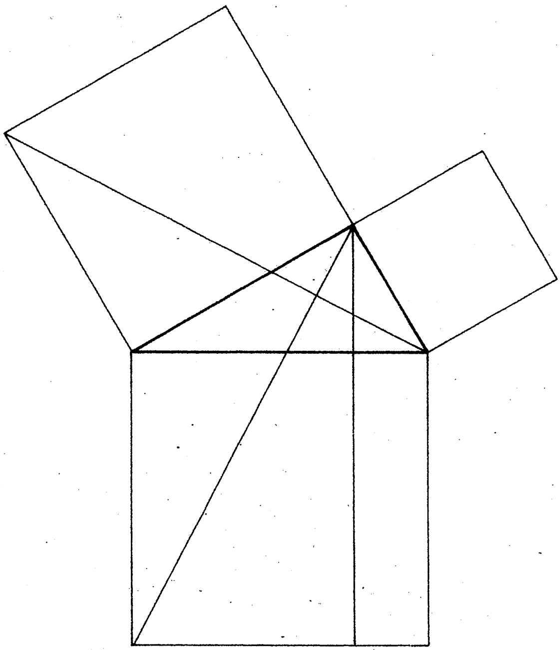
ISSN 0313 - 6825

FUNCTION

Volume 8 Part 1 February 1984



A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
Function,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

The magazine is published five times a year, appearing in February, April, June, August, October. Price for five issues (including postage): $8.00*; single issues $1.80. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

*$4.00 for *bona fide* secondary or tertiary students

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Registered for posting as a periodical - "Category B"

We welcome new and old readers alike with this new issue of *Function* and hope that it contains much of interest and enjoyment. We are grateful to those readers who send us articles, letters and solutions to problems and indeed wish there were more of these "active readers". This issue also has two of Colin Davies' cartoons. (We published another on the back cover of Vol.7, Part 5.)

It is a pleasure to acknowledge a grant of $400 from the School Mathematics Research Foundation (see p.32).

THE FRONT COVER

The diagram shown on the front cover is a famous one, being that used in the "windmill proof" of Pythagoras' Theorem. The "windmill proof" takes its name from the appearance of the diagram and is a relatively poor proof. (Much better is that given in *Function*, Vol.7, Part 3, p.27.) It is, however, the proof given by Euclid and for this reason was widely taught. (Our diagram is based on that in Hall and Stevens' *A School Geometry*.)

Some 60 years ago, when it was thought that there might be intelligent life on Mars, it was proposed to communicate with the Martians in the language of Pure Mathematics, lighting chains of fires in the Sahara Desert to form this pattern. That the Martians might well have taken to their hearts one of the hundreds of other proofs of Pythagoras' Theorem seems not to have been considered.

CONTENTS

$\infty = 572957795\ldots$

Colin Fox,

Swinburne Institute of Technology

Playing with my calculator[†] one day, I noticed something curious about tan 89°, tan 89·9°, tan 89·99°, ...

Before reading on, complete the table below. (Be sure to have your calculator in degrees mode.)

θ	$\tan\theta°$
89	
89·9	
89·99	
89·999	
89·9999	
89·99999	
89·999999	
89·9999999	
89·99999999	

Notice the $\tan\theta°$ values! Do they form a geometric sequence?

Since $\dfrac{\tan 89{\cdot}9°}{\tan 89°} \cong 10{\cdot}001$ and $\dfrac{\tan 89{\cdot}99°}{\tan 89{\cdot}9°} \cong 10{\cdot}00001$ (using calculator) they do not.

But what about the last few $\tan\theta°$ values? Is it true for example that $\dfrac{\tan 89{\cdot}999999°}{\tan 89{\cdot}99999°} = \dfrac{\tan 89{\cdot}9999999°}{\tan 89{\cdot}999999°}$? [The answer is NO! but the calculator will not provide justification this time.]

Is there some limiting sense in which these $\tan\theta°$ values form a geometric sequence? An affirmative answer to this question is provided by the following little theorem.

[†]The calculator in question is the APF Mark 8601. Not all calculators behave in the same way, as they are programmed differently. How does *your* calculator perform?

THEOREM 1. If $\theta_n = 89{\cdot}99\ldots9$ (n nines) then
$$\lim_{n\to\infty} \frac{\tan \theta^\circ_{n+1}}{\tan \theta^\circ_n} = 10 .$$

I leave the proof of this to the reader.

To return to the title of this article, whose most important result reveals to the world the true nature of that most elusive of all numbers: ∞.

THEOREM 2. ∞ is an integer with an infinite number of digits. In fact, $\infty = 572957795\ldots$.

Proof. Using θ_n as defined in Theorem 1, we have

$$\lim_{n\to\infty} \theta_n = 90$$

$$\lim_{n\to\infty} \tan \theta^\circ_n = \tan 90^\circ \quad \text{(since tan is a continuous function).}$$

Now $\tan 90^\circ = \infty$ (a well known fact). Also, as the table you completed shows,

$$\lim_{n\to\infty} \tan \theta_n = 572957795\ldots$$
$$\infty = 572957795\ldots .$$

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

WILL SOMEONE PLEASE TELL THE EDUCATIONISTS?

... the exact sciences [are not] based on an accumulation of statistics. In order to teach the young that three plus four make seven, you do not add four cakes plus three cakes nor four bishops plus three bishops nor four cooperatives plus three cooperatives nor four patent leather buttons with three wool socks. Once the principle has been intuited, the youthful mathematician grasps that three plus four invariably make seven and he does not have to prove it over and over again with chocolates, man-eating tigers, oysters, or telescopes.

Jorge Luis Borges and Adolfo Bioy-Casares,
Chronicles of Bustos Domecq, 1967.

A NUMERICAL COINCIDENCE

$$\frac{100}{81} = 1{\cdot}234567901234568$$

to 15 decimal places. What is

$$1{\cdot}2345678901234567890\ldots\ ?$$

A REMARKABLE FORMULA†

Figure 1 shows a four-pointed star. I want you to find its area. The method for you to use is to divide the star into geometric shapes for which you know the area formula (Figure 1). In this way, you will find the area.

In this article, you will discover a remarkable method which enables you to calculate the area of this star and other simple polygons.

To begin with take a sheet of squared paper and redraw the star so that each vertex is situated at a grid-point (Figure 2).

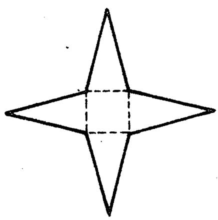


Figure 1

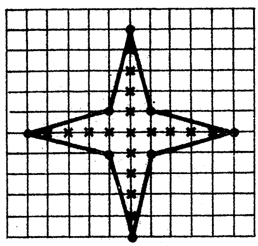


Figure 2

The formula to be used is the following

$$A(P) = i + \frac{b}{2} - 1 \qquad (*)$$

where P is a simple polygon whose vertices are grid-points,
$A(P)$ is the area-measure of P,
i is the number of grid-points *inside* the polygon,
b is the number of grid-points *on the boundary* of the polygon.

(The unit we use is clearly the area of one square of the grid.)

---

†This article is a translation from the French. It first appeared in *Function*'s Belgian counterpart *Math-Jeunes*, Nr.21 (1983), pp.9-15. The translation is printed here under an exchange agreement between *Function* and *Math-Jeunes*. The formula in question is known as *Pick's Formula*.

Let us now calculate the area of our star. Count first the number of interior points (marked $\times$ in Figure 2) to find $i = 17$. Next count the boundary points (marked $\bullet$) to find $b = 8$. Then apply the formula: $A(P) = 17 + \frac{8}{2} - 1 = 20$.

Verify this result and then apply the formula to the shapes shown in Figure 3. If this doesn't work, try to explain why.

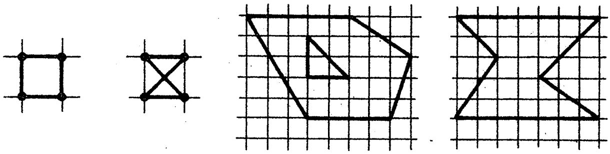


Figure 3

We will now prove the formula (*) by working through several successive stages.

1. Take a rectangle $ABCD$ whose sides are grid-lines (Figure 4) and look for its area. Suppose it is p units long and q units wide. Then there are $p - 1$ points between A and B (these points themselves being excluded); similarly the width contains $q - 1$ points apart from the ends. We find:

$$b = 2(p - 1) + 2(q - 1) + 4 = 2(p + q)$$
$$i = (p - 1)(q - 1).$$

Hence we find

$$i + \frac{b}{2} - 1 = (p - 1)(q - 1) + \frac{1}{2} \times 2(p + q) - 1$$
$$= pq,$$

which is indeed the area of the given rectangle.

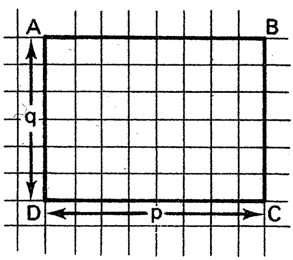


Figure 4

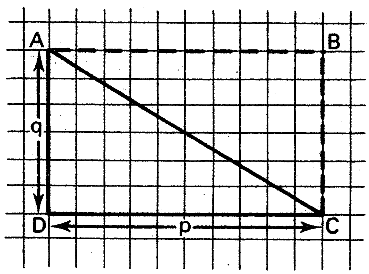


Figure 5

2. We now move on to the case of the triangle. But before we consider general triangles (with vertices at grid-points), begin with the case of a *right-angled triangle*, two of whose sides are grid-lines. Consider, for example the triangle ACD, half of the rectangle $ABCD$ dealt with above (Figure 5). Let d be the number of grid-points on the hypotenuse AC, again excluding the end-points A,C. We find

$$\begin{aligned} b &= (p-1) + (q-1) + d + 3 \\ &= p + q + d + 1 \end{aligned}$$

and

$$i = \tfrac{1}{2}\{(p-1)(q-1) - d\},$$

which gives, when the formula is applied,

$$\begin{aligned} i + \frac{b}{2} - 1 &= \tfrac{1}{2}\{(p-1)(q-1)-d\} + \tfrac{1}{2}(p+q+d+1) - 1 \\ &= \tfrac{1}{2}pq, \end{aligned}$$

which is the area of the triangle ACD.

3. To verify the formula for general triangles, we need an intermediate result. This relates to what happens if a polygon P is cut into two polygons P_1 and P_2 (Figure 6). This cut requires one important remark in the form of a hypothesis H, namely:

> *The boundary between the two polygons must must be a polygonal line whose vertices are grid-points.*

Under this condition, we find

$$b = b_1 + b_2 - 2(d+1) \quad \text{and} \quad i = i_1 + i_2 + d,$$

where b_1 and b_2 are the numbers of grid-points on the boundaries of P_1, P_2 respectively, i_1 and i_2 are the numbers of grid-points in the interiors of P_1, P_2 respectively, and d is the number of grid-points on the dividing line, excluding, as always, its end-points. Let S_1 and S_2 be the areas of P_1, P_2 respectively, and taking Hypothesis H into account, as well as that $S = S_1 + S_2$, the following implications are immediate (S being the area of P):

$$\left(S_1 = i_1 + \frac{b_1}{2} - 1 \text{ and } S_2 = i_2 + \frac{b_2}{2} - 1\right) \Rightarrow S = i + \frac{b}{2} - 1$$

$$\left(S = i + \frac{b}{2} - 1 \text{ and } S_1 = i_1 + \frac{b_1}{2} - 1\right) \Rightarrow S_2 = i_2 + \frac{b_2}{2} - 1 .$$

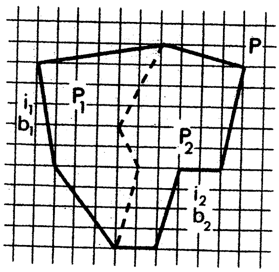


Figure 6

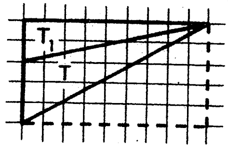


Figure 7

4. We can now verify the formula for a general triangle. Form such a triangle (Figure 7) and call it T. Consider the smallest rectangle containing T. Draw T_1 as shown. Then T and $T \cup T_1$ are two triangles of which two sides lie on the grid; the formula has already been proved for such triangles. Let i_1, b_1, S_1 be the elements of T_1 and i, b, S those of T. Then

$$A(T_1) = S_1 = i_1 + \frac{b_1}{2} - 1$$

and

$$A(T \cup T_1) = (i + i_1 + d) + \tfrac{1}{2}\{b + b_1 - 2(d+1)\} - 1$$
$$= i + i_1 + d + \frac{b}{2} + \frac{b_1}{2} - d - 2$$
$$= i + i_1 + \frac{b}{2} + \frac{b_1}{2} - 2 .$$

But T and T_1 satisfy Hypothesis H, so we can deduce the area of triangle T:

$$S = i + \frac{b}{2} - 1 .$$

So the formula is true for general triangles.†

5. We now generalise the result to *convex polygons*. *A polygon is said to be convex if, when we take any two points from its interior, the line segment joining them lies entirely within that interior.* (See Figure 8.)

†The proof is still not quite complete. T has one of its sides along a grid-line. The proof for the case in which no side lies along a grid-line is similar and is left to the reader. Eds.

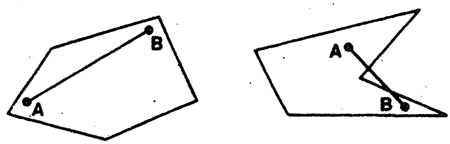


Figure 8

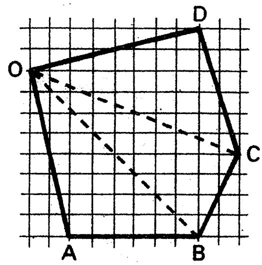


Figure 9

Consider, for example, the convex polygon *OABCD* (Figure 9). Divide it into triangles as shown. The triangles *OAB* and *OBC* satisfy the requirements both of the formula and Hypothesis *H*. This enables us to assert the truth of the formula for the polygon *OABC*. Similarly, considering the polygons *OABC* and *OCD*, we prove that the result holds for the convex polygon *OABCD*.

6. These preliminaries allow us finally to proceed to the *general case*: simple polygons all of whose vertices lie on grid-points. Suppose our polygon P can be decomposed into a finite number of convex polygons $P_1, P_2, \ldots, P_n$ in such a way that the two conditions below hold.

(A)

1. $P = P_1 \cup P_2 \cup \ldots \cup P_n$.
2. For all k between 1 and n-1 inclusive, the polygons $P_1 \cup P_2 \cup \ldots \cup P_k$ and P_{k+1} satisfy Hypothesis H.

If these conditions hold, by an argument similar to that used above we can assert that the formula applies to all simple polygons of this type. (Figure 10 shows some simple polygons; try to find for each a decomposition into convex polygons that satisfies the two conditions (A). Then apply the formula to find their areas.)

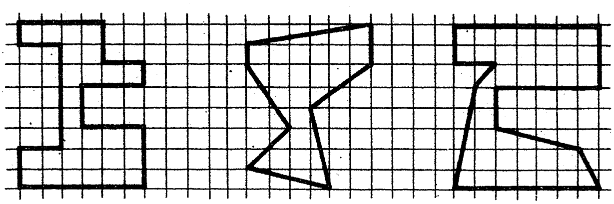


Figure 10

Having proved the formula allowing evaluation of the areas of all simple polygons, let us generalise it to non-simple polygons whose vertices still always lie on grid-points.

Here is the *general formula*:

$$A(P) = i + \frac{b}{2} + \gamma(P),$$

where $\gamma(P)$ is a term depending only on the shape of P. We have just seen that for a simple polygon $\gamma(P) = -1$.

What is the value of this coefficient in the case of a ring-shaped polygon? Consider the polygon of Figure 11. This is made up of two simple polygons P_1 and P_2 and

$$A(P) = A(P_1) - A(P_2) .$$

The two simple polygons obey the formula and so we may write

$$A(P_1) = i_1 + \frac{b_1}{2} - 1$$

$$A(P_2) = i_2 + \frac{b_2}{2} - 1 .$$

But let i,b be the elements of P so that

$$i = i_1 - (b_2 + i_2)$$

$$b = b_1 + b_2 ,$$

so that we get

$$A(P) = i_1 + \frac{b_1}{2} - 1 - (i_2 + \frac{b_2}{2} - 1)$$

$$= i + \frac{b}{2} .$$

Thus for a ring-shaped polygon, $\gamma(P) = 0$. In the same way, you may calculate $\gamma(P)$ for the polygons shown in Figure 12.

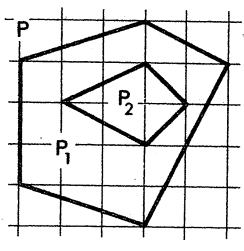


Figure 11

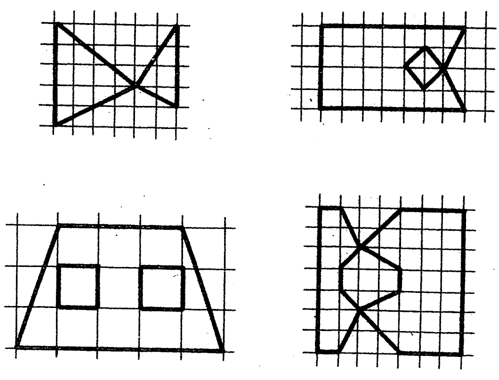


Figure 12

In the case where the edges are all on grid-lines, we can advance another argument to prove our formula, using a technique that might be labelled "ecological".

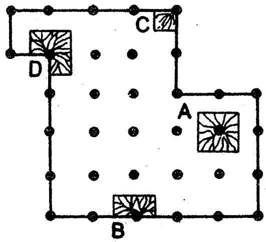


Figure 13

Suppose we imagine our (simple) polygon to be an orchard, each grid-point of which holds a new type of apple-tree so bred that its branches develop to cover a perfect square. (See *A* for example.)

In autumn, as the apples fall, we have four possible situations. Type *A* trees (entirely interior to the orchard) drop all their apples inside the perimeter, type *B* trees drop half their apples inside, type *C* a quarter and type *D* three quarters.

It is not difficult to see that the number of corners of type *C* must exceed the number of corners of type *D* by four. Indeed suppose you were walking round the orchard in the clockwise direction; you would then make four quarter-turns to the right in returning to your starting point. Each quarter-turn to the left would be compensated for by an additional quarter-turn to the right.

So we may note that the boundary trees of types *C*,*D* average out at half a tree, except for these four, for which the half is replaced by a quarter, so that there is a deficit of $4 \times \frac{1}{4}$ of a tree. We have

$$\begin{aligned}\text{Fall of apples} &= \text{no. of interior trees}\\ &\quad + \tfrac{1}{2} \times \text{no. of boundary trees}\\ &\quad - 4 \times \tfrac{1}{4},\end{aligned}$$

which is our formula.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

MAKE IT NON-TRIVIAL!

Mathematical systems and the axioms which define them must have a certain naturalness about them. They must come from the experience of looking at many examples, they should be rich in meaningful results. One does not just sit down, list a few axioms and then proceed to study the system so described. This admittedly is done by some, but most mathematicians would dismiss this as poor mathematics. The systems chosen for study are chosen because particular cases of these structures have appeared time and again, because one finally notes that these special cases were indeed special instances of general phenomena, because one notices analogies between two highly disparate mathematical objects and is led to search for the root of these analogies.

I.N. Herstein.

THE s FORMULA FOR A TRIANGLE

J.A. Deakin,
Shepparton College of TAFE

In a note to Problem 7.1.2, Mr John Barton (Volume 7, Part 4) points out that the area of a triangle ABC with sides of lengths a, b, c can be calculated by means of Hero's formula†

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}\,,$$

where

$$s = \tfrac{1}{2}(a+b+c)\,.$$

This formula does not seem to be well known to the current generation of students; older readers will recall the existence of other 's' formulae, and younger students may be interested in seeing these results, which follow directly from the cosine rule for triangle ABC.

In triangle ABC, $a^2 = b^2 + c^2 - 2bc\cos A$,

so that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}\,.$$

Now

$$\begin{aligned} 2\sin^2\frac{A}{2} &= 1 - \cos A \\ &= 1 - \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{a^2 - (b-c)^2}{2bc} \\ &= \frac{(a-b+c)(a+b-c)}{2bc}\,. \end{aligned}$$

Hence

$$\begin{aligned} \sin^2\frac{A}{2} &= \frac{(a-b+c)(a+b-c)}{4bc} \\ &= \frac{(2s-2b)(2s-2c)}{4bc}\,, \end{aligned}$$

---

† Hero, or Heron, of Alexandria lived in the 1st Century A.D. The formula named after him appears as Proposition I.8 of his book *Metrica*, but was first derived by Archimedes. For more on this point, see T.L. Heath: *A History of Greek Mathematics, Volume II*.

where $s = \frac{1}{2}(a + b + c)$, so that

$$\sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}\ , \qquad \text{(i)}$$

where we take the positive square root, since $\frac{A}{2}$ is acute.

Again $$2 \cos^2 \frac{A}{2} = 1 + \cos A$$
$$= 1 + \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{(b + c)^2 - a^2}{2bc}$$
$$= \frac{(b + c + a)(b + c - a)}{2bc}.$$

Hence $$\cos^2 \frac{A}{2} = \frac{(2s)(2s - 2a)}{4bc}\ ,$$

so that $$\cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}\ . \qquad \text{(ii)}$$

Also, $$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}\ . \qquad \text{(iii)}$$

Hero's formula for the area of a triangle follows from the fact that in any triangle,

$$\sin A = \frac{2}{bc} \sqrt{s(s - a)(s - b)(s - c)}\ .$$

Since $$\sin^2 A = 1 - \cos^2 A = 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2$$
$$= \left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)$$
$$= \left[\frac{(b + c)^2 - a^2}{2bc}\right]\left[\frac{a^2 - (b - c)^2}{2bc}\right]$$
$$= \frac{(b + c + a)(b + c - a)(a - b + c)(a + b - c)}{4b^2c^2}$$
$$= \frac{(2s)(2s - 2a)(2s - 2b)(2s - 2c)}{4b^2c^2}\ ,$$

we have $$\sin A = \frac{2}{bc} \sqrt{s(s - a)(s - b)(s - c)}\ , \qquad \text{(iv)}$$

where again the positive square root is taken since in any triangle ABC, $\sin A$ is positive.

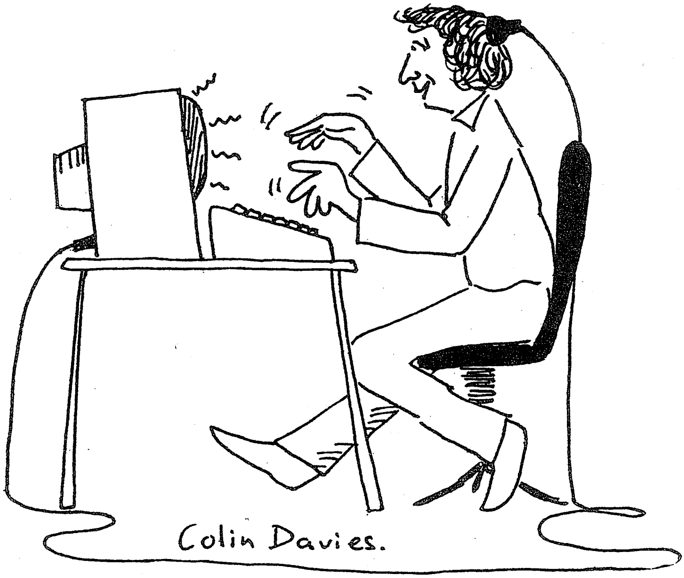
Finally, the area of triangle ABC is given by

$$\text{Area} = \tfrac{1}{2}bc \sin A$$
$$= \sqrt{s(s - a)(s - b)(s - c)} \qquad \text{(v)}$$

The '*s*' formulae (i) - (v) seem to have disappeared quietly from secondary mathematics syllabuses during the 1960s and 1970s. Prior to the advent of the now universal electronic calculator, they were almost invariably used for solving problems in numerical trigonometry in preference to the cosine rule, since they were eminently suited for use with tables of logarithms. Hero's formula, however, should perhaps be more widely known by modern students.

The derivation given above is not the only one available. For a different version, see Z.A. Melzak's *Invitation to Geometry* (Wiley, 1983).

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞



Submitted by Colin Davies.

A TWO-DIMENSIONAL BUILDING?

P.E. Kloeden, Murdoch University

During a recent visit to Boston, I was struck by the appearance of the New Hancock Building. As you can see from Figure 1, it seems to have no thickness. Figure 2 gives the explanation. The ninth and higher floors have a side wall that meets the front of the building at 70° rather than 90°, so that from many angles of view it cannot be seen.

The building won an award for its designer, I.M. Pai, but has been the subject of many lawsuits as the windows are forever blowing out. One theory is that the building's unusual shape leads to high wind-stress.

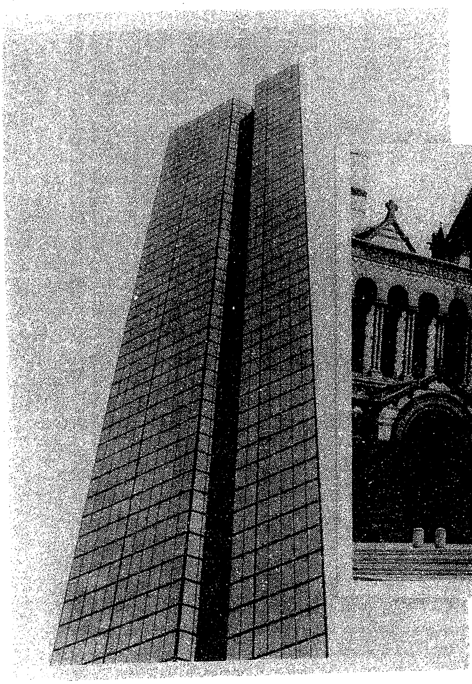


Figure 1

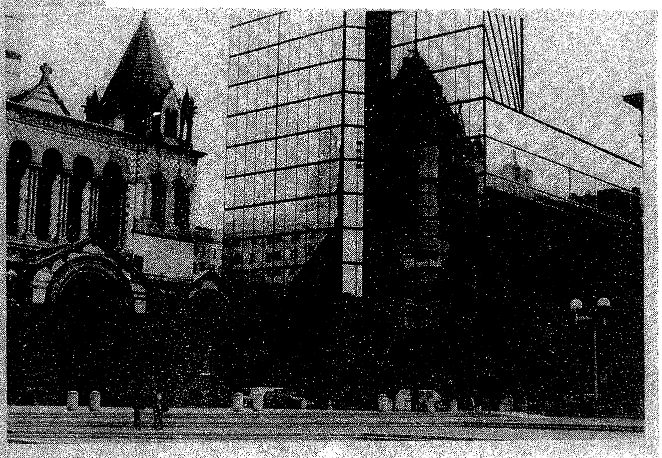


Figure 2

THE INEQUALITY OF THE ARITHMETIC AND GEOMETRIC MEANS†

Ivan Niven, University of Oregon

THEOREM. *Let* A *and* G *denote the arithmetic and geometric means of the nonnegative numbers* $a_1, a_2, \ldots, a_n$ *defined by*

$$A = \frac{a_1 + a_2 + \ldots + a_n}{n}, \qquad G = (a_1 a_2 \ldots a_n)^{1/n}. \tag{1}$$

Then $A \geqslant G$, *with equality if and only if* $a_1 = a_2 = \ldots = a_n$.

In case $a_1 = a_2 = \ldots = a_n$ it is clear from (1) that $A = G$. Henceforth we assume that the a's are not all equal, and prove that $A > G$. Actually we prove the equivalent version $G^n < A^n$: that is,

$$a_1 a_2 \ldots a_n < A^n. \tag{2}$$

This inequality is established by replacing the product on the left by successively larger products, reaching A^n in fewer than n steps. Each step in the process is described by the following *algorithm*, or procedure.

ALGORITHM. In any product of n numbers, replace the smallest number, say x, and the largest number, say y, by two new factors A and $x + y - A$, where A denotes the arithmetic mean of the n numbers.

Example 1. The inequality $2.3.4.6.20 < 7^5$, a special case of (2) above, can be obtained by repeated application of the algorithm:

---

† An extract from Professor Niven's recent book *Maxima and Minima without Calculus*, published by the Mathematical Association of America. We thank Professor Niven and the Association for their permission to reproduce this material in *Function*.

$$\begin{aligned} 2.3.4.6.20 &< 3.4.6.7.15 \\ &< 4.6.7.7.11 \\ &< 6.7.7.7.\ 8 \\ &< 7.7.7.7.\ 7 . \end{aligned}$$

The arithmetic mean of the five numbers in each product is 7. In the first step the smallest and largest numebrs, 2 and 20, are replaced by 7 and 15, because $x + y - A = 2 + 20 - 7 = 15$. In the second step the smallest and largest numbers, 3 and 15, are replaced by 7 and 11, because now we have $x + y - A = 3 + 15 - 7 = 11$. The subsequent steps again follow the algorithm. (Of course, the inequality $2.3.4.6.20 < 7^5$ can be verified by a direct calculation. The general inequality (2) *cannot* be verified in such a simple way, but it can be obtained by use of the algorithm, as we shall see.)

Example 2. The inequality $1.6.7.10.11.19 < 9^6$, another special case of the inequality (2) above, can be obtained by repeated application of the algorithm:

$$\begin{aligned} 1.6.7.10.11.19 &< 6.7.9.10.11.11 \\ &< 7.8.9.\ 9.10.11 \\ &< 8.9.9.\ 9.\ 9.10 \\ &< 9.9.9.\ 9.\ 9.\ 9 . \end{aligned}$$

We note that the algorithm replaces the numbers x and y, the smallest and the largest in any product, by two numbers A and $x + y - A$ having the same sum. Hence the algorithm, when applied to $a_1a_2\ldots a_n$, replaces a product of n factors having arithmetic mean A by another product of n factors also having arithmetic mean A. Why is the new product larger? To answer this question, we prove that

$$xy < A(x + y - A) . \tag{3}$$

Removing the parentheses, and moving the term xy to the right side of the inequality, we see that (3) is equivalent to

$$0 < Ax + Ay - A^2 - xy ;$$

i.e.

$$0 < (A - x)(y - A) . \tag{4}$$

This final inequality is easily verified because $A - x$ and $y - A$ are both positive. The reason for this is that the arithmetic average of n numbers lies between the smallest and the largest of the numbers, that is, $x < A < y$. This verifies (4).

Finally we see that repeated application of the algorithm to the product $a_1a_2\ldots a_n$ leads to the product A^n, because each step brings in one or two more occurrences of the factor A, as illustrated in the examples above. This proves the inequality (2), and so also the theorem.

COROLLARY. *If* $a_1, a_2, \ldots, a_n$ *are nonnegative, then* $a_1^n + a_2^n + \ldots + a_n^n \geqslant na_1a_2\ldots a_n$, *with equality if and only if the a's are all equal.*

This follows by applying the theorem to the numbers $a_1^n, a_2^n, \ldots, a_n^n$, with $G = a_1 a_2 \ldots a_n$ in this case.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

THE COIN TEST

Many readers will find some of the conclusions in this chapter hard to accept. Believing that the odds change with every card played, they will see no advantage in going back, as it were, to study the a priori expectations. To dispel that illusion it may help to make a simple experiment in a medium other than cards.

Suppose that there are five coins, four heads and one tail. They are divided into two piles, three on the left, two on the right. Now you would say that it was 3:2 against the tail being included in the smaller pile. Now take two coins away from the larger pile, with the proviso that neither of them be the tail. (That is what happens in bridge, where the discarding is selective and a player who has the critical honour, a King or Queen, does not play it wantonly.) At this stage there is only one coin on the left and, as before, two on the right; but it remains 3:2 against the tail being on the right.

Terence Reese, *The Expert Game*, p.40.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

MATHEMATICS IN APPLICATION

In criticizing an argument based upon the applications of mathematics to particular matters of facts, there are always three processes to be kept perfectly distinct in our minds. We must first scan the purely mathematical reasoning to make sure that there are no mere slips in it, no casual illogicalities due to mental failure. The next process is to make quite certain of all the abstract conditions which have been presupposed to hold. The chief danger here is that of oversight, namely tacitly to introduce some conditions, which it is natural for us to suppose, but which in fact need not always be holding. There is another opposite oversight that does not lead to error, but only to lack of simplification. It is very easy to think that more postulated conditions are required than is in fact the case. In other words, we may think that some abstract postulate is necessary which is in fact capable of being proved from the other postulates that we have already in hand. The only effects of this excess of abstract postulates are to diminish our aesthetic pleasure in the mathematical reasoning, and to give us more trouble when we come to the third process of criticism which is that of verifying that our abstract postulates hold for the particular case in question.

A.N. Whitehead.

YAN-A-BUMFIT AND ALL THAT†

Michael A.B. Deakin,
Monash University

A brief item in *Function*'s South African counterpart, *Mathematical Digest*, drew my attention to the following passage in Melvyn Bragg's *A Place in England* (Secker and Warburg, 1970).

> [Mr Lenty] handed over [a] piece of paper. On it were the numbers 1 to 20, written out as numerals several times, and beside each numeral, there was the word of the number recorded, as Joseph thought, in several different dialects. 'The sheep-score,' said Mr Lenty. 'Brought to me by my friend Mr Kirkby the schoolmaster after I had mentioned to him your reciting the way the shepherds count here in Cumberland. I was invigorated by that performance and also by the reminder of that particular lump of information,' he went on, ... 'Say them again.'
>
> .... So Joseph rhymed off the count from one to twenty, in his own, the West Cumbrian dialect, singing it almost, as the words demanded:
>
> 'Yan, tyan, tethera, methera, pimp, sethera, lethera, hovera, dovera, dick. Yan-a-dick, tyan-a-dick, tethera-dick, methera-dick, bumfit.'
>
> 'Bumfit!' Mr Lenty interrupted ecstatically. 'Oh, thou Bumfit! My Bumfit! Now why can't we *still* say Bumfit. Fifteen doesn't hold a candle to it. Bumfit! Oh - go *on*, Joseph.'
>
> 'Yan-a-bumfit, tyan-a-bumfit, tithera-bumfit, methera-bumfit, giggot.'

---

† I thank Professor J.N. Crossley, Ms E. Deakin, Dr R. Slonek, Professors C. Probyn and E. Barry and their colleagues in the department of English, Monash University, for their help with aspects of this article. M.D.

'Giggot!' said Mr Lenty. 'Twenty. And-the-days-of-thy-years-are-tethera-giggots-and-dick. Now isn't that better than three score and ten? Tethera giggots and dick. It *sounds* like a lifetime, doesn't it? I could hear you repeat that all evening - but pass the paper back and listen to *my* count.'

He took the paper, ... coughed, smiled most mysteriously at Joseph and began:

'Now Mr Kirkby wrote this out for me. Remember that. Mr Kirkby. I'll take this one. Yes. "Een, teen, tother, fither, pimp, een-pimp, teen-pimp, tother-pimp, fither-pimp, gleeget (yes, Joseph: I too prefer "dick": but forward): een-gleeget, teen-gleeget, tother-gleeget, fither-gleeget, bumfra ("fra" for "fit", you'll observe, but same base - bum): een-bumfra, teen-bumfra, tother-bumfra, fither-bumfra, fith-en-ly." (Twenty. Rather slippery along the tongue.) Well then. So what? - you might ask?'

Here, Mr Lenty really did tremble with excitement, even to wiping his brow, calming the nervousness, unable to bear the strain of it all.

'Joseph,' he said, solemnly. 'Some of those other lists you saw on that piece of paper were sheep-scores taken from different parts of England and one from Wales. You will admit that they were most remarkably similar to the one you say, ours, in Cumberland. But the one *I* read to you, Joseph, and one *other* on that list, Joseph, ... yes, the one *I* read, Joseph - that one is used by the Indians in North America.' He paused to let this revelation have its full effect. 'Indians of the Wawenoc Tribe,' he said, 'and it was recorded there in the year 1717. In a land 3000 miles from our own. Joseph, across that mighty ocean, there, over there,' he pointed, 'are Indians and Cumbrians counting sheep in the same way - give or take dick and giggot. It says something about man, Joseph: but what? That was my immediate question to Mr Kirkby - and he traced it back to the Garden of Eden.

A quite remarkable passage - but what are we to make of it? Bragg is an English author who writes novels (like this one) of the North Country and its customs. We may thus accept the account of the West Cumbrian system, if not that of the "Wawenoc", of which more later.

Now the word "Cumbria" comes from the same Celtic root as *Cymru*, to this day the Welsh name for Wales, so it is plausible to believe that the Cumbrian and the Welsh counting systems might reflect a common origin. This thought drove me to a Welsh dictionary, from which I constructed the following table. (The alternatives in Column 2 are masculine and feminine forms.)

No.	Welsh	W. Cumbrian
1	un	yan
2	dau/dwy	tyan
3	tri/tair	tethera
4	pedwar/pedair	methera
5	pump	pimp
6	chwech	sethera
7	saith	lethera
8	wyth	hovera
9	nau	dovera
10	deg	dick
11	un ar ddeg	yan-a-dick
12	deuddeg	tyan-a-dick
13	tri/tair ar ddeg	tethera-dick
14	pedwar/pedair ar ddeg	methera-dick
15	pymtheg	bumfit
16	un ar bymtheg	yan-a-bumfit
17	dau/dwy ar bymtheg	tyan-a-bumfit
18	deunau	tithera-bumfit
19	pedwar/pedair ar bymtheg	methera-bumfit
20	ugain	giggot

Table 1: The numbers one to twenty in Welsh and West Cumbrian.

The correspondence, though not exact, is impressive, and aspects of pronunciation make it more so. Both the *u* and the *y* of Welsh are pronounced as the French *u* or the German *ü*, so that "pump" is pronounced more like "pimp" than like the last syllable of "waterpump". The *p-m* transition on the numeral 4 is one we shall see again later.

We, speaking English, generally turn cardinal numbers into ordinals by adding the suffix *-th* (e.g. for a "clear" case, "seven" becomes "seventh"). The corresponding suffix in Welsh is *-fed* and this may relate to the *-thera* and *-vera* so prominent in the West Cumbrian. But matters are not simple. We would expect the West Cumbrian *sethera* to correspond to the Welsh word *seithfed* (which means "seventh"), but it doesn't. It means, as the table indicates, six. It may be that some corruption took place between the Welsh and the West Cumbrian. We shall see other examples of this later.

We may ask about the numerical structure of the sequences displayed in the table. Our own system is based on the number *ten*, although words like *eleven* and *twelve* depart from the expected pattern. West Cumbrian starts out quite promisingly along the same lines until we reach *bumfit*. A closer study of the Welsh shows what has happened. *Bumfit* corresponds to *pymtheg* and this is made up of *pym* (i.e. *pum*) and *theg* (from *ddeg* - the *dd* in Welsh is pronounced as the hard *th* in our words "these" and "those"). So *pymtheg* or *bymtheg* is "five-ten" or, as we would say, five-teen, or fifteen.

The Welsh word for "eighteen" fails to fit the pattern; *deunau* is clearly based on the words for "two" and "nine", so they say the equivalent of: one and five-teen, two and five-teen, twice-nine, four and five-teen, twenty. A brief and tantalising glimpse of a base nine system that will appear again later.

Welsh is one of only four Celtic languages that survive today. (The others are Irish, Scots Gaelic and Breton.) Two others are kept half-alive through the efforts of societies dedicated to their preservation, so we have Cornish and Manx as two not quite dead Celtic languages. The other well-documented Celtic language is Gaulish (cf. Asterix and Obelix), which is quite dead, though some scholars claim to have reconstructed parts of it. (Welsh, Cornish and Breton are direct descendents of Gaulish.)†

Welsh, Cornish and Breton form the so-called *Brythonic* sub-group of Celtic. The word "Brythonic" derives from "Brython", which has become "Briton" or "Breton" in our time. Irish, Scots Gaelic and Manx from the other sub-group, the *Goidelic* (cf. the word "Gaelic"). Brythonic languages are classed by linguists as *P*-languages, Goidelic ones are *Q*-languages. This distinction is seen in words like *pump*, whose Irish equivalent is *cúig*, which begins with the sound of the letter *Q*.

---

†Celtic tribes were once widespread throughout Europe and many Celtic languages must have existed. Of these two are worth mentioning.

(a) A group known as the Galatai (Galats) moved into the Balkans and then into Asia Minor in the third century B.C. They appear in the New Testament as the Galatians and retained their language (described by St Jerome as akin to that of the Treveri - the last Gaulish-speaking remnant) till about 500 A.D.

(b) The Greek historian, Herodotus, mentions a group called the Keltoi whose home was the Iberian peninsula (modern day Spain and Portugal). The Romans knew this group as the Celtiberi. Their language (which is preserved in a considerable number of inscriptions - most, however, in a so-far undeciphered script) died out early in the first century A.D.

Both Galatian and Celtiberian place-names survive on the map of Europe today: the last living testimony to the existence of these long-dead languages.

[Greek is a *P*-language, and we see this in the English words *pentagon, pentane* and *pentameter*. Latin, by contrast, is a *Q*-language and has left us *quintuplet, quinquennial* and *quintessence*.]

West Cumbrian is seen to be a *P*-language and thus we may hope to see more of its relatives if we look at the numerals in the Breton and Cornish languages. Here they are.

No.	Breton	Cornish
1	un	idn
2	dou/diu	deu
3	try/teir	try/teir
4	peuar/peder	pajer
5	pemp	pemp
6	huech	wheh
7	seiz	seyth
8	eiz	eyth
9	nau	now
10	dec	deg
11	unnec	idnak
12	douzec	dawdhak
13	trizec	tôrdhak
14	peuarzec	peswôrdhak
15	pempzec	pempthak
16	huezec	whedhak
17	seizdec	seydhak
18	eizdec	eydhak
19	nauntec	nownjak
20	uguent	igans

Table 2: The numbers one to twenty in Breton and Cornish.

Again a masculine, feminine dichotomy appears. What I have omitted, however, are a wealth of alternative forms, particularly in the Breton. There, "eighteen" is given above as "eizdec", which translates quite literally as "eight-teen". Old Breton has "eithnec", which means the same thing, but also "dou nau", which corresponds directly with the Welsh. An alternative "triwec'h" exists in modern Breton - a glance at the table gives the meaning "three sixes", and so here we have another fleeting glimpse of a strange base - in this case, six.

All these investigations had left me as eager an enthusiastic for the subject as Mr Lenty himself. I discussed the matter with my wife, who found a new reference. As an English teacher, she uses an anthology of poetry called *Junior Voices* (Ed. G. Summerfield and published by Penguin, 1970), whose first twenty pages are numbered:

eina, mina, para, peppera, pinn,
chester, nester, nera, nin, dickera,
eina dickera, mina dickera, para dickera, peppera dickera, pumpi,
eina pumpi, mina pumpi, para pumpi, peppera pumpi, ticket.

This clearly is another version of the same system. (It preserves the feature of Welsh and West Cumbrian, absent from the Cornish and Breton, of according a privileged place to the number fifteen.) But look at the start of the sequence - isn't it just that little bit reminiscent of the old playground rhyme *eeny meeny miney mo*? This and other such rhymes are used in "counting out" to determine who shall go "he" in a game of chasies or the like. The origin of such rhymes is discussed by Iona and Peter Opie in *The Oxford Dictionary of Nursery Rhymes* (Clarendon Press, 1951).

These authors mention the various sheep-scores in different North England dialects. These counts, which go to twenty (although the Opies take their examples only to ten) collectively form the "Anglo-Cymric Score" and they are still used by shepherds counting sheep, fisherman tallying their catch, and old women keeping track of their stitches as they knit. Here, from their book, are six such counts (up to ten). The fifth is the West Cumbrian under another name (check it on an atlas).

High Furness	West Riding, Yorks.	Yarmouth	Northumberland	Westmorland	North Riding, Yorks.
aina	eina	ina	eën	yan	yan
peina	peina	mina	tean	tyan	tean
para	paira	tethera	tether	tethera	tithera
peddera	puttera	methera	mether	methera	mithera
pimp	pith	pin	pimp	pimp	mimph
ithy	ith	sithera	citer	sethera	hitter
mithy	awith	lithera	liter	lethera	litter
owera	air-a	cothra	ōva	hevera	over
lowera	dickala	hothra	dōva	devera	dover
dig	dick	dic	dic	dick	dick

Table 3: The numbers one to ten in six English dialects.

Note the *p-m* shift in (e.g.) the words for four and, in the North Riding dialect, five. Opie and Opie also give the numbers one to five in Welsh, Cornish and Breton, although their spelling of the latter two differs from that given here. (Neither language has developed a fixed orthography.)

Some children's counting-out rhymes come very close to the Anglo-Cymric score. The best approximant, again from the Opies, goes:

Ya ta tethera pethera pip
Slata lata covera dovera dick.

Others are more garbled†:

---

†One very garbled version of *hevera, devera, dick* turns up as *hickory, dickory, dock*!

Inty tinty tethera methery
Bank for over dover dick.

One, at least, crossed the Atlantic and turned up in America:

Een teen tether fether fip
Sather lather gather dather dix.

This was recognised as counting and attributed to "the Indians" and so takes us back to the story of the Wawenoc.

Such a tribe does exist. The preferred spelling is *Wewenoc*, and Hodge's *Handbook of American Indians* (Rowman and Littlefield, 1979) states that they once lived on the coast of Maine. Surprisingly, as they must on that account, early and hard have borne the full brunt of European settlement, a remnant still exists, to be found, after a 1747 migration, in Canada.

However, it is most unlikely that they or any one else in America prior to European settlement counted in Celtic numerals. [There *are* crackpots who believe the American Indians to be Celts, but then there are others who believe them to be the lost tribes of Israel, etc.] The Indians did not have sheep and if they ever did count as suggested, the likelihood is that they learned to do so from the settlers (as Joseph in a later passage from Bragg's novel, realises, although he doesn't tell Mr Lenty).

Note also that the "Wawenoc" system, unlike all the others, is a perfect base five system (apart possibly from the words for ten, fifteen and twenty). This begins to look like a deliberate invention, and one of my informants, Professor John Crossley, author of *The Emergence of Number* (∀ Publ. Co., 1980), regards it as a hoax.† Bragg may have been himself taken in by the hoax, or, more probably, allows his characters, Mr Kirkby and Mr Lenty, to be so deceived for the dramatic purposes of his novel.

We may thus dismiss the argument that takes the system back to the Garden of Eden. Nonetheless, it is very old. Opie and Opie report a tradition to the effect that counting-out rhymes were used by the Druids for choosing human sacrifices. Well, the Anglo-Cymric score was, in its ancestral form, certainly used by the Druids. The Druids were the Celtic ruling and priestly caste, akin to the Brahmins of India in social importance and in function and, indeed, quite possibly deriving from the same earlier Indo-European tradition. It was certainly the Druids among the Celts of Julius Caesar's day who would have been able to count and their counting system would obviously have been Celtic. (We see it best preserved, however, in Welsh and Breton, rather than in the more variable and idiosyncratic sheep-scores.)

---

†There are American Indian counting systems that may be thought of as quinary, but none is remotely like this.

The Druids also, so Caesar tells us, resorted to human sacrifice, but whether they chose their victims by ritual counting is a matter that can probably never be known.

And what of Australia? Well, we have *eeny meeny miney mo*, and my mother remembers *indy tindy alligo Mary* (could *Mary* really be *methera*?) and there are other such relics. Perhaps the most convincing is:

Enden deena tucka lucka teena
Sucka lucka ticky tacky enden boom.

This was reported from Melbourne in 1967 and appears in the late Ian Turner's *Cinderella Dressed in Yella* (Heinemann, 1969). It has some affinity with the Anglo-Cymric score although it shows considerable degeneration.

Nonetheless it is interesting to reflect that in the non-sense words of this playground rhyme, we hear the faint echo of a counting system in use, if not in the Garden of Eden, at least well before the time of Christ.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

LETTER TO THE EDITOR

PASCAL'S PYRAMID

The Binomial Theorem, Bamford Gordon, *Function*, Vol.7, Part 5, October 1983, prompts me to examine Pascal's Triangle in more than two dimensions.

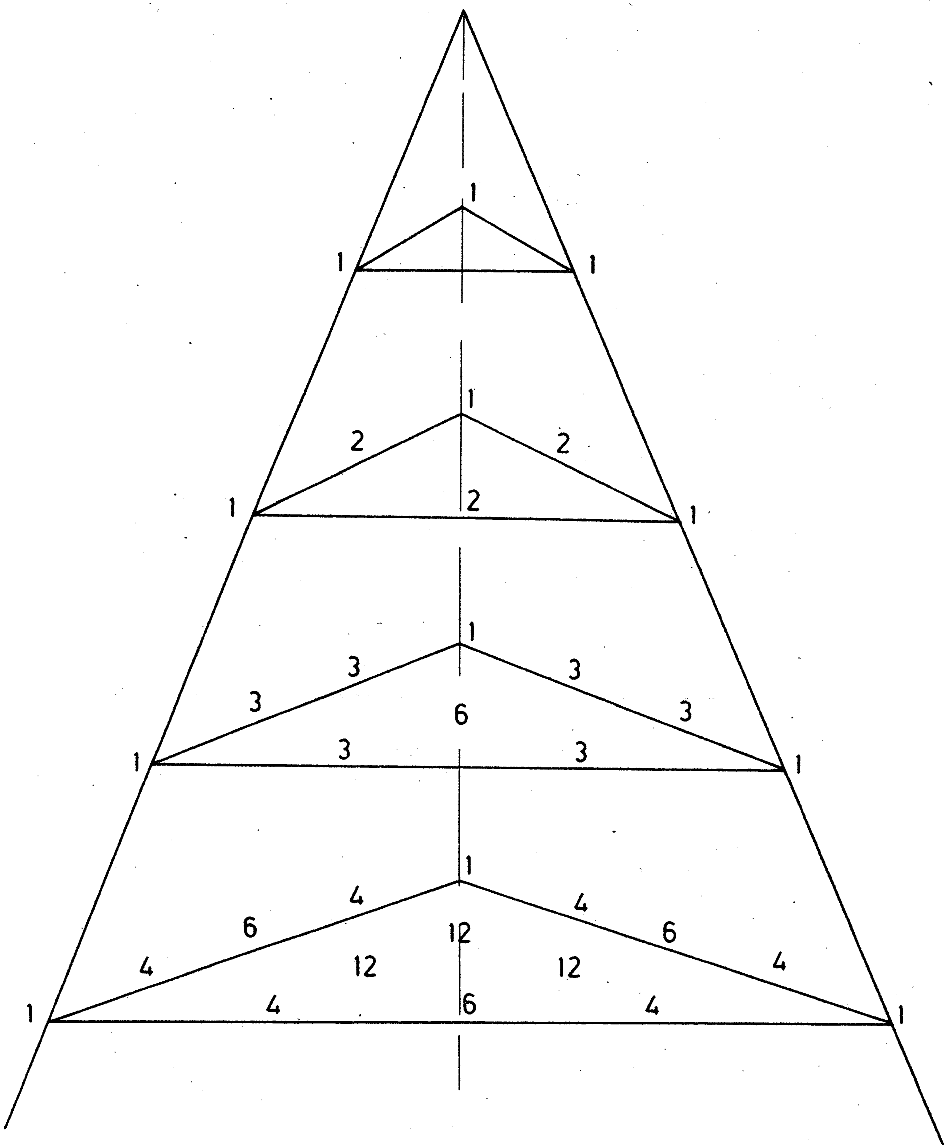
Consider "Pascal's Pyramid", for instance.

Each face is constructed as explained in the original article. The internal numbers in any triangle are found by adding numbers in the triangle immediately above that "surround" the said number.

e.g. $$6 = 2 + 2 + 2$$
$$12 = 3 + 3 + 3 + 3 .$$

This then leads to the diagram overleaf, where the numbers give the coefficients on the different terms in the expansion of $(a + b + c)^2$.

Garnet J. Greenbury,
Brisbane.


1
1
1
1
2
2
1
2
1
1
3
3
3
3
6
1
3
3
1
1
4
4
6
6
12
4
4
12
12
1
4
6
4
1

PROBLEM SECTION

Each issue of *Function* contains a number of problems, either sent in by readers or posed by the editors. Many of our readers find that they can contribute to *Function* by sending solutions to the problems posed. We begin by printing solutions to those problems still outstanding from 1983.

SOLUTION TO PROBLEM 7.5.1.

The problem read:

The recurrence relation

$$f(n) - nf(n-1) = (-1)^n$$

arose in connection with Problem 7.3.1. Derive the solution

$$f(n) = (n!)\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right) .$$

We received solutions from John Percival (Year 12, Elderslie High School, N.S.W.), John Barton (1008 Drummond Street, North Carlton) and David Shaw (Geelong West Technical School). Here is John Percival's solution.

Assuming $f(-1) = 0$, $f(0) = 1$ (from Problem 7.3.1), we see that the formula holds true if $n = 0$. Now assume it true for $n = k - 1$, where n is a natural number. Then

$$f(k-1) = (k-1)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^{k-1}}{(k-1)!}\right) .$$

Consider $n = k$.

By definition, $f(k) - kf(k-1) = (-1)^k$

$$\therefore\ f(k) = kf(k-1) + (-1)^k .$$

$\therefore$ By assumption,

$$f(k) = k\left[(k-1)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^{k-1}}{(k-1)!}\right)\right] + (-1)^k$$

$$= k!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^{k-1}}{(k-1)!} + \frac{(-1)^k}{k!} - \frac{(-1)^k}{k!}\right) + (-1)^k$$

$$= k!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots \frac{(-1)^{k-1}}{(k-1)!} + \frac{(-1)^k}{k!}\right) - k!.\frac{(-1)^k}{k!} + (-1)^k$$

$$= k!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^k}{k!}\right) - (-1)^k + (-1)^k$$

$$= k!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^k}{k!}\right)$$

which is of the form $f(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!}\right)$ where $n = k$.

Since shown true for $n = 0$, and proved true for $n = k$ using the assumption of validity for $n = k - 1$, then by the Principle of Mathematical Induction,

$$f(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right)$$

for all natural numbers n.

John Barton used a somewhat different approach.

Since $f(2) = 2f(1) + 1$, we have

$$f(2) = 2!\left(f(1) + 1 - \frac{1}{1!} + \frac{1}{2!}\right),$$

and then

$$\begin{aligned} f(3) &= 3f(2) - 1 \\ &= 3!\left(f(1) + 1 - \frac{1}{1!} + \frac{1}{2!}\right) - 1 \\ &= 3!\left(f(1) + 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) . \end{aligned}$$

In general

$$f(n) = n!\left(f(1) + 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right) .$$

The given solution has $f(1) = 0$.

He comments:

One could formalize this derivation by presenting it in the standard inductive dress, and it would be a nice question of mathematical aesthetics as to how people would feel about such a demonstration. Would it be more, or less, satisfying than the simple "leaping to the generalisation", as in (1) above? Dare you ask "What do readers think?" ?

David Shaw sent two solutions, the first essentially that supplied by John Barton, but proceeding in the opposite direction. His second solution referred to the origin of the problem: $f(n)$ is the number of complete derangements of the sequence of symbols $1,2,3,\ldots,n$. The solution proceeds as follows.

Firstly, ascertain the number of permutations in which one or more of the symbols are undisturbed (fixed). If we consider one of the symbols to be fixed, then there will be $(n - 1)!$ permutations of the others. There are $\binom{n}{1}$ ways of fixing one symbol. The product $\binom{n}{1}(n - 1)!$ includes the number of permutations in which two or more of the symbols are fixed and, in fact, includes more than once those permutations in which any two of the symbols are fixed. So we have to subtract the product

$\binom{n}{2}(n-2)!$

$$\text{i.e.}\quad \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)!\ .$$

But now we have excluded permutations in which three of the symbols are fixed. So the product $\binom{n}{3}(n-3)!$ must be added. This inclusion-exclusion argument is continued so that the number of permutations which are not derangements is given by

$$\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \ldots + (-1)^n\binom{n}{n}0!\ .$$

$f(n)$ is obtained by subtraction.

$$f(n) = n! - \left\{\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \ldots + (-1)^n\right\}$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \ldots + (-1)^n$$
$$= n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right\}$$

Mr Shaw comments that $f(n)$ can be interpreted as the number of ways in which n rooks may be placed on an $n \times n$ chessboard with none on the main diagonal and such that no rook can take any other.

John Barton also sent a second solution, which, with regret, we judged too difficult for *Function*.

SOLUTION TO PROBLEM 7.5.2.

We asked for (in essence) factors of 1,000,343. John Percival writes:

$$\begin{aligned} 1{,}000{,}343 &= 1{,}000{,}000 + 343 \\ &= (100)^3 + 7^3 \\ &= (100 + 7)(100^2 - 7 \times 100 + 7^2) \\ &= (107)(10000 - 700 + 49) \\ &= 107 \times 9349\ . \end{aligned}$$

Hence 1,000,343 is composite, with prime factors 107, 9349 .

John Barton and David Shaw solved this problem by the same method.

SOLUTION TO PROBLEM 7.5.3.

This problem, from the 1982 South African Mathematical Olympiad, asked for the smallest positive integer n such that if the digit 7 is written after it and the digit 2 in front of it the result is 91 times n.

The same three solvers sent us solutions to this problem also. John Percival's answer begins by seeking a single digit number as n. This leads to

$$200 + 10n + 7 = 91n$$

or $$n = \frac{23}{9}\ , \text{ which is not an integer.}$$

Thus there are no single digit solutions and a two-digit solution $n = 10l + k$ is thus sought. We then require

$$1000 + 100l + 10k + 7 = 91(10l + k)\ ,$$

which reduces to

$$n = 10l + k = \frac{223}{9}\ ,$$

which again is not an integer.

A search for a three-digit solution $n = 100l + 10k + p$ along the same lines is, however, successful, yielding $n = 247$, which is thus the solution.

John Barton used a different approach. He writes as follows.

Let the number be

$$a_p 10^p + a_{p-1} 10^{p-1} + \ldots + a_1.10 + a_0\ .$$

Then

$$2.10^{p+2} + a_p.10^{p+1} + \ldots + a_1.10^2 + a_0.10 + 7$$

$$= 91(a_p.10^p + a_{p-1}.10^{p-1} + \ldots + a_1.10 + a_0)\ .$$

$$2.10^{p+2} + 7 - 81.10^p.a_p - 81.10^{p-1}.a_{p-1} - \ldots - 81.10.a_1 = 81a_0\ .$$

If there be a solution for a_0, we must have $a_0 = 7$.

Subtracting 7 from both sides and then dividing by 10,

$$2.10^{p+1} - 81.10^{p-1}.a_p - 81.10^{p-2}.a_{p-1} - \ldots - 81a_1 = 56\ .$$

If there be a solution, we must have $a_1 = 4$.

Substituting, transposing the term $81a_1$, and dividing by 10:

$$2.10^p - 81.10^{p-2}.a_p - 81.10^{p-3}.a_{p-1} - \ldots - 81a_2 = 38\ .$$

If there be a solution, we must have $a_2 = 2$, and this equation can be satisfied by putting $p = 2$. That is, we can take $a_p = 0$ for all p greater than 2.

This gives the solution $2.10^2 + 4.10 + 7 = 247$.

David Shaw's approach was different again. He used congruences and writes as follows.

Suppose n is composed of d digits;

then $2 \times 10^{d+1} + 10n + 7 = 91n$

$$2 \times 10^{d+1} = 81n - 7\,,$$

i.e. $2 \times 10^{d+1} \equiv -7 \pmod{81}$

$$2 \times 10^{d+1} \equiv 74 \pmod{81}$$

$$10^{d+1} \equiv 37 \pmod{81}.$$

Now $10^2 \equiv 19 \pmod{81}$

$$10^3 \equiv 190 \equiv 28 \pmod{81}$$

$$10^4 \equiv 19^2 \equiv 361 \equiv 37 \pmod{81}.$$

So $d = 3$ and

$$\begin{aligned} 2 \times 10^4 + 10n + 7 &= 91n \\ 20007 &= 81n \\ 247 &= n\,. \end{aligned}$$

247 is the required number.

SOLUTION TO PROBLEM 7.5.4.

A leaf is torn from a paperback novel. The sum of the remaining page numbers is 15000. Which pages were torn out?

Here is John Percival's solution.

Let the book have n pages, with the kth and $(k - 1)$th pages removed with the leaf torn out. Then

$$15000 = 1 + 2 + \ldots + n - (k + k - 1) \qquad (k < n)\,.$$

Since $1 + 2 + \ldots + n$ is the sum of an arithmetic series,

$$15000 = \tfrac{n}{2}(1 + n) - 2k + 1\,.$$

Since $k < n$, $\tfrac{n}{2}(1 + n) > 15000$.

By trial, $n \geqslant 173$, since when $n = 172$, $\tfrac{n}{2}(1 + n) < 15000$.

When $n = 173$, $\tfrac{n}{2}(1 + n) = \dfrac{173 \times 174}{2}$

$$= 15051.$$

So $51 = 2k - 1\,,$

i.e. $k = 26\,.$

If the novel has 173 pages, the pages 25 and 26 were removed.

If $n = 174$, $\tfrac{n}{2}(1 + n) = \dfrac{174 \times 175}{2}$

$$= 15225\,.$$

So $225 = 2k - 1$, i.e. $k = 113$.

If the novel has 174 pages, the 112th and 113th pages could have been removed. However, if the odd-numbered page occurs on the right hand leaf of the opened book (as they usually are), then the removal of these two pages with the tearing out of one leaf is not possible. I.e. we require the odd-numbered page to be the smaller of the two removed. Hence this is not a valid solution.

$$\text{When} \quad n = 175, \ \frac{n}{2}\left(1 + n\right) = \frac{175 \times 176}{2}$$

$$= 15400 .$$

So $400 = 2k - 1$, giving $k > 175$ which is not possible.

Hence the only valid solution is the removal of page numbers 25 and 26 from a 173-page book.

John Barton and David Shaw solved the problem using similar arguments.

This disposes of all the outstanding problems. Here are some new ones. Please send in your solutions.

PROBLEM 8.1.1. (Submitted by David Shaw, Geelong West T.S.)

How many permutations are there of the digits 1,2,3,...,8 in which none of the patterns 12,34,56,78 appear?

PROBLEM 8.1.2.

Ten people form the queue at a bank. The first has a bank balance of one cent, while the tenth has a little over $5 million. The accounts of the others are each computed by adding ten elevenths of the account of the person ahead to one eleventh of the account of the person behind. Can the sixth person afford to buy a new car?

PROBLEM 8.1.3. (From *Mathematical Spectrum*, Vol.16, No.1.)

Evaluate

$$\left(9 + 4\sqrt{5}\right)^{1/3} + \left(9 - 4\sqrt{5}\right)^{1/3} .$$

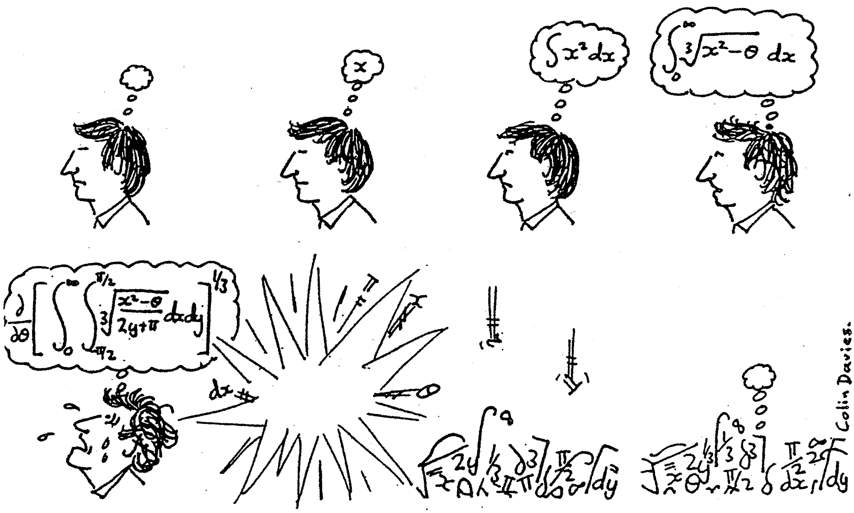
PROBLEM 8.1.4.

111 players enter a tennis tournament. Allowing for byes, first round matches, second round matches, etc., how many matches must be played altogether?

∞ ∞ ∞ ∞ ∞ ∞ ∞

THANK YOU!

We are pleased to acknowledge with grateful thanks the receipt of a grant of $400 from the School Mathematics Research Foundation. This will help to defray the cost of publishing *Function* and has meant that we do not need to increase our price in 1984.


x
$\int x^2\,dx$
$\int_0^\infty \sqrt[3]{x^2-\theta}\,dx$
$\frac{\partial}{\partial\theta}\left[\int_0^\infty \int_{\pi/2}^{\pi/2} \sqrt[3]{\frac{x^2-\theta}{2y+\pi}}\,dx\,dy\right]^{1/3}$
Colin Davies.