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FUNCTION

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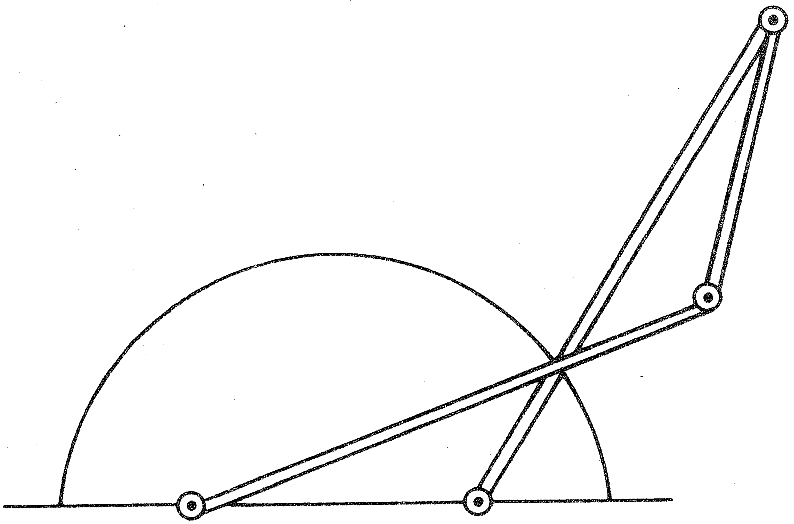

A SCHOOL MATHEMATICS MAGAZINE

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Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
Function,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

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We thank, in particular, several people for their contributions to this issue of *Function*. Anne-Marie Vandenberg has sent us more translations from our Netherlands counterpart, *Pythagoras*. These all appeared originally in Volume 22, Part 3 (January 1983). The cartoon on p.22 was the front cover of that issue. The remaining cartoons, all but one, were sent us by Brian Morearty, Year 12, Mt Tamalpais H.S., California.

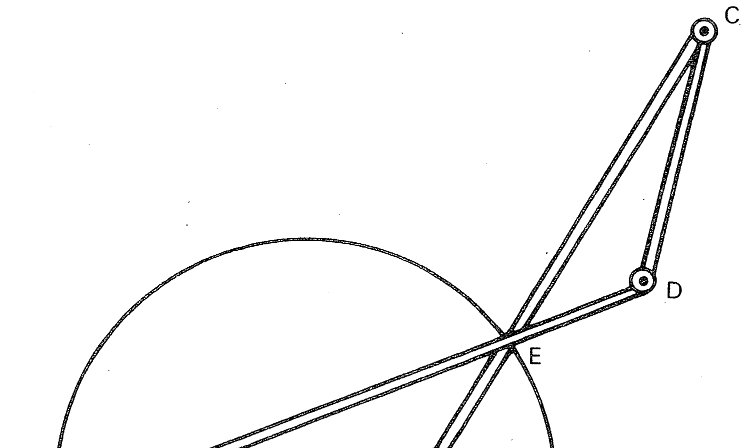
The news story on p.24 deserves mention also. *Function* has long encouraged girls to take up the study of Mathematics. It is pleasing indeed to be able to tell of three who have done this so successfully. The photograph is by Rick Crompton and it first appeared in *Monash Reporter* 3-83. We thank the Monash information office for making it available to us.

CONTENTS

THE FRONT COVER

We show an instrument, devised in 1886, for drawing an ellipse. There are more practical ones available now, but this one has certain features of interest. It was invented by George B. Grant of Maplewood, Massachusetts, and first described in the *Franklin Journal* of August 29 of that year. The brief description (which does not include a proof of the claimed result) was reprinted in *Scientific American Supplement*, October 23, 1886.

The lengths *AD*, *BC* are equal, as also are the lengths *AB*, *CD*. *A* and *B* are held fixed while *D*, *C* rotate about them. The point *E* then traces out an ellipse. The original article does not describe how to fix a pencil to the point *E*, which slides along the bars *AD*, *BC*, and this may well have led to practical difficulties in actually constructing the instrument.



To prove that it works, consider the figure below, in which the instrument has been reduced to an outline and DB has been inserted. In the triangles ABD, CDB,

	$AB = CD,\ AD = CB$	(by the nature of the instrument)
and	$BD = DB$	(obviously).

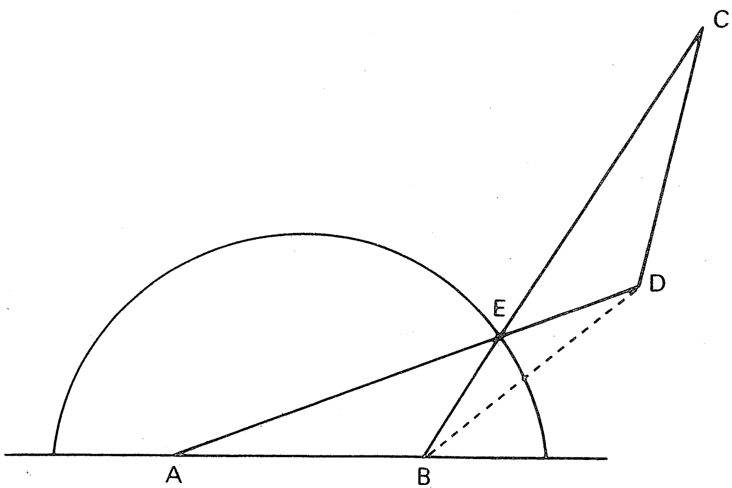
Thus these triangles are congruent, and $\measuredangle BAD = \measuredangle DCB$.

Now consider the triangles ABE, CDE.

	$AB = CD$	(as before)
	$\measuredangle BAD = \measuredangle DCB$	(just proved)
and	$\measuredangle AEB = \measuredangle CED$	(by symmetry at E).

Thus these triangles are congruent, and $BE = DE$.

It follows that $AE + BE = AE + DE$, which is constant. But this now gives a relation for E equivalent to the more familiar "pin and string" method, so that E traces out an ellipse whose major axis has length AD.



CONES AND CONIC SECTIONS II

John Mack, University of Sydney

[In our previous issue we printed the first half of a talk addressed to talented students by Dr Mack in 1982. That article concentrated on properties of the ellipse. For the conclusion of the talk, read on.]

The hyperbola

The two spheres which touch the cone and the plane of the hyperbola lie one in each nappe, on the side of the plane containing the vertex. An analogous discussion to that carried out for the ellipse results in

$$|PA - PB| = \text{constant}.$$

Directrices are obtained as before and each point P on the hyperbola satisfies

$$\frac{\text{distance of } P \text{ from a focus}}{\text{distance of } P \text{ from the corresponding directrix}} = \text{a constant } e,$$

where $e > 1$ (because now $C\hat{P}D < B\hat{P}D$). The x-axis and origin can be chosen so that the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1.$$

The parabola

Because the plane of the curve is parallel to a generator, there is only one touching sphere. The focus and its corresponding directrix now satisfy

$$\frac{\text{Distance of } P \text{ from focus}}{\text{distance of } P \text{ from directrix}} = 1$$

(because $C\hat{P}D = B\hat{P}D$). With appropriate choice of x-axis and origin, the equation of the parabola becomes

$$x^2 - 4ay = 0.$$

Some applications

Rather than discuss the Latin meaning of 'focus', the use of hyperbolas in radio navigation or parabolas as collectors or transmitters of parallel light, I shall mention two applications of conics which arise in connection with famous geometrical problems of ancient Greece.

Duplication of the cube: The problem is to construct the edge of a cube of volume 2, using straight-edge and compasses in the "approved" fashion. It is known that this cannot be done. We can do it if we cheat and allow ourselves to draw parabolas. In fact, suppose we construct the parabolas $x^2 = y$ and $y^2 = 2x$. The two curves meet in two points, $O(0,0)$ and $P(X,Y)$, say. At P,

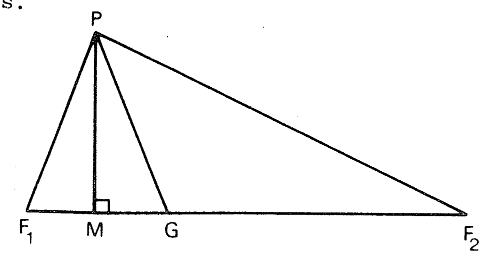
$$X^4 = Y^2 = 2X \quad \text{and} \quad X \neq 0 \quad \text{so that} \quad X^3 = 2.$$

Thus the abscissa of P is the desired edge length. Can you design an instrument that will draw these parabolas?

Descartes (after whom cartesian coordinates are named) showed that only one parabola will do. For if the parabola $x^2 = y$ and the circle $x^2 + y^2 = y + 2x$ (which can be drawn with ruler and compasses) intersect at $Q(S,T)$ with $S \neq 0$, then

$$S^2 + S^4 = S^2 + 2S \quad \text{and so} \quad S^3 = 2 \quad \text{again.}$$

Trisection of an angle: the problem is to trisect any given acute angle using ruler and compasses in approved fashion. Again it is known that this cannot be done. We again cheat, this time by using one hyperbola. As a preliminary, we derive a result about triangles.



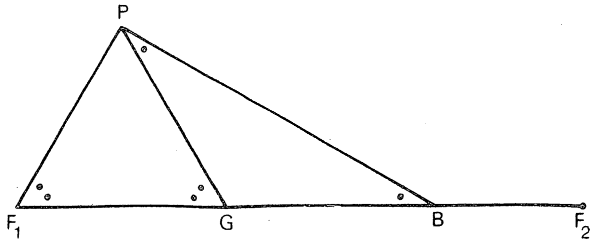
ΔPF_1G is isosceles and F_2 is any point on F_1G produced.

By the properties of isosceles triangles, if $PM \perp F_1G$ then $MF_1 = MG$. Apply Pythagoras' theorem to Δs PMF_1 and PMF_2:

$$PF_1^2 = PM^2 + MF_1^2$$
$$PF_2^2 = PM^2 + MF_2^2 .$$
$$\therefore\ PF_2^2 - PF_1^2 = MF_2^2 - MF_1^2 = (MF_2 - MF_1)(MF_2 + MF_1)$$
$$= (MF_2 - MG)(MF_2 + MF_1)$$
$$= GF_2 . F_1F_2 .$$

We now solve the following problem: Given two points F_1 and B, what is the locus of a point P which moves so that

$P\hat{F}_1B = 2P\hat{B}F_1$?



Let P be such that the condition is satisfied. Draw PG as shown so that $B\hat{P}G = P\hat{B}G$. Then $P\hat{G}F_1 = 2P\hat{B}F_1 = P\hat{F}_1G$, so ΔPF_1G is isosceles. Hence the result proved above holds for any point F_2 on F_1G produced:

$$\begin{aligned} PF_2^2 - PF_1^2 &= GF_2 . F_1F_2 \\ &= (GB + BF_2)F_1F_2 \\ &= (PF_1 + BF_2)F_1F_2 \qquad (PF_1 = PG = GB). \end{aligned}$$

Choose F_2 so that $F_1F_2 = 4BF_2$. Then

$$PF_2^2 - PF_1^2 = (PF_1 + BF_2)4BF_2 ,$$

i.e. $$PF_2^2 = (PF_1 + 2BF_2)^2 .$$

Take the positive square root and obtain

$$PF_2 - PF_1 = 2BF_2 , \text{ a fixed distance.}$$

Hence the required locus is one branch of a hyperbola with foci at F_1 and F_2. Assume that this hyperbola has been drawn. The following is a construction to trisect a given acute angle θ.

Step 1. Construct an isosceles triangle F_1BO on base F_1B, with $O\hat{F}_1B = O\hat{B}F_1 = 90° - \theta$, such that O lies below F_1B.

Step 2. With centre O, draw the arc F_1B above F_1B intersecting the given hyperbola at P.

Claim. $F_1\hat{B}P = \frac{1}{3}\theta$.

Exercise. Verify the claimed result.

Finally, here is a problem taken from a recent British Mathematical Olympiad. Take an ellipse on a cone with horizontal base. Let L be the lowest point of the ellipse and H the highest point. One half of the ellipse provides a path p' joining L to H on the surface of the cone. Another path (on the same "half" of the cone as this path) is the *shortest* path p joining L to H on the cone. Recall that the surface of a cone is 'developable' - we may cut the cone along a generator and spread the surface out flat on a plane. The shortest path is clearly the line segment joining L to H on the flat surface.

Question: Under what conditions will the paths p and p' meet one another somewhere between L and H? (i.e., as the plane of the ellipse is tilted and the angle of the cone is varied, p will sometimes meet p' and sometimes not. Find a simple condition that will determine exactly when they meet.)

Solutions to this problem may be sent to the Editor.

Note. The first complete description of the use of focal spheres to establish the focus-directrix properties of the conic sections was apparently given by Pierce Morton, an English mathematician, in 1829. Dandelin, a Belgian mathematician, had used them to determine the foci of the sections in 1822.

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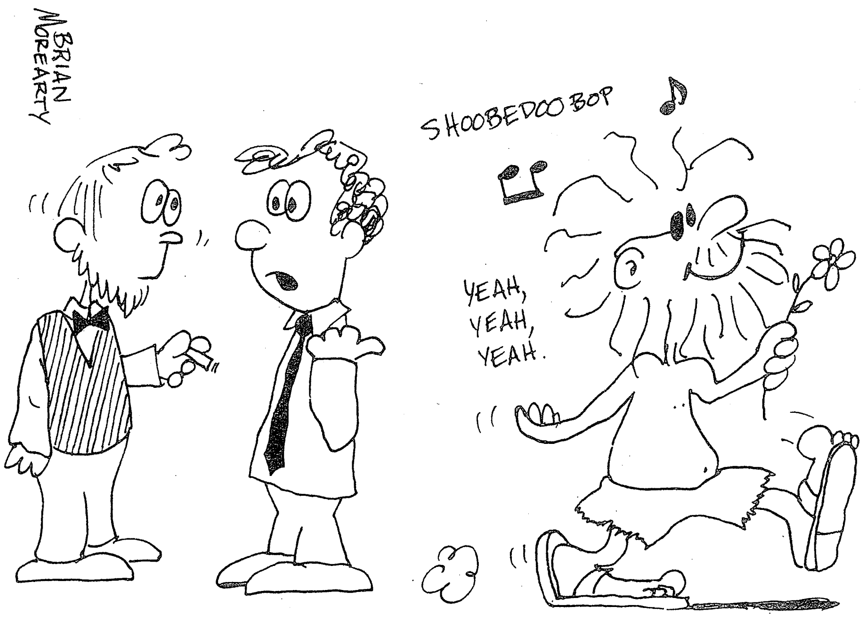


"HE ONLY LIKES NATURAL LOGARITHMS."

LEWIS CARROLL IN HIS PROFESSIONAL LIFE

M.A.B. Deakin, Monash University

Lewis Carroll is best remembered as the creator of *Alice in Wonderland* and other works of fantasy. The name "Lewis Carroll" is a pseudonym derived from the first two names of the actual author, an Oxford cleric and academic, Charles Lutwidge Dodgson.

Dodgson was born in 1832, the third child of Charles Dodgson, a clergyman, and his wife and cousin (née Francis Jane Lutwidge). The young Charles first attended Richmond School, in Yorkshire, and came to the attention of his headmaster for the mathematical ability he displayed. From Richmond he went to Rugby, of *Tom Brown's Schooldays* fame, which he found less congenial. After three years, aged 19, he "escaped" (his own word) to Christ Church, Oxford, where he remained till his death in 1898.

He seems to have been extremely shy, stuttering, it is said, except in the presence of the various nymphets he befriended. Of these the best known was Alice Liddell, the original Alice; she was the daughter of a colleague, Dean Liddell, with whom Dodgson fell out when he requested permission to photograph the pre-adolescent Alice in the nude (albeit from behind).

Dean Liddell and almost an entire post-Freudian generation took rather a dim view of this request, although we now know that Dodgson was puritanical to a fault. (He nurtured thoughts of expurgating further the already expurgated Shakespeare of Dr Bowdler, and expressed the hope that the illustrators of his books would not ply their trade on a Sunday.) His interest in photography was genuine and deep. The quality of his work in this area is high.

It is paradoxical that Dodgson, who by profession was a mathematician (he never practised as a parson, although he took holy orders in 1861), is deservedly better remembered for his creative writing and, indeed, his photography. The aim of this article, however, is to discuss his mathematical achievements.

His mathematical work lies in five main areas, only two or three of which he recognised as respectably mathematical. Most of his writing in mathematics was to do with Euclidean geometry, and the best-known of his strictly mathematical books, *Euclid and His Modern Rivals*, the only one still quoted to any extent, lies in this area. However, he did in his 30's also occupy himself with determinants, numbers arising in the study of matrix algebra. His other mathematical interest was the theory of tournaments and elections, of which more later. Beyond these three interests, he wrote mathematical recreations and works on Symbolic Logic under his pseudonym.

This distinction is important. Dodgson and his alter ego, Carroll, shared many concerns and their writing styles are similar. It has been said that Dodgson had two personalities, his own and Carroll's. This seems not to be so. Rather, he used the pseudonym for his works of fantasy, thus distinguishing them from his serious writing. The pseudonym gave him, shy as he was, some protection from the fame that *Alice* brought him.

Of his mathematical achievements, Carroll himself was wont to say that they lie "chiefly in the lower branches of mathematics". No "Dodgson's Theorem" exists, and few would try to point to any lasting mathematical advance due to his insight. He was a pedantic (i.e. tediously finicky) teacher, obsessed with his own idiosyncratic notations, such as ⋒ for sine and Ω for cosine. He attracted very few students to his lectures, which were, surprisingly perhaps to us, regarded as very dull. He published numerous, now forgotten, pamphlets, most of them divorced from the mainstream mathematics of his day, and even less relevant to us.

He probably approached that mainstream most closely in his work on determinants, which arise in the theory of matrix algebra. This was a relatively new branch of mathematics in 1866, when Dodgson published a brief note in the *Proceedings of the Royal Society*. A determinant associates a single number with a square array of numbers. Determinants arise particularly in the solution of simultaneous equations, and their evaluation is an important problem in this area. Efficient methods for this evaluation needed to be developed, and it was to this question that Dodgson's paper addressed itself.

Unfortunately, it is almost incomprehensible, and to see what is meant, one does best to turn to his subsequent book: *Elementary Treatise on Determinants*.

(It was, incidentally, this book that followed most immediately on the heels of *Alice in Wonderland*. The story has it that Queen Victoria, enchanted by *Alice*, asked the publishers - Macmillan - for a copy of the author's next work, and was unamused to receive a copy of *Elementary Treatise on Determinants*. The story is probably apocryphal (i.e. of doubtful authenticity), but it makes a good yarn, and I follow convention in repeating it here.)

Elementary Treatise on Determinants is a reasonable introductory textbook, rather less original than its author supposed, but extremely systematic. Its eccentric notation is such that one would not recommend it to a modern reader. (For example, he refuses to use the word "Matrix", preferring "Block".) Some of the problems he set himself, but could not solve, now seem trivially easy as more advanced matrix algebra has become widely known. Appendix II of that work gives an expanded version of the *Royal Society* paper. It appears that what Dodgson had invented was a minor variant on what we now call *Gaussian elimination*, but its full details eluded him.

Dodgson's most extensive mathematical work lay in the field of euclidean geometry. It is in this field that his only mathematical work still available - *Euclid and His Modern Rivals* - is to be found.

One might imagine that this refers to the rise, in the years preceding the book's publication, of the non-euclidean geometries†, but this is not so. Apart from knowing that they stem from a denial of Euclid's parallel postulate, Dodgson shows little acquaintance with these. What the book, a dialogue in five acts, attempts to do is to show the superiority of Euclid, as an expositor of *euclidean* geometry, over modern rivals such as, in particular, J.M. Wilson, a text-book writer of the time.

The following excerpt on the *pons asinorum* (i.e. the theorem that the angles at the base of an isosceles triangle are of equal magnitude)†† gives the flavour of Dodgson's writing. He is discussing Pappus' proof, which, in essence, proceeds by turning the triangle over and superposing it on its previous position.

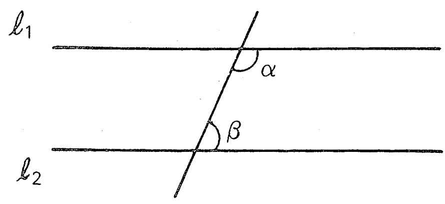
Minos: It is proposed to prove 1.5 [i.e. the *pons asinorum*] by taking up the isosceles Triangle, turning it over, and then laying it down again upon itself.

Euclid: Surely that has too much of the Irish Bull about it, and reminds one a little too vividly of the man who walked down his own throat, to deserve a place in a strictly philosophical treatise?

Minos: I suppose its defenders would say that it is conceived to leave a trace of itself behind, and that the reversed Triangle is laid down upon the trace so left.

Nowadays we dismiss such metaphysical questions from mathematics, omitting the actual motion from the argument. Dodgson's criticism does not apply to modern accounts at all††.

Dodgson did recognise that, when it came to the parallel postulate, Euclid's account might not be the best. Refer to the diagram. Euclid's version of the postulate is that ℓ_1, ℓ_2 are parallel if and only if $\alpha + \beta = \pi$; if $\alpha + \beta < \pi$ they meet when extended to the right, otherwise to the left.



† See *Function*, Vol.3, Part 2.

†† See *Function*, Vol.3, Part 3.

By Dodgson's day, this rather cumbrous form had been replaced by the more illuminating *Playfair's Axiom:*

> *If* P *is a point not on a line* ℓ, *then exactly one line may be drawn through* P *parallel to* ℓ.

Dodgson resisted this new approach, though later, after he had retired on the proceeds of *Alice* (to devote his life to mathematical writing), he produced a convoluted alternative best passed over in silence.

In the field of symbolic logic, Dodgson wrote for publication under his pseudonym, which probably implies that he saw the subject as essentially recreational. His two books in the area are *Symbolic Logic* and *The Game of Logic*, both in print today. Opinions differ on their significance.

On the one hand, W.W. Bartley III (*Scientific American*, July 1972) can write "his work on logic was highly original", but N.T. Gridgeman (*Dictionary of Scientific Biography*) finds that "although he was not ignorant of the new trends [in mathematical logic], their importance either escaped him or was discountenanced".

Both books (and I include the second part of *Symbolic Logic*, reconstructed by Bartley) are original, quirky, and, to my mind, ultimately sterile. They both post-date Boole's *Laws of Thought*, which Dodgson is known to have possessed, but neither shows the slightest acquaintance with that work.

The predominant concern of these books is not really modern symbolic logic, but a rather baroque outgrowth from the puzzle world - the *sorites* (pronounced sore-*eye*-teas).

I quote but one very simple example. Three premisses are given:

(1) No potatoes of mine, that are new, have been boiled;

(2) All my potatoes in this dish are fit to eat;

(3) No unboiled potatoes of mine are fit to eat.

These all concern "my potatoes", which may be: a (boiled), b (eatable), c (in this dish), d (new). The object is to construct a valid conclusion from the premisses. The method is perfectly mechanical and one of several equivalent techniques employed by Carroll proceeds as follows.

Write $\rightarrow$ for "implies" and $'$ for "not". The premisses now translate as:

(1) $d \rightarrow a'$

(2) $c \rightarrow b$

(3) $a' \rightarrow b'$.

Four letters are involved, of which two (a,b) occur twice and the others (c, d) only once. (More generally, if n premisses are involved, and some of Carroll's soriteses involve over 50, there are $(n + 1)$ letters, $(n - 1)$ of them occurring twice and two occurring once.) The problem is to find a logical connection between the two which are listed only once.

In the above example, write (1) and (3) in the alternative (and equivalent) forms:

(1) $a \to d'$

(3) $b \to a$.

We thus find

$$c \to b \to a \to d' \text{ whence } c \to d'$$

which translates as "My potatoes in this dish are not new".

For more on these topics, see *Function*, Vol.1, Part 5.

There are some nice things in *Symbolic Logic* and *The Game of Logic*. Venn diagrams are used in an elegant way with coloured counters and some subjects are raised which still occupy some (usually less mathematically inclined) logicians today.

Of Carroll's other mathematical recreations, the best known is his proof that all triangles are equilateral. This featured in the April Fools' Day column of our previous issue. Although Carroll did not realise it, this is a significant result, for the proof is not technically incorrect. Where it goes wrong is in its translation into reality. Of the points G, H there constructed, one necessarily lies *inside* and the other *outside* the triangle and the proof fails (in almost its last line). Euclid's axioms do not, however, refer to the "inside" or "outside" of a triangle, and thus what Carroll had done was to prove the inadequacy of Euclid's system of axioms, a conclusion he would not have liked at all!

This example is found in a collection called *Curiosa Mathematica*, as is this next (the relevant part has also been published as *Pillow Problems*). The kindest thing that can be said about Carroll's error here is that it may serve to show how easy it is to make mistakes in elementary probability theory.

Problem "A bag contains 2 counters as to which nothing is known except that each is either white or black. Ascertain their colours without taking them out of the bag."

Now this is nonsense, but Carroll confidently gives the answer "one white, one black" and moreover argues for it by means of a specious probability argument. We leave it to the reader to find the error, noting merely with one commentator (Eperson, *Mathematical Gazette*, Vol.17 (1933), p.99), that a similar argument, applied to the case of 3 counters, shows that there were not 3 after all. Here is Carroll's "solution".

"We know that if a bag contains 3 counters, 2 being black and one white, the chance of drawing a black one is $\frac{2}{3}$, and that any other state of things would not give this chance.

Now the chances that the given bag contains, (1) *BB*, (2) *BW*, (3) *WW*, are respectively $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.

Add a black counter.

Then the chances that it contains (1) *BBB*, (2) *BBW*, (3) *BWW*, are as before, $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.

Hence the chance of now drawing a black one

$$= \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{2}{3}.$$

Hence the bag now contains *BBW* (since any other state of things would not give this chance).

Hence before the black counter was added, it contained *BW*, i.e. one black and one white counter."

Regrettably, one encounters other such lapses in Dodgson's writing. He clearly enjoyed mathematics as a recreation and kept a journal of mathematical thoughts.

"31st October, 1890. This morning, thinking over the problem of finding two squares whose sum is a square, I chanced on the theorem (which seems true, though I cannot prove it) that if $x^2 + y^2$ is even, its half is the sum of two squares. A kindred theorem that $2(x^2 + y^2)$ is always the sum of two squares also seems true but unproveable."

"True but unproveable" would seem to presage Gödel's Theorem, but this is not what Dodgson had in mind. He found a proof five days later. It is one line long and we print it on p.32, but try first to discover it for yourself. On the 5th of November, he proved also the related theorem: "Any number whose square is the sum of two squares is itself the sum of two squares."

The result, as stated, is, in fact, false. E.g. $15^2 = 12^2 + 9^2$, but 15 is not the sum of two squares. However, it is true that if

$$z^2 = x^2 + y^2$$

and if z, x, y have no common factor, then there exist integers u, v, such that $z = u^2 + v^2$. This result was known to the Babylonians, and proofs had been available for hundreds of years before Dodgson. See Problem 7.3.5. See also for more background *Function*, Vol.4, Part 1, p.27 and the article "Pythagorean Triples" by F. Schweiger in Vol.6, Part 3.

Gridgeman's assessment of Dodgson's mathematical work takes the least-known of it (the work on tournaments and elections) to be the best. (Although there are those, like Bartley, who rate his logical works more highly than I have done.)

This is contained in a number of pamphlets, letters and broadsheets, all very rare, and one so rare that it survives merely as a single copy. The best account of this work is to be found in the book *The Theory of Committees and Elections* by Duncan Black and I draw on this.

There is an extensive mathematical theory concerned with the organisation of tournaments, elections and fair decision-making procedures. Black's book is a good introduction to an area we can only touch on here. Some of this theory predates Dodgson, but Black shows quite conclusively that Dodgson did not know of this.

His sources were his organisation of tennis tournaments and his work on committees at Christ Church. In this latter capacity he used his work to further his quarrel with Dean Liddell (Alice's father).

Again, I would hardly consider Dodgson's work in the area earth-shattering, but he does consider a number of unusual and imaginative voting schemes, methods for multiple decision-making and allowance for the expression of degrees of preference in a ballot. There are many numerical examples, which at least serve to show the limitations of methods in vogue.

(To those new to this field - consider the differences between
(a) a "first-past-the-post" system as used in Britain,
(b) the Australian Federal lower house system,
(c) the Australian Senate system,
(d) Tasmania's Hare-Clark system,
(e) the system used to decide the *Age* footballer of the year.)

Gridgeman, rather generously, remarks that Dodgson was the first to use matrices in multiple decision-making, and, if the tabulation of results in a rectangular array is to be called a use of matrices, so be it. No use is made of *matrix algebra*, of which, apart from the relatively elementary theory of determinants, Dodgson seems to have been ignorant.

The picture that emerges of Dodgson the mathematician is one of a pedant (his *Notes on Euclid* includes definitions of "problem" and "theorem", and *Symbolic Logic* has a definition of "definition"), original enough, but out of touch with the mathematics of his day. He was a mediocre mathematician in other words.

Of course, his talents in both literature and photography were much greater and for these he is justly famous. He is deservedly best remembered for the things he did best.

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INCITEMENT TO CRIME?

[The judge] told her the penalty for armed robbery can range from 6 to 30 years in prison. 'And if you can't assure me that you won't behave properly,' he added, 'you won't be in the courtroom for your trial.'

Chicago Sun-Times, 30.4.80.

BOOK REVIEW

INSIDE RUBIK'S CUBE AND BEYOND

Christoph Bandelow, Birkhäuser, 1982

Available from D.A. Book Depot, Station Street, Mitcham

Price $10

Reviewed by E.A. Sonenberg

Many "cube books" have been published in the last couple of years - see *Function*, Vol.6, Part 1 for a review of some. Bandelow's recent book is a worthy addition to the library of anyone keenly interested in the cube.

The book, of course, includes a systematic procedure for "solving" the cube. But more importantly there is a presentation of a number of concepts and theorems from a branch of mathematics known as group theory through which one can develop an *understanding* of the cube. By such theoretical considerations one can answer questions like: Which positions can be reached from the start position by turning the layers? How are we restricted if only a part of all the moves is permitted? Is there any non-trivial operation for which it makes no difference whether we perform it before of after another operation?

The theoretical part of the book (2 chapters of 6) is not easy reading, but for those curious about the mathematics of the cube it provides a thorough, clearly written treatment of material not found in any other book.

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PARABOLAS IN LAPLAND†

Laplanders build conical huts of logs or beams (Figure 1). These in fact resemble a large open fireplace or hearth around which you can sit with an entire family. A fire is lit on the ground in the centre and the smoke disappears through the vent at the top.

Figure 1: A conical Lapp hut in northern Sweden.

Figure 2: The double parabolic framework.

There is also another door and a hole close to the ground to draw in the necessary air.

How are they constructed?

In the village of Fatmomakke in northern Sweden they were just building a new hut (Figure 2). Four bent pieces of wood had been put up in the clearing, connected in pairs to make parabolas. They form the framework of the hut. Logs or tapering beams are then nailed to this.

It is not surprising that parabolas are chosen as framework for the hut. If a cone is intersected by a plane which is parallel to a generator of the cone, the intersecting curve is a parabola. (See Figure 3 opposite.)

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†Translated from the Dutch by Ms A.-M. Vandenberg. The original is from *Pythagoras*, a Netherlands counterpart of *Function*, and we publish the article under an exchange agreement between *Pythagoras* and *Function*. This article relates very closely to John Mack's articles in *Function* (see pp.4 – 7 and our previous issue).

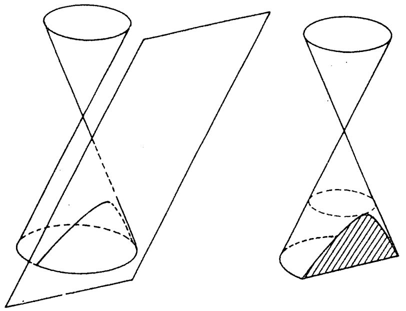

Figure 3: A parabola shown as a section of a cone.

A NON-RIGID 14-HEDRON[†]

Here is a plan from which to construct a remarkable object. As you can see from Figure 1, it has to become a multi-sided structure with 9 vertices, 14 triangular faces, and 21 edges (lines around the perimeter count as half!). The triangles occur as 4 different types, the edges in 5 lengths.

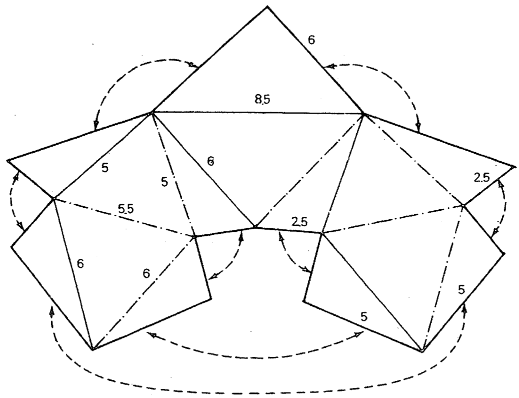

Figure 1

In plan, the figure exhibits nice left-right symmetry, but as soon as you start folding this no longer applies. Note the two types of folding lines in the drawing. Construction may well cause some headaches, for the result is a rather weird monster with two deep indentations. You will only perceive some symmetry again by looking for the longest edge and its opposite corner point (the "top") and marking it in colour.

What is so remarkable about it?

Of course you can think of an endless number of other weird many-sided structures. But the case described here has the following very distinctive feature: *If you hold the ends of the longest edge with one hand, you can move the top a little, back and forth.* (See Figure 2.) The many-sided construction is not fixed, not rigid. It can "wobble" a bit, without forcing.

[†]Translated from the Dutch by Ms A.-M. Vandenberg. The original occurs in *Pythagoras* and appears here under an exchange agreement between *Pythagoras* and *Function*.

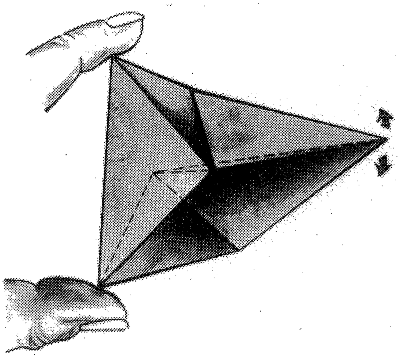

Figure 2

"So what", you may say, "is that so remarkable?" Probably not if you have the thing in your hand and see it happen. But it *is* if you consider that, until very recently (a few years ago), such a structure had never been made. And therefore everybody thought such a construction, consisting of rigid faces and hinged edges, should always be "rigid". Moreover, quite some time ago it had been proved beyond doubt that *convex* many-sided structures are definitely always rigid. (Convex means: "without dents or indentations".)

Thus this is indeed a quite remarkable many-sided sturcture. It demonstrates again how dangerous it is in mathematics to say something does not exist as long as such a statement has not been conclusively proved.

Construction guidelines.

- Use good stiff drawing paper or thin cardboard for the model.
- In copying the figure, choose at least 1 cm as the unit of measure for the measurements indicated (in the drawing).
- Pinch along the solid lines from the front, and the pecked lines from the back; make the folds in the corresponding direction.
- Join the free edges together with sellotape as indicated by the arrows. *Or* work out *before* cutting out where pieces of tape should go.
- Leave the free edges of the uppermost triangle in the model loose at first in order to see what happens inside.
- Do your work with the utmost precision! Good luck.

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LETTERS TO THE EDITOR

We have had three separate letters all on Peter Higgins' proof that $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1)$. This is a record for *Function*, and speaks well for the interest that the article generated.

AN ALTERNATIVE GEOMETRIC PROOF

The formula $1 + 2 + 3 + \ldots + n = \frac{1}{2}\,(n + 1)$ may be proved geometrically without reference to the trapezia used by Higgins (*Function*, Vol.7, Part 2). See Figure 1. The total area of the entire rectangle is clearly $n(n + 1)$ and the shaded portion, representing the required sum is clearly half of this.

J.M. Mack,
University of Sydney.

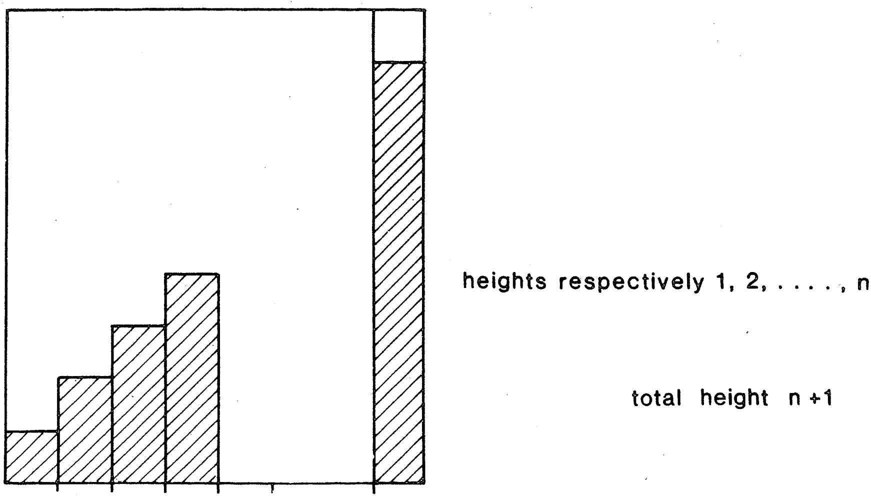

Figure 1

THE SAME ALTERNATIVE PROOF

The formula for the sum of an arithmetic progression is proved algebraically by writing

$$S_n = a + (a + d) + (a + 2d) + \ldots + (\ell - 2d) + (\ell - d) + \ell,$$

where a is the first term, ℓ is the nth (or last) and d is the common difference. Writing this series backwards so that the last term comes under the first, etc., we have

$$S_n = \ell + (\ell - d) + (\ell - 2d) + \ldots + (a + 2d) + (a + d) + a.$$

Add the corresponding terms to find

$$2S_n = (a + \ell) + (a + \ell) + (a + \ell) + \ldots + (a + \ell) + (a + \ell) + (a + \ell)$$

where there are n terms in the sum. Thus

$$2S_n = n(a + \ell)$$

and

$$S_n = \tfrac{1}{2}n(a + \ell).$$

In Higgins' example, $a = d = 1$, $\ell = n$.

I have long made a practice of demonstrating this case by means of *Cuisenaire rods* and this gives a simple geometric proof (see Figure 1 opposite).

K. McR. Evans,
Scotch College, Melbourne.

A RELATED RESULT

Higgins' proof that $1 + 2 + 2 + \ldots + n = \frac{1}{2}n(n + 1)$ and his remark that it is related to concepts used in integral calculus prompted me to prove similarly that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1).$$

Replace Higgins' straight line by a parabola $y = x^2$ as shown in Figure 2. Then the trapezium of Higgins' proof is replaced by a figure whose top is a parabolic arc and whose area is

$$\int_{r-1}^{r} x^2\,dx.$$

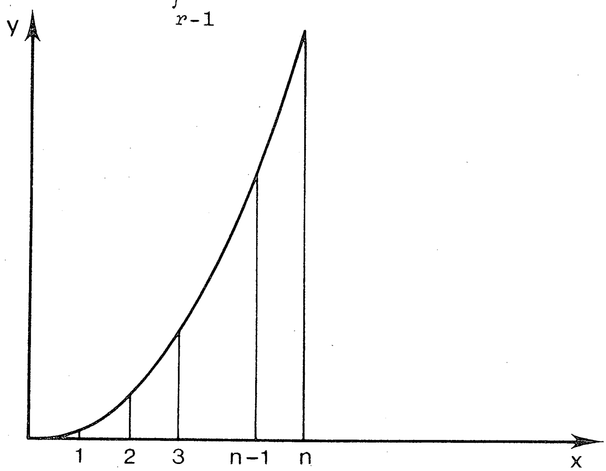

Figure 2

But $\displaystyle\int_{r-1}^{r} x^2 dx = \frac{1}{3}\left\{r^3 - (r - 1)^3\right\} = r^2 - r + \frac{1}{3}$. Now add all these contributions for $r = 1, 2, 3, \ldots, n$, knowing that the total area is

$$\int_0^n x^2 dx \qquad \text{or} \qquad \tfrac{1}{3}n^3 .$$

This gives the result claimed, after some algebra.

T.C. Brown,
Monash University.

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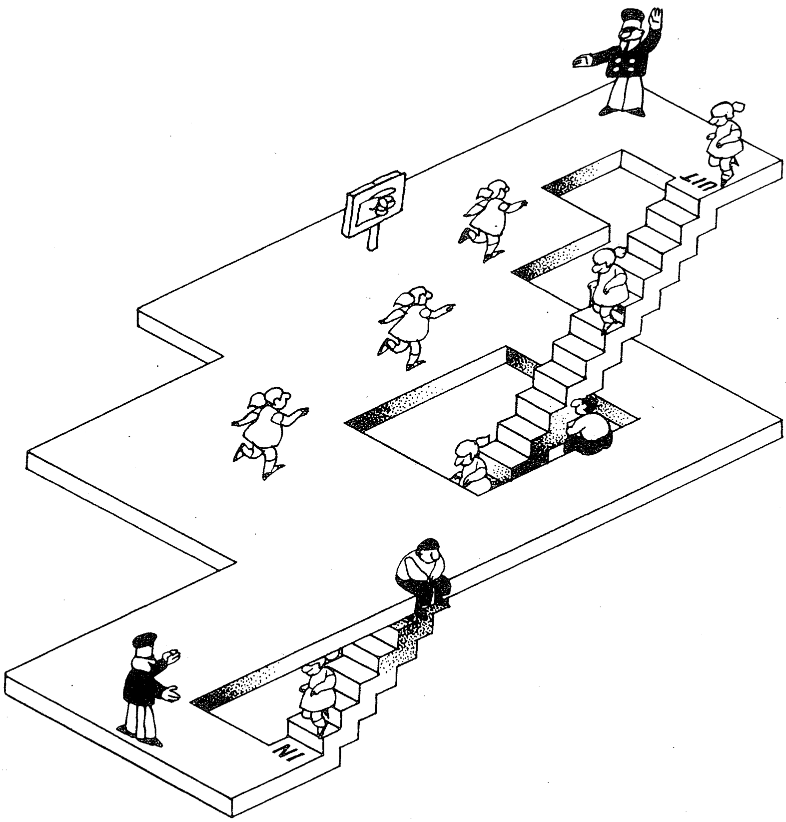

Reprinted, under an exchange agreement, from *Pythagoras* (January, 1983).

PROBLEM SECTION

The only outstanding problems are those posed in our last issue, to which we have already received some solutions, but our experience of the previous issue has decided us not to publish any of these as yet, in order to allow more time for other solvers to send us material.

We acknowledge a solution to Problem 7.1.1 submitted by George Karaolis, 66 Cruikshank Street, Port Melbourne, that arrived too late for inclusion in our previous issue.

For those who wished to read, rather than themselves produce, solutions to the problems posed in April, a further two months' wait will thus be necessary. Here are even more problems to work on. The first three were submitted by J. Ennis, Year 11, M.C.E.G.S.

PROBLEM 7.3.1.

There are $n!$ possible permutations of the symbols $1,2,3,\dots,n$. Of these $f(n)$ are such that *no* symbol remains in its original position, and $g(n)$ are such that *exactly one* symbol is undisturbed. Prove that

$$f(n) = g(n) + (-1)^n .$$

PROBLEM 7.3.2.

Express each of the integers from 1 to 20 in terms of the symbols π, $+$, $-$, $\times$, $/$ (or $\div$), $\sqrt{\ }$, (,), [ ], where the square brackets indicate the largest integer less than a given number (e.g. $[\pi] = 3$). A "rule of the game" is to strive to be as economical as possible.

$$3 = (\pi + \pi + \pi)/\pi ,$$

but $[\pi]$ expresses this integer more economically.

PROBLEM 7.3.3.

A deck of 52 playing cards is shuffled and placed face down on a table. Cards are removed from the top of the pile until a black ace is encountered. In which position is this ace most likely to be found?

PROBLEM 7.3.4. (See the article on Lewis Carroll.)

Let x, y, z be relatively prime, and satisfy $x^2 + y^2 = z^2$. Prove that there exist integers u, v such that $z = u^2 + v^2$. Find x, y in terms of u, v.

PROBLEM 7.3.5.

Solve for x: $\sqrt{x - 1} - \sqrt{x - \sqrt{3x}} = \frac{1}{2}\sqrt{x}$.

A MATHEMATICAL TRIAD

The three graduates pictured opposite all graduated with first class honours from Monash University in April, 1983. They are (from left) Helen Pongracic (Applied Mathematics), Caroline Finch (Mathematical Statistics), and Julie Ann Noonan (Applied Mathematics). Julie topped her class at the end of 1982 and was awarded the L.J. Gleeson Memorial Prize, awarded annually by Monash to the best student in Applied Mathematics.

The three have been associated for some time. Caroline and Julie were at school together, and Caroline and Helen have known each other since childhood.

All three are continuing their studies as research students. Helen is at Monash and is investigating the effects of asteroid impact on the earth. It is now widely thought that a catastrophe of this kind exterminated the dinosaurs (and thus led, indirectly, to our emergence).

Caroline has gone to La Trobe University and is investigating medical uses of statistics - with particular reference, she hopes, to the evaluation of various cancer treatments.

Julie is also continuing at Monash, working on medium scale meteorological phenomena - notably a strange, but regular, occurrence in Queensland's Gulf Country. This is known as the *Morning Glory* and consists of regular lines of cloud that appear over Burketown and its environs.

We wish all three success in their future, and hope that other young women will follow them.

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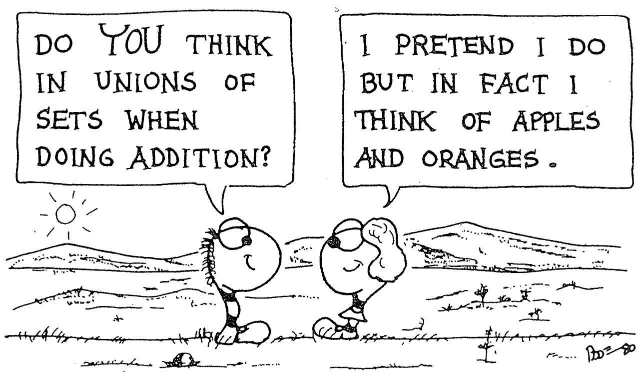
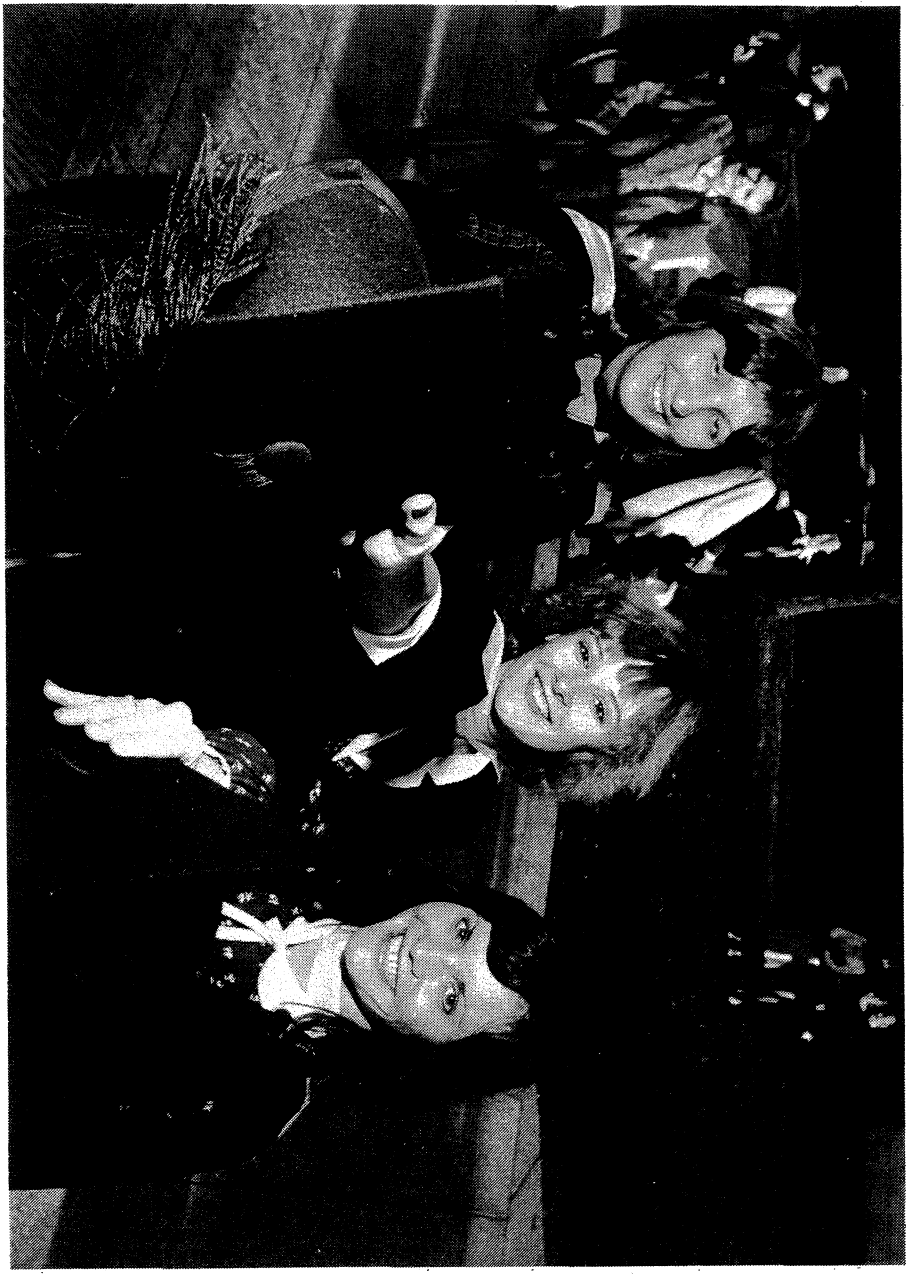


Andrejs Dunkels, Department of Mathematics, University of Luleå, Sweden. Reprinted under an exchange agreement from *Mathematical Digest* (RSA), No.50, (January, 1983).



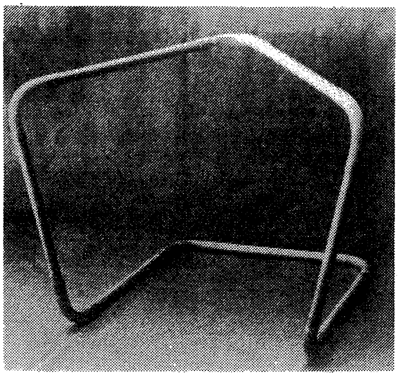
A THREE-DIMENSIONAL HEPTAGON

In our last issue, we included (p.11) a problem as to whether a heptagon (a seven-sided polygon) can be constructed in three dimensions in such a way that:

(a) all its sides are equal,

and (b) all its angles are right angles.

The answer is "yes", and the picture at the right (from *Pythagoras*, Volume 22, Part 3) shows how this is done.



The principle of construction used here allows us to demonstrate the existence of such regular n-gons for all $n \geqslant 6$.

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A STILL UNSOLVED PROBLEM

In Volume 5, Part 3 we introduced readers to a "Viennese Puzzle" and speculated that it was unsolved.

Let a_0 be an odd number (say 3). Multiply it by 3 and add 1 (to get 10). Now divide out by b_0, the maximum power of two that will divide this new answer (here $b_0 = 2$). Call the result a_1 (here 5). Repeat the process to get a_2 ($3 \times 5 + 1 = 16$, $b_1 = 16$, $a_2 = 1$), and so on. The general formula is

$$a_{n+1} = \frac{3a_n + 1}{b_n} \quad . \tag{*}$$

The question is: Does the process always result (ultimately) in a string of ones? Clearly the number 1 generates itself by the formula (*) ($b_0 = b_1 = \ldots = 4$), and also (fairly clearly) no other number can be self-generating. We might envisage, however, some sequence of a_n increasing without limit, or locking into a cycle.

We explored the behaviour of the sequences for various a_0, and these ultimately all yielded the string of 1's, although the case $a_0 = 27$ takes a long time to settle down.

The problem is, as we suspected, unsolved. It (or rather a problem equivalent to it) was one of a number of related unsolved problems discussed by Professor Richard R. Guy in the January, 1983 number of *American Mathematical Monthly*. It seems to be extremely difficult to resolve problems of this type.

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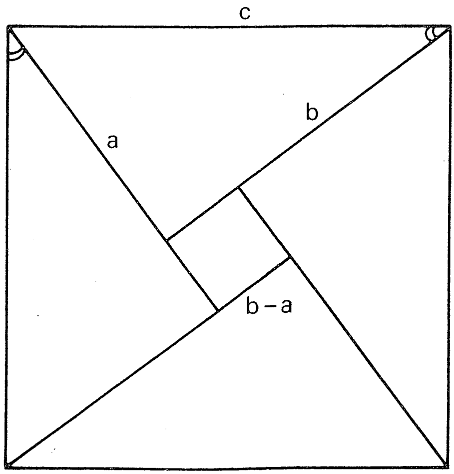
THE BEST PROOF OF PYTHAGORAS' THEOREM

We have had several requests for a (we thought) well-known proof of Pythagoras' Theorem - essentially that given by Bronowski some years ago in the television series *The Ascent of Man* - so here it is.



In the figure, a, b, c are the sides of a right-angled triangle, whose other two angles thus add up to a right angle. It follows that if the figure drawn is made up as shown the outer perimeter is a square of side c, as the marked angle and the other angle at top left must add to a right angle. We thus have

$$c^2 = 4(\tfrac{1}{2}ab) + (a - b)^2 ,$$

which simplifies to

$$c^2 = a^2 + b^2 ,$$

that is to say, Pythagoras' Theorem.

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THE SPARSENESS OF PRIMES

Is there a number N such that $N+1, N+2, \ldots, N+k-1$ are all composite (i.e. not primes) for a fixed k no matter how large k is?

The answer to this question is, incredibly, YES. The number N is in fact given by $N = k! + 1$.

Then $N+1 = k! + 2$ which is a multiple of 2.
$N+2 = k! + 3$ which is a multiple of 3.
$N+q = k! + q + 1$, which is a multiple of $q+1$ $(q<k)$.

Thus we can form a set of $k-1$ consecutive numbers, all of which are composite.

This demonstrates how "thin" the primes become as their size grows. Since $100!$ has 157 digits, there is a 157-digit number such that the next 99 numbers contain no primes.

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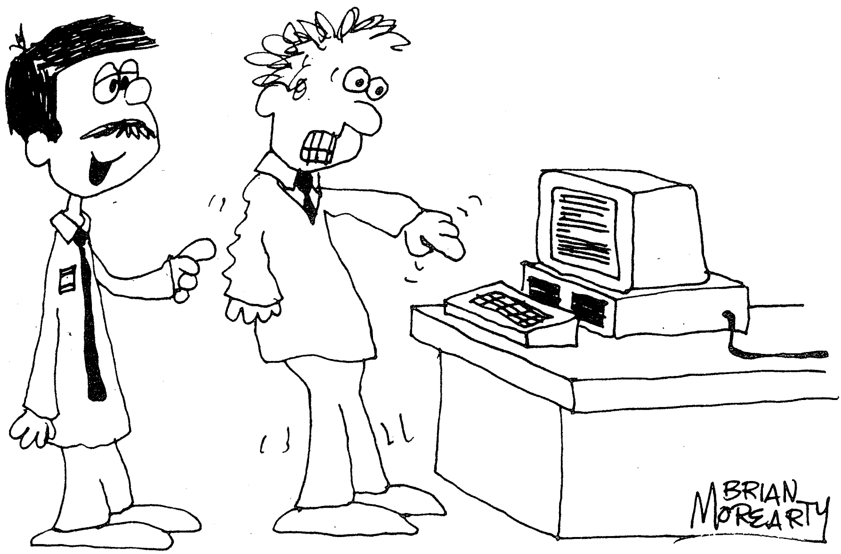

"IT'S ALREADY HAD ITS RABIES SHOTS."

BECOME A MATHEMATICS TEACHER?

The Mathematical Association of America, which is normally concerned with mathematics at the tertiary level, has begun a campaign to attract good graduates into mathematics teaching at the secondary level. There is, they say, a critical nationwide shortage of high school mathematics teachers in the United States.

This has led to that decline in standards in mathematics at entry to college level that has been in the news so much lately. This in turn has led to the vast increase in college level remedial mathematics courses (196% in five years), reported in *Function*, Vol.6, Part 3.

One problem, however, as the Guest Editorial in *Focus*, the Association's newsletter makes clear, is pay. The author, Peter Hilton, writes "... the material rewards are so paltry compared with what a talented student could earn in industry that it is not easy, in all conscience, to justify [influencing] a student to choose a teaching career".

In Australia, the situation is somewhat different. What we do share with the U.S. is a shortage of mathematics teachers. Two trends are likely to exacerbate this. First, students, due to the depressed economy, are staying in school longer, and second, there is a tendency evident now to return to the "harder" subjects like Mathematics and Physics in the belief (largely justified) that these offer better employment prospects.

However, Hilton's doubts do not apply here. Lionel Parrott, head of Monash' Careers and Appointments Service, states that the starting salaries, opportunities for further study, and promotional prospects in the teaching profession all compare favourably with those offered by industry in this country.

Of course, such a career will not suit everyone. Student readers of *Function* are well placed to realise that not everyone is cut out to be a teacher and also to note the fact that, although there are some fringe benefits like short hours and long holidays, there are other aspects, such as correction and preparation, that make these less advantageous than at first appears. Students might also reflect that the classroom aspect of teaching can be difficult and stressful.

Nonetheless the jobs are there and are unlikely to disappear in the foreseeable future.

Function has been informed that the prospects for mathematics teachers, at least in Victoria, are further improved if the teacher, at the start of his/her career is willing to work in country areas. This can be rewarding in itself. The country teacher may miss many of the attractions of the "big smoke", but this can be compensated by the warmth and friendliness of rural communities and the enhanced status that a teacher may enjoy as a part of such a community.

WHAT APPLIED MATHEMATICIANS DO

[We reprint an extract from an article by Hirsch Cohen, President of SIAM (Society for Industrial and Applied Mathematics), and published in full in the SIAM Newsletter (May, 1983). Eds.]

One of the things applied mathematicians find themselves doing often is explaining what they do. Sometimes to general audiences, like wives or husbands, kids, and neighbours but, often as not, to other scientists. I'd like to have a try at it because I believe here is a change coming about in what we do, or really, an important addition. I've observed that [recently] it has been extremely important to explain what mathematics does for the world. What applied mathematicians do counts a great deal in these public explanations because it's relatively easy to understand the applications we work on and their benefits to society.

I believe there are three major things that applied mathematicians do. The first is to create a better understanding of phenomena by describing them mathematically; the second is to create and to teach the techniques and methods for solving the mathematical problems that result from those descriptions. These two are familiar to all of us. We have studied laminar and turbulent flows in fluid mechanics. We've described nerve impulse propagation and blood flow in physiology. The understanding of galaxy formation and the motion of planetary bodies has been importantly aided by mathematical analysis. The list goes on and on and includes the physical, biological, and the social sciences. In almost all areas of nature and society, mathematical representations have been of use. Obviously it's a scholarly activity that we share with other scientists in these fields who are also intent on mathematical understanding of phenomena.

This attention to description, formulation, and then problem solving has often led to the development of new techniques and methods. The techniques that have been developed for solving application problems have played a stimulating role within mathematics itself. ... Obviously one major preoccupation of applied mathematicians is to improve on these methods and to pass them on as teachers. This is not so much a shared activity with others, but is more central to our own discipline. The interest and knowledge that applied mathematicians have in the formulation of problems and the utility of their solutions often provides them with special insights into approximations that make hard or large problems tractable and resolvable.

So, understanding how things work, unraveling this understanding in mathematical terms and communicating the understanding and the technology are two things we do.

But there's a third function. Mathematics has also always been used to design and to operate. For many years ship hulls, bridges, airfoils, and other engineering products have been designed for strength, size, weight, durability, cost and other values using mathematics. Mathematics has also been an essential ingredient in the handling of manufacturing lines, agricultural projects, electrical networks, controlling chemical processes, and many other operational activities. However, with the increase in computational power we have seen a swift rise in the amount and the complexity of the mathematics used in design and operations. New fields of applied mathematics have been created to make use of the computational facilities available. Mathematical programming is one example, used both in product development and operations design, for manufacturing, chemical processing, traffic flow and many scheduling and routing problems. Oil recovery procedures are another example of operational calculations. Nuclear reactors, magnetic disk heads, transonic airfoils, semiconductor chips are but a few of the many examples from a growing number of industrial and military products in which numerically based design procedures have become one of the essential steps. For these, new methods of numerical analysis are being developed.

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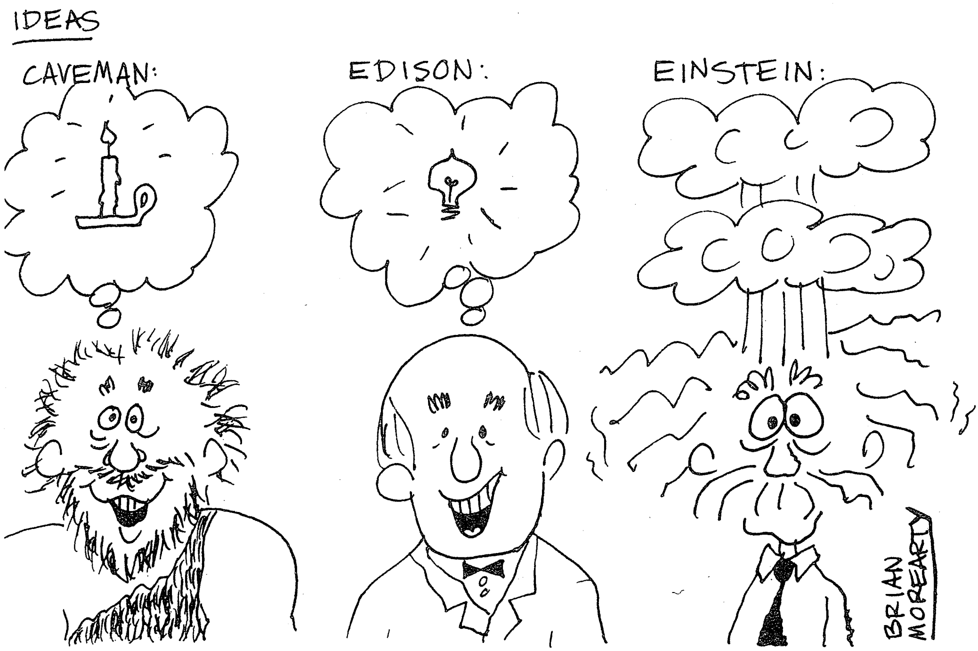

A BASIC ASPECT OF NUMERACY

Statisticians are often appalled at the many misuses of statistical methods, from elementary data analysis, which might be found in newspapers or magazines, to statistics presented in journal articles summarizing experimental research. Frequently, correct statistical summaries do not require a great deal of mathematical training, but rather some good common sense associated with some basic instruction in statistical thinking, namely "making sense out of numbers". Accordingly, many of us in the statistical profession believe that not enough has been done to introduce students (not only those in college, but those in elementary and secondary schools also) to good statistical reasoning.

Robert V. Hogg, *Studies in Statistics*, 1978.

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COINCIDENCE?

In Tampa, Florida, so the New York *Herald Tribune* reported, A Mr Earl M. Lofton sank an "ace" on the 119-yard first hole of the Palma Ceia golf course. The next two players of the foursome were so excited that they fluffed their shots. But the fourth player, Gilbert Turner, said that "a little thing like a hole-in-one" wouldn't bother him. It didn't. He got one too.

Warren Weaver, *Lady Luck*, 1963.

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THE ONE-LINE PROOF

The challenge on p.13 was to show that $2(x^2 + y^2)$ was the sum of two squares. Here is the proof.

$$2(x^2 + y^2) = (x + y)^2 + (x - y)^2.$$

The other statement ("if $x^2 + y^2$ is even ...") is a corollary.

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REMARKABLE!

Not only does Holding have a long run-up; he also walks an equal distance back.

Trevor Bailey, BBC TV Cricket Commentary, 12.7.1980.

The Canadian score of 7 was scored on six ends, and, what's more, five of those were singles.

ABC TV Lawn Bowls Commentary, Commonwealth Games, 1.10.1982.

"THAT'S THAT MATH TEACHER WHO ALWAYS ORDERS BANANA CREAM PI."