquad (1)$$

This is the solution of our problem. Notice that the probabilities in Table 1 follow immediately from this formula.

There is a close relationship between Equation (1) and the well-known binomial distribution

$$Pr\{j \text{ heads in } n-1 \text{ coin tosses}\} = \binom{n-1}{j} \Big/ 2^{n-1}. \qquad (2)$$

Equation (2) follows easily from the observations that each



4, 5
Elimination Final
L
W
L
First Semi-Final
L
L
W
2,3
Qualifying Final
Preliminary Final
L
L
W
Second Semi-Final
Grand Final
H
H
W
W
1
W
W
T
T
Second Semi-Final
Grand Final
W
W
L
L
Qualifying Final
Preliminary Final
L
L
First Semi-Final
W
L
Elimination Final
L

Figure 3

sequence of j heads and $n-1-j$ tails has probability $(1/2)^{n-1}$ and that there are

$$\binom{n-1}{j}$$

such sequences, the latter because of the above result involving the A's and B's. Now, it is a rather interesting fact that the n probabilities in Equation (2) $(0 \le j \le n-1)$, when arranged in descending order of magnitude, coincide with the n probabilities in Equation (1) $(1 \le k \le n)$.

We close with two topics for discussion.

The 28-team National Football League (in the U.S.) consists of two conferences, each of which consists of three divisions. At the conclusion of the regular season, the top five teams in each conference meet in a (single-elimination) series of matches leading to the conference championship. The two conference champions subsequently meet in the Super Bowl. The top five teams in each conference are comprised of the three division winners and two so-called wild-card teams, which are the two teams with the two best records among all non division winners. In the first round, the two wild-card teams meet, and the three division winners receive byes. The four remaining teams play two semi-final matches in the second round, and the two winners meet in the third round for the conference championship. Should the NFL adopt a pair of VFL-style five-team Finals Series (one in each conference) to determine its conference champions?

With the addition of teams in Brisbane and Perth, the VFL has expanded from 12 to 14 teams this year. Should a six-team Finals Series be introduced in order to maintain the proportion of teams admitted to post-season competition?

Reference

VFL Year Book 1985. Seventh Revised Edition. Victorian Football League, Jolimont, Vic.

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THE AXIOM OF INEQUALITY

Michael A.B. Deakin, Monash University

GOTTFRIED WILHELM LEIBNIZ (or *LEIBNITZ*) (1646-1716) was, with *NEWTON*, the co-discoverer of calculus. While this is his main claim to fame, he was active in other areas of mathematics, science and philosophy and would merit the attention of historians even had he not succeeded in the research which led to the calculus.

This being the case, you may perhaps find it unfair of me to begin my story with one of his mistakes. But great thinkers' mistakes are often particularly instructive. Cricket commentators sometimes say of a particular ball that only a good batsman would go out to it. And so it is here. Some mistakes are such that only the great can make them.

In 1715, Leibniz wrote to Dr Samuel Clarke, who translated his letter into English, an argument on the impossibility of there being atoms. (The Philosophical Library, New York, reprinted this in 1956 in *The Leibniz-Clarke Correspondence*, edited by H.G.Alexander - the quoted passage is on p.36.) He wrote:

> "There is no such thing as two individual objects indiscernible from each other. An ingenious gentleman of my acquaintance, discoursing with me in the garden of Herrenhausen in the presence of Her Electoral Highness the Princess Sophia thought he could find two leaves of grass perfectly alike. The Princess defied him to do it. And he ran all over the garden for a long time to look for some. But it was to no purpose. Two drops of water, or milk, viewed in the microscope, will appear distinguishable from each other. This is an argument against atoms, which are confuted ... by the principles of true metaphysics."

There are some details left out, which I will now fill in, following *PHILIP MORRISON*, an American physicist, writing in 1958.

> "[Suppose a] blade of grass consists of a finite number of discrete particles ... . These can at most be arranged in a finite number of ways. Therefore there is a number N of possible blades of grass (below a certain maximum mass). If we examine carefully $N+1$ blades of grass, at least two must be alike. But we *never* find two alike. Therefore, said Leibniz, there cannot be atoms."

I leave it to readers to detect where the fallacy lies, because I want to take a different direction, although using this story as my starting point.

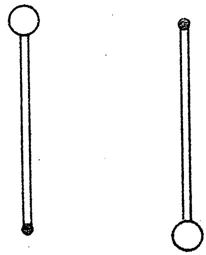
Clearly, Leibniz was not out merely to make a simple botanical point about leaves of grass. Indeed he later refers to drops of water and of milk. These all have the status of examples illustrating a general principle: that in the natural world we do not find things exactly alike. (How like are two peas in a pod, really?)

The principle has been expressed by the biologist GARRETT HARDIN as "No two things or processes in a real world are precisely equal". Hardin's discussion comes in the course of his analysis of some (mostly) ecological questions and it is he who gives the principle the title of *The Axiom of Inequality*. He remarks that, in respect of *things*, the matter is often trivial - does it matter, for many purposes, that no two blades of grass are alike? - but in respect of *processes* it often leads to important and far-reaching conclusions. (Hardin's article was published in 1960 in Volume 131 of the journal *Science*; regrettably, because of its reliance on verbal rather than mathematical argument, it contains some very contentious claims and indeed some clear errors.)

Consider one important case. For most of human history, the birth-rate has exceeded the death-rate and so the world population has steadily increased. Until very recently this was hardly a matter for concern and it only emerged as such with the analysis of THOMAS MALTHUS at the time of the industrial revolution. We are now at a point of history where it has become of very great concern indeed.

In this climate, what many people hope for is a "steady state population" in which the population remains constant, birth-rate in exact balance with death-rate. (Compassionate observers hope for a decline in birth-rate rather than an increase in death-rate as a means of bringing this about.) What the axiom of inequality tells us is that this simple equilibrium is unlikely to occur exactly like that. At any given time either the birth-rate will exceed the death-rate or vice versa. When the birth-rate is the greater, the population will increase; otherwise it will decrease. So the population will be a fluctuating one. It is to be hoped that it is not one which fluctuates too wildly - sudden famine or epidemics alternating with times of high increase.

Consider now a second example. Figure 1 shows a rigid pendulum pivoted about a fixed point *O* , and capable of resting in either of the two positions shown. Either its bob may be above the pivot or it may be below it.



Now if the pendulum is to rest with its bob above the pivot, then it must be *precisely* above the pivot, for if it were not, the pendulum would swing around toward the lower position. Now the axiom of inequality tells us not to expect this precise equality to occur, so we do not see this case. An equilibrium in which the bob rests above the pivot is said to be *unstable*. A corollary of the axiom of inequality is that unstable equilibria (or other behaviours) do not occur in nature.

Contrast this with the case when the bob lies below the pivot. Again it is most unlikely to lie precisely below it, but now if it is displaced slightly it will tend to go back toward equilibrium, and will in fact oscillate about its equilibrium position. Indeed were the friction forces precisely zero, it would repeat precisely its oscillations (which are approximately but not exactly simple harmonic) for ever, or at least until it were disturbed again. However, friction (by the axiom of inequality) cannot be exactly zero (and the second law of thermodynamics tells us it can't be negative), so frictional forces will ultimately retard the pendulum, so that, apart from other disturbances, it ultimately returns to rest.

One of the pioneers of mathematical ecology was the Italian mathematician *VITO VOLTERRA* (1860-1940). He produced a mathematical analysis of the interaction between a predator and its prey. In this analysis, suppose that both species are initially present in small numbers. The prey are then relatively free to multiply and they do so, but they thereby create favourable conditions for the predators whose numbers then expand, at great cost to the prey. Eventually, the prey are depleted and the predators go hungry and begin to die off. This process continues till there are only a few predators left and the cycle begins again.

Volterra's original theory had the system returning periodically to *precisely* its initial state, a situation at odds with the axiom of inequality. Later it was found that if certain terms in the equation are not exactly constant, but vary with the actual numbers of predators and of prey, then the oscillations are not repeated precisely, but tend either to become less noticeable, to lock into one especially preferred cycle, or to lead to the extinction of the predator (and possibly the prey as well).

Volterra's system is now said to be *structurally unstable.* By this is meant that small changes to the form of the equations involved produce large changes in the predicted outcome.

In Volume 2, Part 5 of *Function*, *PETER KLOEDEN* wrote on Stability and Chaos in Insect Populations and drew attention to the possibility that some systems seem to be chaotic. This describes a state in which any disturbance whatsoever, no matter how small, to a system results in its being altered completely in the long run. In one example, attempts at predicting the size of ~~super~~ cicada plagues from the last such, would soon become very inaccurate if we overlooked even one cicada in our count. Weather systems seem to be essentially chaotic in this sense, and that is why forecasting (particularly over long times) is so difficult. These instabilities result from another application of the axiom of inequality - the data we measure are never precisely accurate.

This is a form of instability that seems to be forced on us by nature. While we would very much like to be able to predict the weather with more certainty or to know what will happen if we disturb an ecosystem, it is beginning to emerge that there may be limits to human knowledge - that there may be some things we will never be able to know because our initial data can never be exactly perfect.

A not dissimilar insight was proposed in 1873 by the physicist *JAMES CLERK MAXWELL* (1831-1879) in a discussion of the freedom of human action. In those days, a perceived dilemma arose between the determinism of Newtonian mechanics and the freedom of our wills. It would seem to be possible, given the laws of Physics and a knowledge of the position and velocity of every particle in the universe, to predict the position and the velocity of each of those particles for all future time. Such an enterprise could never, of course, be carried out in practice, but it would seem to be possible in theory and this would seem to say that there is no room for me to decide whether to do this or that, how to spend my time, what to eat, how to fill my evenings, etc. The position and velocity of each particle that goes to constitute me is pre-ordained and there's nothing I can do about it. Or so it would seem.

Of course, this is a nonsense. What Maxwell sought was a way in which the apparent fact of human free will could coexist with the deterministic principles of Newtonian physics. His solution was to use the axiom of inequality and say that a knowledge of the initial positions and velocities can never, even in principle, be *precisely* exact, and so it may well be that the predictions made from such a basis allow enough uncertainty to give space to human freedom of choice. Maxwell wrote before the modern discovery and exploration of chaotic phenomena and this new development greatly reinforces and adds substance to the point he was concerned to make.

Quite recently, the French mathematician *RENÉ THOM* considered the possiblity of applying mathematics to (among other things) embryology. Although no two embryos develop (axiom of inequality) to be precisely identical, each develops into either a male or a female, and these are recognisable categories. (This statement slightly over-simplifies, but even with intersexes, they fall into separate and identifiable classes.) This is a *structurally stable* (or as Thom called it *generic*) feature of the system. Thom proposed that in the biological sciences, good theories had to be structurally stable, that the important features were those that were generic. He was thus lead to a detailed study of the property of structural stability (or genericity). He found that mathematical functions, when we considered that we might not know them exactly, could be classified into different categories - in fact, unless the functions became quite complicated, surprisingly few categories. His book *Structural Stability and Morphogenesis*, in which he explained these ideas, is widely seen as the work that initiated a new branch of mathematics, called *Catastrophe Theory*.

It is not my purpose here to explain this theory; for that see *Function*, Vol.1, Part 2. But it is in order here to note that Thom envisioned a new methodology for the biological sciences, rather different from the traditional one. In this, the way to analyse a biological phenomenon was to relate it to the simplest of the available structurally stable categories that was consistent with the experimental facts.

This approach is new and controversial. What future lies ahead for it is not certain; it is less than 20 years old and so it is too early to judge. So far, most attempts to apply it are rather unconvincing, but a medical application to thyroid gland malfunction has attracted some favourable attention.

So these are some of the things that have been done with the axiom of inequality. Left unanswered, so far, is one very basic question.

Is it true?

And the answer to this is "No, not always." Even in the biological sphere, there can be exceptions.

In most, but not quite all, instances, higher animals have two copies of each of their genes. Very often such genes exist in two forms called *alleles*. So an individual could have each of two alleles of the first type and be classified as *AA* (say), or there could be one of each, so that we could classify the individual as *Aa*, finally both alleles could be of the second type and the individual would be classed as *aa*.

Now very often we cannot tell *AA* individuals from those of type *Aa*. In such a case the allele *A* is said to be *dominant* and the allele *a recessive*. Very often, the combination *aa* corresponds to some genetically transmitted disease. When *A* is dominant and *a* recessive, an individual of the type *Aa* does not manifest the disease, but can transmit it to subsequent descendants.

In some cases, as with the disease *sickle-cell anemia*, for example, such carriers can be identified (by, in this instance, microscopic examination of their red blood cells). In others this is not possible.

Now we would imagine that an individual's carrying such an allele ought to make *some* difference. That is what the axiom of inequality would lead us to believe. Why doesn't it work like that?

The answer to this question comes from molecular biology. In such cases, the allele *A* is a sequence of DNA that codes for a protein, which forms a normal constituent of the body. Such proteins are referred to as *enzymes* and they act as catalysts for specific biochemical reactions. The allele *a* by contrast codes for an ineffective form of the enzyme. So the effective enzyme is present in *AA* and *Aa* individuals, but absent in *aa*. The absence of the enzyme is what leads to the disease.

So here is a case in which the axiom of inequality breaks down. But notice an important point of principle. It was the falsity of the axiom that required explanation. What philosophers call the *burden of proof* lies with the person who would deny the axiom.

One philosopher, *KARL POPPER*, put forward the thesis that this is characteristic of good science. In a book, translated into English as *The Logic of Scientific Discovery*, he argued that while science can never prove its statements (it differs from mathematics in this regard), it can make surprising statements that are susceptible of *disproof*. The more implausible the statement seems at first sight, the greater the scientific advance. Thus, in a nutshell, Popper. (I should add that Popper himself was forced to amend this view later, in particular to accommodate *DARWIN*'s theory of evolution, and that although influential, the account is by no means universally accepted.)

But let me close with a story of a grand violation of the axiom of inequality - a very surprising statement indeed, that takes us, in a way, back to Leibniz.

Every electron has the same charge and mass as every other electron.

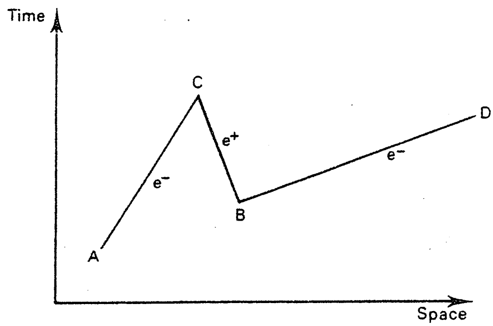
One physicist, J.A.WHEELER, saw this as very remarkable and requiring heroic explanation. He rang up one of his students, RICHARD P. FEYNMAN, who later told the story in his Nobel Prize acceptance speech. According to Feynman, the conversation went like this.

Wheeler: Feynman, I know why all electrons have the same charge and mass.

Feynman: Why?

Wheeler: Because they are all the same electron.

To see what Wheeler had in mind, consider the following diagram, (Figure 2). The track *AC* represents the motion of an electron (e). Somewhat later than *A*, at *B*, spontaneous pair production occurs (as it can) resulting in the production of an electron (e) and a positron (e^+). Their tracks are respectively *BD*, *BC*. At *C*, the positron interacts with the first electron and they annihilate each other.



What Wheeler is saying is that we can see the line *ACBD* as one zig-zag path. An electron travels from *A* to *C* where it reverses direction in time, becoming a positron in the process. At *B* it reverses time-direction again, reverting as it does so to being an electron. Wheeler saw such graphs (now called Feynman diagrams) as all connecting up in a sort of knot, presumably at the instant of the big bang or shortly thereafter.

Feynman did not accept Wheeler's idea that all electrons are really the same electron, and indeed there are problems with it, but the view of a positron as an electron travelling backwards in time he did use. It became one of the elements of his theory of quantum electrodynamics that earned him his Nobel Prize.

The concept can work only if all electrons and positrons have the same mass and (up to sign) charge. (Did you notice that Leibniz made a similar assumption about the atoms whose existence he set out to disprove?) The fact that this is so, despite what the axiom of inequality would incline us to believe, makes this a significant scientific statement.

From "The Age", 1 June 1987.

SUPERCOMPUTERS PLAY AT SLICING THE PI

By Surendra Verma

What do supercomputers do for recreation? Some play chess, others compute value of pi. NEC SX-2, a supercomputer at the University of Tokyo, recently took a break from its routine research job for a bit of fun. It spent two leisurely days pushing the computation of the value of pi beyond 134 million digits and thus eclipsed the record set last year by Cray-2, a supercomputer at the NASA Ames Research Centre in California.

If numbers are your recreation, then reciting the pi's 134,217,700 digits, one digit every second, would take you about four years.

Like square and cube roots of 2, pi is an irrational number: it takes a never-ending string of digits to express pi as a decimal number.

The value of pi - the ratio of a circle's circumference to its diameter - has captured the imagination of mathematicians since antiquity. From the Ahmes papyrus, an ancient papyrus roll found last century, we learn that, around 2200 B.C., Egytian mathematicians used 3.16 as a value of pi.

Archimedes's major mathematical contribution was the calculation of the 'correct' value of pi. He developed a method for approximating it by nesting a circle between a pair of polygons whose perimeters were easy to calculate. He proved that the value of pi lay between 223/71 and 220/70 (3.140845 and 3.142857, in the modern notation).

In 1650, John Wallis, an English mathematician, worked out unlimited series for the calculation of the value of pi. This opened a new crazy field of mathematics - calculation of the value of pi to an unlimited decimal place.

Even Newton was tempted. He calculated the value of pi to 15 decimal places but was never proud of this achievement. 'I am ashamed to tell you to how many figures I have carried out these computations, having no other business at the time,' he wrote to a friend.

In 1873, an Englishman named William Shanks spent years working out the expansion of pi to more than 700 digits. In 1949, a primitive computer pushed the expansion fo 2037 digits.

These days computation of pi is used to demonstrate publicly the capabilities of rival supercomputers. In the most recent demonstration, conducted by Yasumasa Kanada and his colleagues, two computer algorithms were used to check the result Each took about two days each time on the SX-2.

But why does one compute pi to 134 million digits, when even for designing a satellite one need not to know the value for more than a few digits? 'Part of the reason is that pi is the most naturally occurring of the nonalgebraic numbers,' says Peter Borwein of Dalhousie University in Halifax, Nova Scotia. 'And it's a number we know a little bit about but not a great deal about.'

Peter and his brother Jonathan developed the computer algorithm used in recent supercomputer pi wars. The Borwein algorithm is considered 'close to the theoretically best possible algorithm for computing pi'.

'There is a very small gap between what is known and what is possible,' says Peter Borwein. On that basis he predicts that no one will ever know 10^{1000} digits of pi.

Pi war games are nearly over for supercomputers.

* * *

RENÉ DESCARTES (continued from page 72)

Descartes is, of course, as well known for his philosophy as his analytic geometry. The *Geometry* was originally an appendix to his main philosophical work, the *Discourse on Method*. The other appendices were the *Dioptrics*, a treatise on optics, and the *Meteorics*, the first attempt to give a scientific theory of the weather. In the *Dioptrics*, Descartes did not inform his readers that Ptolemy, Alhazen, Kepler and Snell had already discovered the main principles of optics, nevertheless he presented the subject with greater clarity and thoroughness than before, and undoubtedly advanced both the theory and practice of optical instrumentation. As for the *Meteorics*, we now know how premature it was to attempt a theory of the weather in 1637, so it is understandable that this treatise has more misses than hits. His big hit was a correct explanation of rainbows (except for the colours, whose explanation was completed by Newton), which Descartes was able to give on the basis of his optics. More typical, unfortunately, was his explanation of thunder - caused by clouds bumping together, and not related to lightning. An excellent survey of Descartes' scientific work and philosophy, with a particularly detailed analysis of the *Geometry*, is given by Scott [1952].

* * *

PERDIX

There has been some interesting correspondence about question 2 of the 1987 Australian Mathematical Olympiad and I give below three separate approaches to a solution to this question, the second two of which progressively strengthen the result proved.

* * *

Question 2.

If p is an odd prime,

$$0 = (1-1)^{2p} = 1 - \binom{2p}{1} + \binom{2p}{2} - \dots - \binom{2p}{p} + \dots \qquad + 1$$

$$= 2 - \binom{2p}{p} - 2\left[\binom{2p}{1} - \binom{2p}{2} + \dots - \binom{2p}{p-1}\right]$$

since $\binom{2p}{k} = \binom{2p}{2p-k}$.

Hence $\binom{2p}{p} - 2 = -2\left\{\binom{2p}{1} - \binom{2p}{2} + \dots - \binom{2p}{p-1}\right\}$

Now, for $0 < k < p$,

$$\binom{2p}{k} = \frac{2p.(2p-1)\ \dots\dots\ (2p-k+1)}{1.\ 2.\ 3\ \dots\ k} .$$

By definition, $\binom{2p}{k}$ is an integer , and, since p is prime, no one of the numbers $1,2,3, \dots , k$, of the denominator divides p.

Hence $\binom{2p}{k}$ is a multiple of p . It follows that , since each term in $\{\ \dots\ \}$ is a multiple of p , so is $\{\dots\}$ itself. The result follows, on noting that, for $p = 2$,

$$\binom{4}{2} - 2 = 4 \text{ , a multiple of } 2 .$$

John Barton

I am writing to present a new angle on Question 2 of the 1987 Australian Mathematical Olympiad. Given a prime number p, we are to show that the integer

$$n = \binom{2p}{p} - 2$$

is a multiple of p. I claim it is a multiple of p^2.

Consider a club with p men and p women. It is desired to select a committee of size p which contains at least one person of each sex. Clearly there are n such committees. If $1 \leq k \leq p$, there are

$\binom{p}{k}\binom{p}{p-k} = \binom{p}{k}^2$ committees with exactly k men and $p-k$ women.

So we have

$$n = \sum_{k=1}^{p-1} \binom{p}{k}^2 .$$

Hence, to establish my claim, it is sufficient to show that $\binom{p}{k}$ is a multiple of p whenever $1 \leq k < p$. But

$$\binom{p}{k} = \frac{p(p-1)\ \dots\ (p - k+1)}{k!} = \frac{(p)_k}{k!}$$

is an integer, and so each prime factor of $k!$ will cancel a prime factor of $(p)_k$. Since $p > k$, the prime p is *not* a prime factor of $k!$, and so the p in (p_k) is not cancelled.

J.G.Kupka
Monash University

* * *

The relation of 'congruent mod p' may be extended from integers to rational numbers (i.e. fractions): then $a/b \equiv c/d \pmod p$ means $a/b - c/d$ is an integral multiple of p. Then from

$$a/b \equiv c/d \pmod p$$
$$e/f \equiv g/h \pmod p$$

follow $$a/b + e/f \equiv c/d + g/h \pmod p$$

and $$a/b \,.\, e/f \equiv c/d \,.\, g/h \pmod p .$$

When p is a prime, then for any fraction a/b, where $b \not\equiv 0 \bmod p$ (i.e. b is not an integral multiple of p), we have

$$p.a/b \equiv 0 \pmod p.$$

To see this consider the residues modulo p: each is congruent to 0, 1, 2, ..., or $p-1$, mod p. *Let* b be one of these with $b \not\equiv 0$. Then, since p does not divide b, i.e. $b \not\equiv 0 \bmod p$, then each of the powers $b^2, b^3, \ldots$ is also $\not\equiv 0 \bmod p$. *Since there are only* $p-1$ residues, namely 1, 2, ..., $p-1$, modulo p, then two powers of b must be equal modulo p. Suppose that $0 < r < s$ and that

$$b^r \equiv b^s \pmod p ,$$

i.e. $b^s - b^r = b^r(b^{s-r} - 1)$ is a multiple of p. Since $p \nmid b$, it follows that $b^{s-r} - 1$ is divisible by p, i.e.

$$b^{s-r} \equiv 1 \pmod p .$$

Thus, either $b = 1$ or there is a positive integer k such that $b.b^k \equiv 1 \pmod p$. In the first instance $1/b = 1$ and in the second instance $1/b \equiv b^k \pmod p$. Thus a/b is equal to an integer modulo p. Hence it follows that

$$p.a/b \equiv 0 \pmod p ,$$

for $p.a/b$ is simply an integral multiple of p, modulo p.

This leads to a short proof of the original question. For

$$\binom{2p}{2} = \frac{2(2p-1)(2p-2) \ldots (p+1)}{(p-1)!}$$

$$\equiv \frac{2(-1)(-2)\ \dots\ (-(p-1))}{(p-1)!} \pmod p$$

$$= 2(-1)^{p-1} \quad = 2 \text{ , if } p \text{ is odd,}$$

* * *

Mark Kisin makes use of the above property $p.a/b \equiv 0 \bmod p$, in the letter from him that follows.

Problem: Show $p^3 \left| \left(\binom{2p}{p} - 2 \right) \right.$ for $p \geq 5$ and prime .

1. Consider the set of numbers $1, 2, \dots, 2p$. We can choose i from the set $\{1, 2, \dots, p\}$ and $p-i$ from the set $\{p+1, p+2, \dots, 2p\}$. This will give us p numbers in all. There are $\binom{p}{i} \binom{p}{p-i}$ ways of doing this, and i takes values $0, 1, 2, \dots, p$. Therefore

$$\binom{2p}{p} = \sum_{i=0}^{p} \binom{p}{i} \binom{p}{p-1} = \sum_{i=0}^{p} \binom{p}{i}^2 =$$

$$= \sum_{i=1}^{p-1} \binom{p}{i}^2 + 2 \, .$$

Therefore

$$\binom{2p}{p} - 2 = \sum_{i=1}^{p-1} \binom{p}{i}^2 = \sum_{i=1}^{p-1} \left(\frac{p!}{i!(p-i)!} \right)^2$$

$$= \sum_{i=1}^{p-1} \left(p^2 \left(\frac{(p-1)!}{i!(p-i)!} \right)^2 \right) = p^2 . \sum_{i=1}^{p-1} \left(\frac{(p-1)!}{i!(p-i)!} \right)^2 .$$

Therefore to prove that $p^3 \left| \left(\binom{2p}{p} - 2 \right) = p^2 . \sum \left(\frac{(p-1)!}{i!(p-i)!} \right)^2 \right.$ it is sufficient to prove that

$$p \left| \sum_{i=1}^{p-1} \left(\frac{(p-1)!}{i!(p-i)!} \right)^2 \right. .$$

2. First we note that since $\binom{p}{i} = \dfrac{p!}{i!(p-i)!}$ is integral $\dfrac{(p-1)!}{i!(p-i)!}$ is an integer. This is because $(p, i!(p-i)!) = 1$, as p is prime. Next since by Wilson's theorem $(p-1)! \equiv -1 \pmod p$, $((p-1)!)^2 \equiv 1 \pmod p$. Therefore we may multiply the final expression in para. 1 by $((p-1)!)^2$ to get

$$\sum_{i=1}^{p-1} \left[\frac{(p-1)!}{i!(p-i)!} \right]^2 \equiv \left[(p-1)!\right]^2 \cdot \sum_{i=1}^{p-1} \left[\frac{(p-1)!}{i!(p-i)!} \right]^2$$

$$\equiv \sum_{i=1}^{p-1} \left[\frac{(p-1)!(p-1)!}{i!(p-i)!} \right]^2 \pmod p .$$

3. Let $t_k = \dfrac{(p-1)!(p-1)!}{k!(p-k)!}$. Clearly $t_1 = (p-1)! \equiv -1 \bmod p$ (Wilson). Also $t_{k+1} \equiv t_k \left[\dfrac{-k}{k+1} \right] \pmod p$.

Suppose, for some k, $(t_k)^2 \equiv 1/k^2 \pmod p$. Then

$$(t_{k+1})^2 \equiv (t_k)^2 \left[\frac{p-k}{k+1} \right]^2 \equiv {}^1/k^2 \cdot \frac{(-k)^2}{(k+1)^2} \equiv {}^1/k+1)^2 \pmod p .$$

Therefore since $t_1^2 \equiv (-1)^2 \equiv 1 \pmod p$, by induction, $t_k{}^2 \equiv {}^1/k^2 \pmod p$. Therefore $\sum_{i=1}^{p=1} \left[\dfrac{(p-1)!(p-1)!}{i!\ (p-i)!} \right]^2$

$$\equiv \sum_{i=1}^{p-1} t_i^2 \equiv \sum_{i=1}^{p-1} {}^1/i^{\cdot 2} \pmod p .$$

4. From para 1, 2 and 3, $\binom{2p}{p} - 2 \equiv 0(\text{mod } p^3)$ ($p \geq 5$ and prime)

if and only if $\displaystyle\sum_{i=1}^{p-1} {}^1/i^2 \equiv 0 \ (\text{mod } p)$.

Now ${}^1/i^2(\text{mod } p)$ means the square of the inverse of $i(\text{mod } p)$ squared. Since if p is prime the inverses of the numbers are all distinct, as are the numbers themselves, every number (except 0) will appear mod p , as ${}^1/i$ takes various values for $i = 1, 2, \ldots, p-1$. . Similarly every number will appear (except 0) as i takes values for $i = 1, 2, \ldots, p-1$. Therefore

$$\sum_{i=1}^{p-1} {}^1/i^2 \equiv \sum_{i=1}^{p-1} i^2 \equiv \frac{(p-1)(p)(2p-1)}{6} \quad (\text{mod } p) \quad .$$

Since $p \geq 5$, 2 and 3 do not divide p, whence 6 does not divide p.

Therefore $\dfrac{(p-1)p(2p-1)}{6} \equiv 0(\text{mod } p)$.

Mark Kisin
M.C.E.G.S.

The improved version of the problem observed by Dr Kupka was in fact the problem intended for the Australian Mathematical Olympiad. The version that appeared was due to a typing error.

The result that Mark Kisin gives in his letter is due to Marta Sved. Mark was asked by Dr Lausch if he could prove it.

* * *

STOP PRESS

1987

INTERNATIONAL MATHEMATICAL OLYMPIAD

This year the IMO took place in Cuba in the week of July 6-10. There were 42 teams competing. Australia was 15th overall (we were 11th out of 38 teams in 1985 and 15th out of 37 teams in 1986).

Romania was 1st, West Germany was 2nd, and Hungary was 3rd.

Australia won no gold medals or bronze medals, but won three silver medals. The silver medallists were:

Terence Tao	Blackwood High School, S.A.
Ben Robinson	Narrabundah College, A.C.T.
Chung Kim Yan	Duncraig Senior High School, W.A.

Congratulations from *Function* to them and to all the team.

The questions set in the IMO are selected from questions proposed by each competing country, each country being invited to offer at most six problems. Each team competing is accompanied by a team leader and a deputy team leader, who bring their country's proposed problems with them. These problems are of course kept completely confidential, and the accompanying team has no knowledge of them. A panel consisting principally of the team leaders selects the six questions to be set for the competition.

This year 32 competitors obtained perfect scores (42 marks with 7 marks for each question). Of course this means that the panel were not sufficiently clever in selecting questions of sufficient difficulty to ensure that there was adequate differentiation between competitor's scores. I do not under-estimate the difficulty of making such a selection. The questions must be such that they are not so difficult that no-one can give a complete answer, but also difficult enough to ensure that the whole set of six questions can be completed only by very few. Perhaps an impossible task.

However this year it seems they got it wrong. One effect of this was that you needed a near perfect score to get a silver medal. Terence Tao obtained 40 marks out of 42, a sure gold medal score in any normal year. Unfortunately this year he had to be content with a silver. Nevertheless he scored brilliantly.

PROBLEMS

PROBLEM 11.3.1 If a and b are positive and $a+b = 1$, show that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geqq \frac{25}{2}.$$

[From G.H.Hardy, *A course of Pure Mathematics*, 9th Ed. 1946, C.U.P., p.34.]

PROBLEM 11.3.2 Suppose you have 12 coins one of which is of different weight from the other 11, all of which have the same weight. Suppose you have a pair of scales, i.e. a balance, but no weights.

Show how to determine which is the coin different from the rest and whether it is lighter or heavier than the rest, by using the balance just three times to compare the weights of selected groups of the coins.

The problem is easier to solve with the additional information as to whether the different coin is lighter or heavier than the rest.

It is also easier to solve if you use the results of each act of weighing to determine which coins you select for the next weighing. However it is possible to specify the coins to be used in the weighings in advance and then, when all three weighings have been completed, to determine which is the odd coin and whether it is lighter or heavier than the rest.

[Problem 11.3.2 first appeared on the scene, so far as I know, in 1944, and has since been generalized in various ways to deal with n weighings.]

PROBLEM 11.3.3 Let a, b and c be positive integers such that

$$c = a(a^2 - 3b^2)$$

and

$$b(3a^2 - b^2) = 107 .$$

Find the value of c .

[Adapted from a question in the American Invitational Mathematics Examination 1985.]

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