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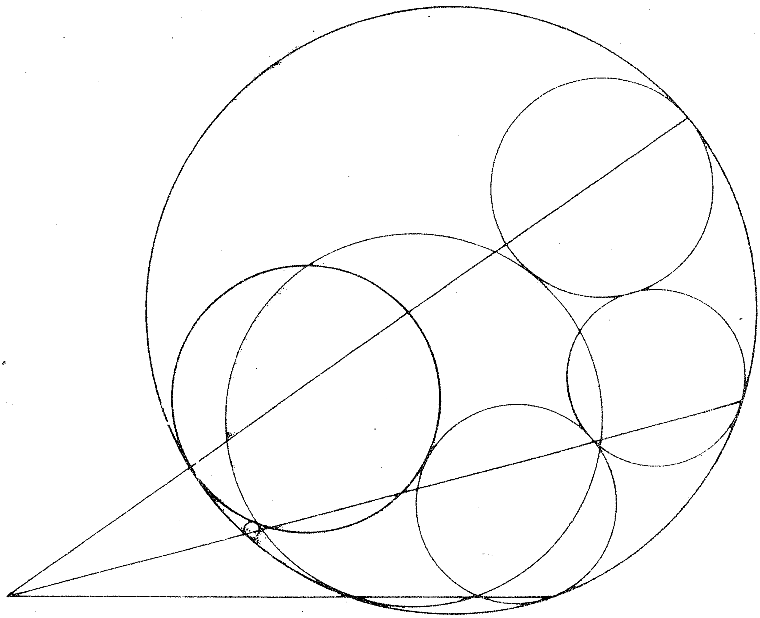

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Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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This issue inaugurates *Function's* seventh volume and we welcome new readers. *Function* is a magazine of school mathematics for those with interests in the area. Each issue has several main articles, as well as problems, news items, short contributions and a cover story. We urge our readers to submit material to us and to let us have their comments, favourable and unfavourable. We are always glad of feedback from our subscribers.

Our articles this time deal with prime numbers and their digit patterns, some remarkable extensions of Pythagoras' Theorem and a knotty problem of law. We include yet another article on the Rubik cube, this time not so much a "how to unscramble your cube" recipe as an entry to the mathematics behind these recipes.

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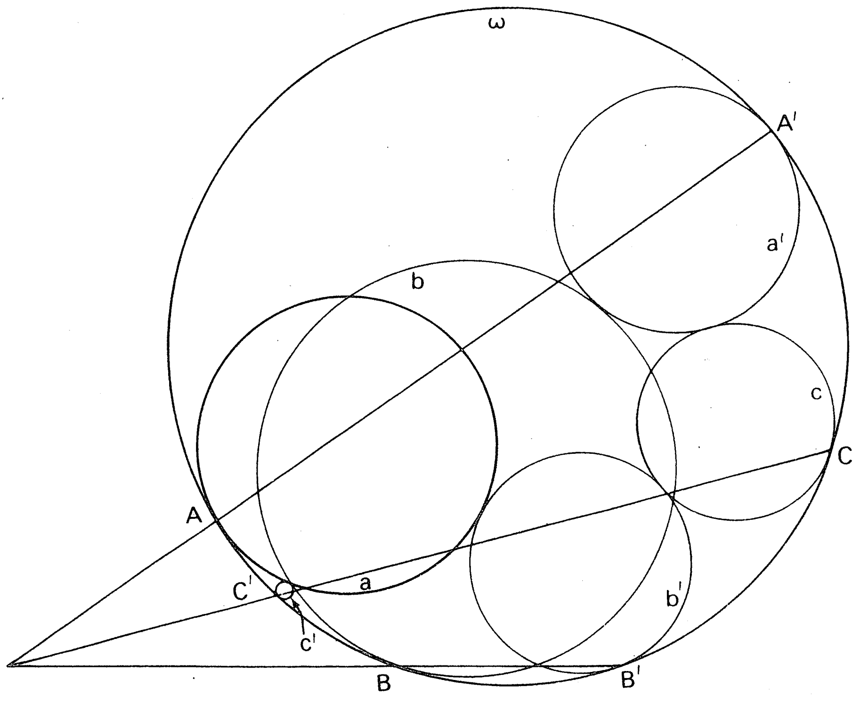
THE FRONT COVER

M.A.B. Deakin, Monash University

Take a circle ω and draw another circle a which is tangent to it, meeting ω at a single point A. Now draw a third circle (called b' for reasons that will become clear) tangent to both a and ω. Let this circle touch ω at B'. Another circle c is now drawn. c is tangent to both ω and b' and its point of tangency with ω is called C.



We draw two more circles: a', which touches both c and ω and meets ω at A', and b which touches both a' and ω and meets ω at B.

We now have a *chain of circles* a, b', c, a', b, each of which is tangent to the circles preceding and following it in the chain, and all of them tangent to ω. We now set about completing the chain. This is done by putting in a seventh circle c', which is to be tangent to all three of a, b and ω, and whose point of tangency with ω will be called C'.

This, it turns out, can be done in exactly two ways (discounting, as above, degenerate cases in which three or more circles are all tangent at the same point). The proof of this statement is not particularly difficult, but there are many cases to consider and we do not give it here; you could explore the matter for yourself.

We now find the following remarkable result. Join AA', BB', CC'. Two such diagrams are, of course, possible, depending on the choice of c' (and thus of C'). *For exactly one of these diagrams, these lines are concurrent - i.e. pass through a common point.*

On the front cover, we show a case where they do. We leave it to readers to find the other choice for c' and check that the three lines then do not possess a common point. Overleaf, we show the other case. Here AA', BB', CC' are not concurrent, but you may check that the alternative choice of C' gives concurrency.

This theorem is termed the "Seven Circles Theorem" and was first published in a book of that name in 1974 by C.J.A. Evelyn, G.B. Money-Coutts and J.A. Tyrell. Because only one choice of C' yields the concurrency, the seven circles theorem has been described as a result with a 50% chance of being true, but this was, of course, said tongue in cheek.

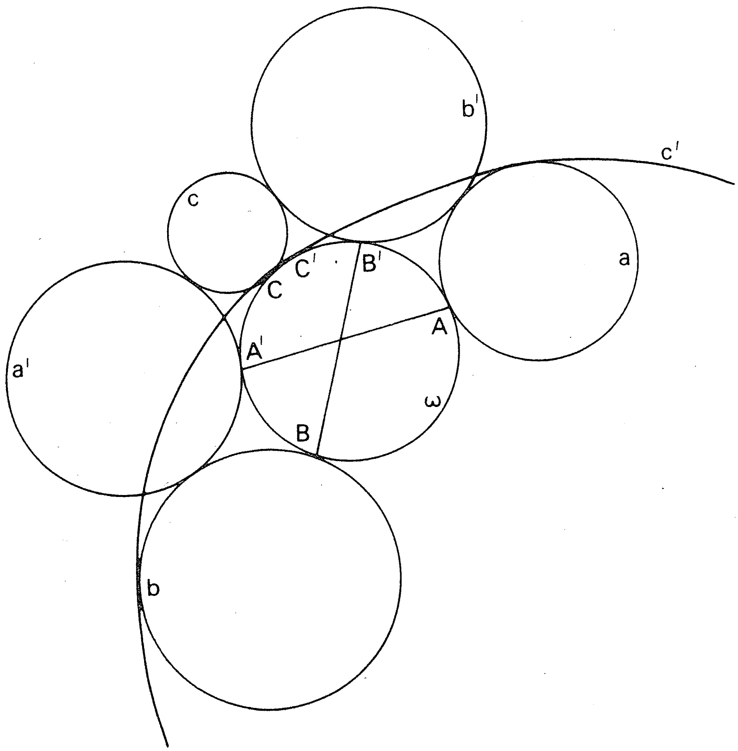
There are many other possible configurations, including some in which the circles degenerate into straight lines. They are worth exploring, but very accurate drafting is required.

Although geometry, in particular euclidean geometry, has a history that dates back thousands of years, significant new results such as this continue to be produced. A theorem such as the seven circles theorem is difficult to prove - which is why we do not attempt to present a proof here, but nonetheless it can be described and explored at an elementary level because the objects to which it refers are familiar to us.

Less difficult, but still significant, results, such as Pythagoras' Theorem, can be understood more fully at the school level. This has led many commentators to regret the extent to which euclidean geometry has been removed from school syllabuses. It is hard to think of another branch of mathematics in which results of comparable significance can be reached so readily and systematically. It also offers a good context in which to introduce the concept of rigorous proof.

The international mathematical olympiads, while, of course, they do draw questions from other areas of mathematics, often involve challenging problems in euclidean geometry. David Chalmers, one of our 1982 olympians (and a bronze medallist), reporting on the last competition in our South Australian counterpart, *Trigon*, makes a point similar to those made above:

"It has often been said that Australia ... is deficient in geometry teaching, and I think this was borne out by the result of the two geometry questions .... In European countries, a large part of the mathematics course is devoted to deductive geometry, whereas in Australia it is hardly ever mentioned."



DIGIT PATTERNS OF PRIME NUMBERS

K. McR. Evans, Scotch College, Melbourne

A *prime number* (a prime) is a natural number which has exactly two distinct factors. A *composite number* is a natural number with more than 2 distinct factors. Thus 2, 3, 5, 7 are examples of primes because each has 1 and itself as distinct factors; 4, 6, 8, 9 are examples of composite numbers because each has more than 2 factors; 1 is the only natural number which is neither prime nor composite since it has one factor.

The distribution of primes in the set of natural numbers does not appear to follow any pattern, but 'on average' primes seem to get further apart as they get bigger. The following question then arises. Is the number of primes finite or infinite? The answer was known to Euclid around 300 B.C. The substance of his argument is as follows.

Denote the primes 2,3,5,7,... by $p_1, p_2, p_3, p_4, \ldots$, so that p_n is the nth prime. Assume that the number of primes is finite. Then there exists a largest prime, p_m say. Now consider the numbers $q_1, q_2, \ldots, q_m$ defined by

$$q_n = p_1 p_2 p_3 \ldots p_n + 1, \quad n \in \{1,2,3,\ldots,m\} .$$

$q_1 = 2 + 1 = 3$ which is prime

$q_2 = 2 \times 3 + 1 = 7$ which is prime

$q_3 = 2 \times 3 \times 5 + 1 = 31$ which is prime

$q_4 = 2 \times 3 \times 5 \times 7 + 1 = 211$ which is prime

$q_5 = 2 \times 3 \times 5 \times 7 \times 11 + 1 = 2311$ which is prime

but $q_6 = 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30\ 031 = 59 \times 509$

which is composite.

Notice, however, that q_6 is not divisible by any of the primes $p_1, p_2, \ldots, p_6$; its smallest prime factor is bigger than p_6. Finally consider

$$q_m = p_1 p_2 p_3 \ldots p_m + 1 .$$

q_m is not divisible by p_1 or p_2 or ... or p_m since, in each case, there is a remainder of 1. Hence q_m (which is larger than p_m) is either prime or has all its prime factors larger than p_m. In either case there exists a prime larger than p_m. This contradicts the assumption. Hence the assumption is false and the number of primes is infinite.

Since there is only one even prime, Euclid's theorem may be stated in the following form.

Theorem 1. The set of odd natural numbers, $\{x: x = 2n - 1, n \in N\}$, contains an infinite subset of primes. [N is the set of natural numbers: $N = \{1,2,3,\ldots\}$.]

The following question may now be asked. Does the set of numbers of the form $3n - 1$, i.e. $\{x: x = 3n - 1, n \in N\} = T$ (say), contain an infinite subset of primes? Before proving an answer to this question we need a preliminary theorem.

Clearly each natural number belongs to one of the following disjoint sets:

$$S = \{3,6,9,12,15,\ldots\} = \{x: x = 3n, n \in N\}$$
$$T = \{2,5,8,11,14,\ldots\} = \{x: x = 3n - 1, n \in N\}$$
$$U = \{1,4,7,10,13,\ldots\} = \{x: x = 3n - 2, n \in N\}.$$

Now S contains only 1 prime, so either T or U or both contain an infinite subset of primes.

Theorem 2. U is *closed* under multiplication, i.e. the product of any two elements of U is an element of U. (T is not closed under multiplication since, for example, $2 \times 5 \notin T$).

Proof. Any two elements of U can be put in the form $3x - 2$ and $3y - 2$ for some natural numbers x, y. Also

$$\begin{aligned}(3x - 2)(3y - 2) &= 9xy - 6x - 6y + 4 \\ &= 3(3xy - 2x - 2y) + 4 \\ &= 3(3xy - 2x - 2y + 2) - 2 \\ &= 3m - 2, \text{ where } m = 3xy-2x-2y+2 \in N, \\ &\in U.\end{aligned}$$

The following theorem can now be proved.

Theorem 3. $T = \{x: x = 3n - 1, n \in N\}$ contains an infinite subset of primes.

Proof. Assume that the number of primes in T is finite and hence that there is a largest prime, say p, in T.

$$\begin{aligned} \text{Let } q &= 2 \times 3 \times 5 \times \ldots \times p - 1 \\ &= 3(2 \times 5 \times \ldots \times p) - 1 \\ &= 3m - 1 \qquad \text{where } m = 2 \times 5 \times \ldots \times p \in N \\ &\in T. \end{aligned}$$

Now q is not divisible by 2 or 3 or 5 or ... or p since addition of 1 would make it so. Hence q, which belongs to T, is prime or has prime factors all of which are larger than p. In the latter case at least one of the prime factors belongs to T. If this were not the case, all factors would belong to U. Hence T contains a larger prime than p, a result which contradicts the assumption. Hence the number of primes in T is infinite.

U also contains an infinite subset of primes. This result and Theorem 3 are special cases of a more general theorem due to Peter Gustav Lejeune-Dirichlet (1805-59). Dirichlet was one of a series of extraordinarily able professors of mathematics at the University of Göttingen in Germany. From this position he influenced mathematicians in both France and Germany. His theorem, published in 1837, is much more difficult to prove (and is not proved here) but is easy to state as follows:

Theorem 4. If a, b are natural numbers and if the highest common factor of a, b [HCF(a,b)] is 1, then $\{x\colon x = an - b,\ n \in N \cup \{0\}\}$ and $\{x\colon x = an + b,\ n \in N \cup \{0\}\}$ both contain an infinite subset of primes.

Consider again the numbers $q_1, q_2, q_3, \ldots$. Notice that an infinite number of these (viz. $q_3, q_4, \ldots$) have 1 as last digit. Can you see why? Does this mean that there is an infinite number of primes with last digit 1? The latter question can be answered using Dirichlet's theorem without considering whether or not $\{q_1, q_2, \ldots\}$ contains an infinite subset of primes.

Let $V = \{x\colon x = 10n + 1,\ n \in N \cup \{0\}\} = \{1, 11, 21, 31, \ldots\}$. Each element of V has 1 as last digit. Since HCF(10,1) = 1, V has an infinite subset of primes (Theorem 4). Hence an infinite number of primes end in 1. It is left to the reader to say how many primes end in 2,3,4,...,9.

If you wish to know how many primes have 69 (say) as last two digits, let

$$W = \{x\colon x = 100n + 69,\ n \in N \cup \{0\}\} = \{69, 169, 269, 369, \ldots\}$$

Since HCF(100,69) = 1, W has an infinite subset of primes so an infinite number of primes end in 69.

Furthermore if p is a prime other than 2, 5 and if m is any natural number, then HCF(10^m,p) = 1. Hence

$$X = \{x\colon x = 10^m . n + p,\ n \in N \cup \{0\}\}$$

has an infinite subset of primes. m may be chosen sufficiently large that all elements of X end with all the digits of p. Thus, for example, if $p = 65537$ and if we choose m to be 5, the number of digits in p, then

$$X = \{x: x = 10^5.n + 65537,\ n \in N \cup \{0\}\} = \{65537, 165537, 265537, ..\}$$

contains an infinite subset of primes all ending in the digits 65537.

As a final mind-boggling example, let p be the largest known prime, $2^{44497} - 1$, which has 13395 digits. By Dirichlet's theorem

$$X = \{x: x = 10^{13395}.n + p,\ n \in N \cup \{0\}\}$$

contains an infinite subset, Y, of primes all of which end with the same 13395 digits as p. Of course p is the only element of Y which is known or, at present, likely to be known!

We conclude this article by noting that it contains two different kinds of results. Dirichlet's theorem is a theorem about numbers and is independent of the base in which they are written, whereas our application of the theorem is dependent upon the numbers being written in base 10.

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HIGHER-DIMENSIONAL ANALOGUES OF PYTHAGORAS' THEOREM

W.E. Olbrich, Monash University

Pythagoras' Theorem is well-known and widely used inside as well as outside mathematics. However, this contrasts markedly with the fact that the analogous result relating to the squares of areas does not seem to be generally known, even though it has some practical significance. Even less known is a corresponding result for volumes.

One can of course think of a number of reasons why this might be. Firstly, quantities such as the square of an area which has units of length to the fourth power, or the square of a volume the units of which are length to the sixth power, are not easily visualized, unlike the square of a length. Secondly, and perhaps more importantly, the results to be discussed below depend for their proof on the validity of Pythagoras' theorem and hence are not independent. Nevertheless their simplicity and the intriguing pattern of progression with increasing order of dimension does, I would argue, justify spending some time on the consideration of these results. And (may I add *sotto voce*) they're fun.

<u>The Three-Dimensional Case</u>

There are several ways of approaching this case; of these we choose a purely geometric account.

Consider a right-angled triangle *ABC* of (possibly) unequal shorter sides, and attached to these shorter sides two other right-angled triangles such that the three 90-degree angles are at the common vertex. Let us impose on the two shorter sides of the attached triangles, that do not touch the original triangle, the condition that they be equal. This is illustrated in

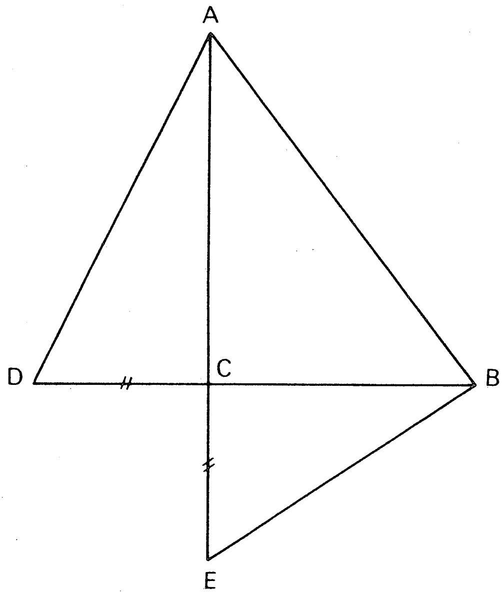

Figure 1

Figure 1. Thus if *DC* equals *CE* and the attached triangles were rotated about *AC* and *BC* respectively, out of the plane *ABC* till sides *DC* and *CE* touched, then points *D* and *E* would coincide at a point vertically above plane *ABC* at *C*. Thus by having "folded" the two attached triangles out of the original two dimensions into the third dimension the straight lines *AB*, *AD* and *BE* now lie in an oblique flat surface which together with the three right-angled triangles encloses a tetrahedron of which three faces are mutually perpendicular. This solid is illustrated in Figure 2. It may be tought of as the "corner" sliced off a box by an oblique flat cut.

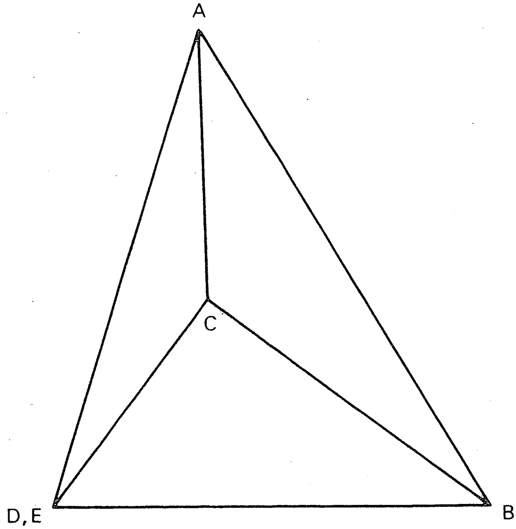

Figure 2

Note that the area of the oblique face of this solid could have been constructed while remaining in the original two dimensions, by pivoting triangle *ACD* about vertex *A* and triangle *BCE* about the vertex *B*. Let the two points, *D* and *E* coincide as shown in Figure 3 at the point *E'* the triangle *ABE'*, which will always contain three acute angles, will in fact be the triangle of the oblique face.

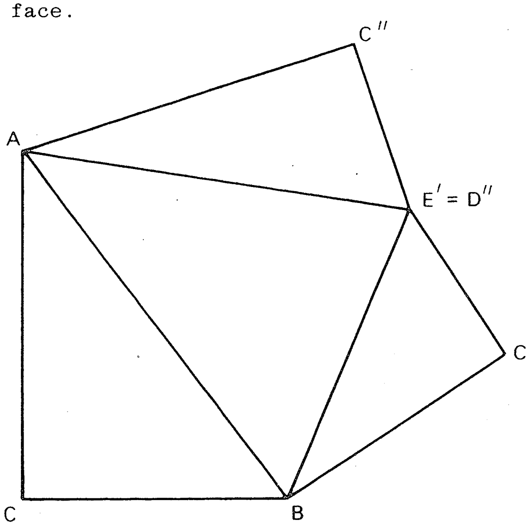

Figure 3

Let us now look at the relevant area relationship. Of the four triangles, only one has an area that is not immediately obvious from the lengths of its sides. Consider for a moment the general triangle shown in Figure 4. The perpendicular from

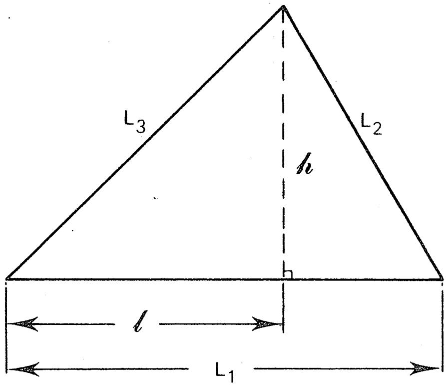

Figure 4

the side of length L_1 passing through the opposite vertex introduces two further dimensions: h and ℓ. Application of Pythagoras' theorem twice allows us to eliminate these and so compute the area from the lengths of the sides.

Let A be the area of the triangle shown in Figure 4. Then

$$A = \tfrac{1}{2}hL_1$$

and so

$$A^2 = \tfrac{1}{4}h^2L_1^2 \ .$$

But

$$h^2 + \ell^2 = L_3^2$$

and

$$h^2 + (L_1-\ell)^2 = L_2^2 .$$

Thus

$$(L_1-\ell)^2 - \ell^2 = L_2^2 - L_3^2 ,$$

so that

$$\ell = \frac{1}{2L_1}\left(L_3^2 - L_2^2 + L_1^2\right) .$$

But now

$$h^2 = L_3^2 - \ell^2 = L_3^2 - \frac{1}{4L_1^2}\left(L_3^2 - L_2^2 + L_1^2\right)^2$$

so that

$$A^2 = \tfrac{1}{4}L_1^2L_3^2 - \frac{1}{16}\left(L_3^2 - L_2^2 + L_1^2\right)^2 \quad .$$

This last expression may be rearranged to give

$$A^2 = \frac{1}{16}\left\{L_1^2\left(L_2^2 + L_3^2 - L_1^2\right) + L_2^2\left(L_3^2 + L_1^2 - L_2^2\right)\right.$$

$$\left. + L_3^2\left(L_1^2 + L_2^2 - L_3^2\right)\right\} \; .$$

This form relates directly to the tetrahedron of Figure 2. Put $AC = a$, $BC = b$, $EC = DC = c$, and let $L_1 = AB$, $L_2 = BE$, $L_3 = AD$.

This gives:

$$L_1^2 = a^2 + b^2$$

$$L_2^2 = b^2 + c^2$$

$$L_3^2 = c^2 + a^2 \quad .$$

Consequently the expression for A^2 yields:

$$A^2 = \frac{1}{16}\left\{\left(a^2 + b^2\right)2c^2 + \left(b^2 + c^2\right)2a^2 + \left(c^2 + a^2\right)2b^2\right\} \; ,$$

which simplifies to

$$A^2 = \left(\frac{ab}{2}\right)^2 + \left(\frac{bc}{2}\right)^2 + \left(\frac{ac}{2}\right)^2 \quad ,$$

the relation between the squares of the areas of the faces of the right-tetrahedron, that is analogous to the relation between the squares of the lengths of the sides of the right-angled triangle about which Pythagoras speaks.

The Four-Dimensional Case

We have proved the three-dimensional analogue of Pythagoras' Theorem by reducing a three-dimensional picture to a two-dimensional one. With rather more work, it is possible to reduce the four-dimensional analogue to a three-dimensional form and so prove it to be true.

This result falls into a pattern with the earlier cases. We have

$$L^2 = a^2 + b^2$$

in two dimensions,

$$A^2 = \left(\frac{ab}{2}\right)^2 + \left(\frac{bc}{2}\right)^2 + \left(\frac{ca}{2}\right)^2$$

in three, and

$$V^2 = \left(\frac{abc}{6}\right)^2 + \left(\frac{bcd}{6}\right)^2 + \left(\frac{cda}{6}\right)^2 + \left(\frac{dab}{6}\right)^2$$

in four, where V is now the volume of a tetrahedron (analogous with the oblique triangle of the three-dimensional case) and the terms on the right give the volumes of tetrahedra analogous to the other triangles in the earlier case.

The denominators involved in the three equations above are precisely 1!, 2! and 3!. These results apply in a sense to the simplest cases of closed figures. Thus, a triangle has the minimum number of straight sides with which it is possible to enclose a two-dimensional region; a tetrahedrom is the solid enclosed by the minimum possible number of flat surfaces and so on.

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RUBIK'S MAGIC CUBE†

The magic cube was invented in 1975 by the Hungarian architect and designer Ernö Rubik. It looks like an ordinary cube built up of $3 \times 3 \times 3$ small sub-cubes, all of equal size. The surprise is revealed when we take the cube into our hands: each face can turn wholly around its centre without the cube falling apart. It seems almost magic to devise a trick like that. Even with the cube in your hands it's hard to understand how it is put together. There is no question of magnetism, electronics or other gadgetry. It is a purely mechanical construction.

Rubik turned his Magic Cube into a hellishly difficult brainteaser by giving the surface of each face a different colour. Giving the cube a few twists (about five is usually sufficient) results in the colours getting completely mixed up and it seems virtually impossible to return the thing to its correct position if we have failed to remember exactly which sides we turned in which sequence. Each new turn will then only confuse us further and we then, of course, face the task of restoring it to the original position.

Astronomical numbers

This turns out to be a terribly difficult puzzle. It is tricky enough to turn *one* of the six faces to the right colour. But now we're in real trouble. Just about every turn upsets the correct side! Don't imagine that, by just turning the cube at

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†This article first appeared in the Netherlands journal *Pythagoras* and is reprinted here under an exchange agreement. We thank A.-M. Vandenberg for the translation.

random, chances are you will hit upon the correct position. The number of possibilities is far too great for that. There are more than 4×10^{19} different configurations. This is a 20-digit number. To give an idea of its magnitude: if we put an equally large number of standard cubes in a row, it would cover a distance of more than 60 times the distance to Proxima Centauri, which is our nearest star, apart from the sun. The same number of cubes would be enough to cover the entire area of Holland† with a layer of cubes 200 km thick!

The cube has three types of visible small blocks. When the cube is turned the *centre cubes* of the six sides don't change position, although each can rotate about its centre.

Then there are 8 blocks, each having 3 coloured sides, and always situated on a corner point of the cube, and 12 blocks with *two* coloured sides always located on an *edge* in between two corner blocks.

Solution methods

The cube's puzzle is the task of restoring a cube, which has been put into disarray, to its original position, that is, with each side having a single colour. Actually the idea is to find a *method* by which the cube can be turned back to the original position from *any* position. Experienced "cubists" achieve this within a couple of minutes and champions can even do it from any position *within half a minute*.

That is of course the result of a lot of puzzling, thinking and practising. For a beginner it is quite an achievement to restore *one* side to a single colour in a few hours. Then one realizes it is not enough to have the colours on such a side all the same but also to have the sides of the respective blocks correctly coloured. Once that has been laboriously achieved and, for example, the entire *bottom* layer is in order, there are only four wrong edge blocks remaining in the *middle row*. The central blocks of the sides are automatically correct as they, of course, never change position. So one can put this in order with some insight and perseverance. But this is usually the point where most get stuck.

The bottom and middle layers are now in order but the top row is still in disarray. At least usually. Of course one can be lucky enough to have a few blocks or even *all* blocks, by coincidence, already in the right place in the top layer. This is not impossible but the chance is rather remote; even with two rows correct there are still no less than 62 208 different configurations possible. However, the true cubist must not let chance help him. He has to have a method to correct *each* situation.

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†The area of Holland is about half that of Tasmania. (Eds.)

The systematic approach

Trying to find a way out of this maze is no longer a question of simple manipulation. The matter must be handled systematically. The idea is to devise sequences of turns which effect small changes in the top side without affecting the remainder of the cube. It is therefore usually necessary to record these sequences of turns and their effects. Few people indeed have such good memories that they can do so without pen and paper.

Another effective trick is to dismantle the cube and then put it back together correctly. In this way we can experiment to our heart's content with small sequences and thus try to crack the big puzzle by solving smaller problems first. Some people may find this objectionable. It is of course a greater achievement to find a method of solving the cube puzzle without ever dismantling it. But after weeks or even months of fruitless turning one may look at matters in a different light. In principle there is no difference whatsoever between trying out sequences on a "clean" cube (so that the outcome of such a sequence is immediately clear), and recording on paper the difference between "initial state" and "final state" with a sequence on a "dirty" cube. Dismantling the cube merely saves a lot of time and paper.

Dismantling

Of course one needs a certain amount of nerve to use this brutal way out. However, rough and careless turning of the cube causes it more damage than dismantling and restoring it (with due care) time after time. One dismantles the cube as follows: turn the top layer by one-eighth (i.e. at an angle of 45°). Take a small block situated between two corner blocks in the top layer. This can be carefully levered up a bit. You can lift it up further with a screwdriver, key or some such implement, until it pops out of its own accord. Be careful not to damage the plastic. When *one* block has come out, it is easy to dismantle the entire cube. It is useful occasionally to twist a side to release the blocks more readily.

When the cube has been entirely dismantled, we see the middle part with the three axes to which the revolving centres of the sides are fixed. Then there are two types of loose blocks: the eight *corner ones* and the twelve *edge ones*. They have interlocking indentations and protruberances which grip each other during turns. Each corner block can be set in *three* positions on each corner, each edge block in *two* positions on each edge. But for each block there is one and only one correct position. The red-white-blue corner block, for example, belongs to the corner where the red, white and blue sides of the cube meet, and the colour of a side is indicated by its central block. Thus there is only *one* way to put together the cube so that each side is made up of *one* colour. That is the initial position, the position it had when we got the cube. We will henceforth call this position START.

Reassembling

In putting the cube together again it makes sense to do it in such a way that the START position is restored. We shall explain later why. With due care, reassembling the cube is relatively easy but we must be careful with the last block. Make sure this is an *edge block* from the top layer. Return this layer to the 45° position. The block can then be tilted and slotted into its place and forcibly pushed inward. It will then click into its position.

Of course we can reassemble the cube in an immense number of other positions. It is interesting to calculate exactly in how many ways this can be done. Let us first look at the corner blocks. We can insert the first corner block into 8 places and in 3 positions in each of these places. Seven places remain for the next corner block, each again with 3 positions. For the next corner block there are still six places available, each again with 3 positions. And so on. So altogether there are $(8 \times 3) \times (7 \times 3) \times (6 \times 3) \times \ldots \times (1 \times 3) = 264\ 539\ 520$ different ways in which the corner blocks can be inserted into the cube.

The twelve edge blocks can be inserted in two ways in each place. So there are $(12 \times 2) \times (11 \times 2) \times \ldots \times (2 \times 2) \times (1 \times 2) = 1\ 961\ 990\ 553\ 600$ ways to put the edge blocks into the cube. Summarising, there are thus

$$264\ 539\ 520 \times 1\ 961\ 990\ 553\ 600 = 519\ 024\ 039\ 293\ 878\ 272\ 000$$

different ways of putting the cube together.

But you cannot reach all these positions by turning around from START, and you can't go back to START by turning from *all* of these positions. It can be shown there are twelve "paths", each with as many turning positions. Each path has

$$\frac{1}{12} \times 519\ 024\ 039\ 293\ 878\ 272\ 000 = 43\ 252\ 003\ 274\ 489\ 856\ 000$$

turning positions and this is, more particularly, also the number of positions to which the cube can be turned from START. We have seen above how unimaginably large this number is.

When is a solution method complete?

Suppose someone has found a method of always turning the cube back to START. How does he know if his method is complete? In other words, how does he know he may never face a situation he can't handle? And if we want to be mean and give him a cube deliberately put in disarray with a *wrong* path (so that he can never get to START by turning), how does he discover this? Even if he has heard of the twelve different paths, how does he know whether he really *is* on a wrong path or simply has an insufficient solution method? One should only call oneself a consummate cubist if one knows the answers to all these questions.

It is beyond the scope of this article to deal with this question in greater detail, but we will raise the veil of mystery a little. Figure 1 shows a characteristic position for each of the 12 paths: START or a position close to it. If one meets one of the eleven other positions, when turning the cube, one must clearly be on a wrong path. Of the twelve figures, the first shows the START position. In the second one corner block has been turned 1/3 to the left, in No.3 1/3 to the right, in No.4 one rib block has been reversed and in No.7 two rib blocks have been interchanged. All other figures are combinations of the above.

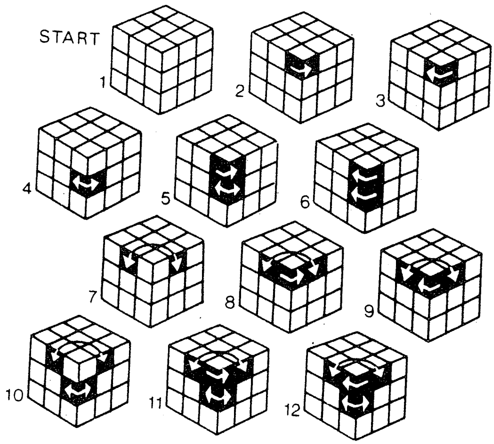


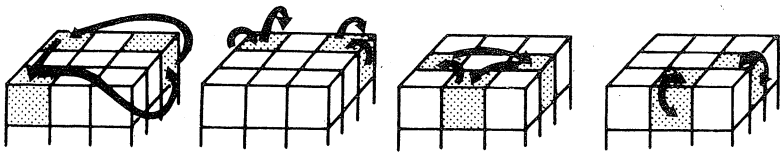
Figure 1.

None of the above eleven variations of the START position can be achieved by turning from the START position. Nor can they lead from one to the other. Only the screwdriver can solve the situation here!

Useful sequences

The cube puzzle has been the rage in many schools and no doubt many readers have solved the riddle. Daily and weekly newspapers have also published methods of solving it. But it is more fun, of course, to devise a method yourself. Many don't get further than a "nearly complete" method: they are short of just a couple of tricks in restoring the cube to correct colour.

If you have got stuck likewise, you can get further with one or more of the turning series shown in Figure 2. Sequences I to IV only affect the *top layer*; they don't touch the bottom and middle rows of the cube. And their effect in the top part is strictly limited.



II III IV

Figure 2

Sequence I: three corner blocks in the top side are pushed around anticlockwise, two of them tilting.

Recipe: $BLFL^{-1}.B^{-1}LF^{-1}L^{-1}$.

Sequence II: all blocks keep their positions but two corner blocks are tilted.

Recipe: Sequence I, then $L^{-1}B^{-1}R^{-1}B.LB^{-1}RB$.

Sequence III: three edge blocks are pushed around clockwise without tilting.

Recipe: $F^2ULR^{-1}.F^2.L^{-1}RUF^2$.

Sequence IV: all blocks keep their positions but two edge blocks are tilted.

Recipe: Sequence III, then $L^{-1}RB^{-1}LR^{-1}.U^2.L^{-1}RB^{-1}LR^{-1}$.

With these sequences, you could even design a complete solution method for the puzzle, but we would spoil your fun if we published it here. It would also be a rather cumbersome method. So we are giving you these sequences with their limited effect only to help you overcome some obstacles if you get stuck. You will then have to refine your method yourself with combinations and simplifications.

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THE PITY OF IT ALL

"... we find ourselves in the company of astute and intelligent businessmen, men of culture whose conversation may range from the paintings of Kokoschka to the intricacies of international finance, but whose minds have been closed to the stimulating challenge of mathematics. Yet here is a subject which stretches from the metaphysical extremes of pure philosophy to the hard facts of costing on the shop floor."

Albert Battersby,
Mathematics in Management, 1966.

THE RODEO

G.A. Watterson, Monash University

The owner of the rodeo charges $4 admission fee to see the show. In fact 499 people pay to go through the turnstiles. However, when the show commences, the owner counts the number of people watching - 1000! So 501 people have climbed the fence and got in for nothing.

The owner identifies as many spectators as he can, and takes each of them to court in an attempt to recover his losses. Now read on.

Owner: Your Honour, I plead that John Smith entered the rodeo without paying.

Judge (to J. Smith): What have you to say to that?

J. Smith: I did pay!

Owner: The probability that John Smith did not pay is 501/1000, so that on the balance of probabilities I should win my case.

Judge: Yes, I find that J. Smith, more likely than not, did not pay. I order him to pay the owner the $4 entrance fee. Next case!

Owner: Your Honour, I plead that Bill Brown entered the rodeo without paying.

Judge (to Bill Brown): What have you to say to that?

Etc., etc., etc.

[Shakespeare didn't write plays like that!]

So, if the owner took all 1000 spectators to court, they might each be found guilty on the balance of probabilities and the owner would collect

$$1000 \times \$4 + 499 \times \$4 = \$5996 \text{ total,}$$

including the originally-paid entrance fees.

"The absurd injustice of this suffices to show that there is something wrong somewhere. But where?" Thus asks L.J. Cohen in his book "The Probable and the Provable", Oxford University Press. However, Sir Richard Eggleston (in an article in the *Criminal Law Review*, 1980) pointed out that as the court cases proceed, the number of spectators who have paid increases, and the number who remain to be dealt with decreases. The probability of a person not having paid drops below $\frac{1}{2}$ and subsequent cases would have to be dismissed.

What would be fair to both owner and spectators? Suppose that it were possible (as it might be in damages cases) for the judge to order each of the 1000 spectators to pay an amount of $\$x$ to the owner, with $\$x$ possibly being less than the full \$4 entrance fee. Then, the outcome would be that the owner would have $\$(499 \times 4 + 1000x)$, the 499 honest spectators would have paid $\$(4 + x)$ each, and the 501 dishonest spectators would have paid $\$x$ each.

The owner should have received \$4000, so with the above scheme he loses

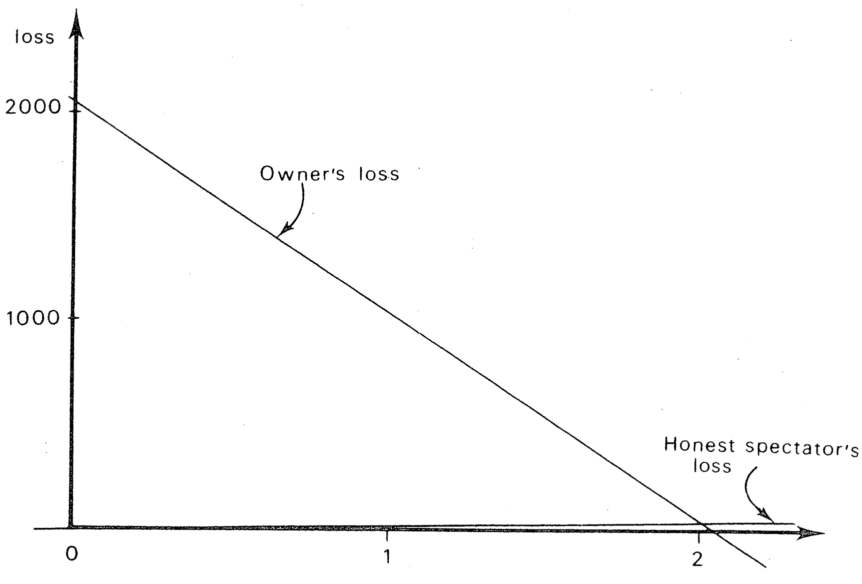
$$\$4000 - (499 \times 4 + 1000x) = \$(2004 - 1000x).$$

Of course this "loss" would actually be an additional profit if it were negative, which occurs if $x > 2{\cdot}004$. The loss to the honest spectators, over what they should have paid, is $\$x$ each. Perhaps it would be fair to minimize the *worst* loss to anybody - owner or spectator alike. This is clearly achieved when all the losses are equal, so that the judge should choose x to be the solution of

$$x = 2004 - 1000x,$$

that is,

$$x = 2004/1001 \approx 2{\cdot}00.$$



Under this scheme, the dishonest customers end up paying only half the amount they should have paid. They are being subsidised by the honest customers, who have to pay $1\frac{1}{2}$ times the original fee.

Strictly speaking, rounding x off to 2 does not give the best answer. When $x = 2$, the worst loss is to the owner, who loses $4 then. If we took $x = 2{\cdot}01$ (the next nearest cent), the worst loss is to the honest customers ($2·01) while the owner makes a profit of $6.

What do *you* think *should* happen in this problem?

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NEW H.S.C. GUIDELINES

The Victorian Higher School Certificate is to alter its method of allocating marks. Previously an elaborate scaling procedure was adopted. This certainly had its defects, but it did attempt to recognise intrinsic differences between the levels of difficulty of the various subjects, and to make allowance for these.

Draft proposals for the new scheme suggest that, in all but a few subjects with very small (< 60) enrolments, marks be awarded on a simple percentage basis. The top 7% of students in *any* subject would receive the grade of A and be given marks in the range 80 - 100. The next 18% would be awarded the grade of B and marks in the range 70 - 79, etc. (A similar scheme is proposed for N.S.W.)

In practice, because of roundoff errors, there is some variation from this simple picture. In theory, 80% of students would pass each subject - i.e. be awarded a mark of 50 or more. The actual percentage is typically between 79·5 and 80·5, with even wider variation in the case of subjects like (e.g.) Czech with very small enrolments.

This gives some grounds for concern. Some students will clearly be penalised. Chinese (*not* a small enrolment subject) will be forced to fail 20% instead of the current 4%. Other students will benefit. Accounting and Computer Science currently have failure rates of about 35%, so more of these students will pass. The effects on the three mathematical subjects are very small, although the new scheme will, to some extent, penalise candidates in Applied Mathematics.

Beyond this, there is more reason for concern. We direct attention to a letter published in *The Age* (1.2.83). The authors, Tim Brown and Aidan Sudbury, suggest that these figures will cause a flight to weaker subjects like Accountancy and Computer Science, from harder courses such as languages, music and (to a lesser extent) Applied Mathematics. They question whether this erosion of genuine skill is in the national interest.

The impact of the proposed scheme on tertiary selection is, as yet, little investigated. Much depends on how tertiary institutions respond.

REVIEW

WOMEN IN MATHEMATICS AND SCIENCE

A Kit For Use In Classrooms; Written by Jenny Pausacker

Available from VISE

Reviewed by Susan Brown

This kit consists of a series of biographies of women mathematicians and scientists, ranging from the Greek mathematician Hypatia (c. 370-415) to an Australian scientist, Suzanne Cory (b. 1942). It is divided into two sections - Women in Science and Women in Mathematics, each covering the lives and achievements of nine women. The biographies are grouped in threes, and each group is in a separate booklet for easy lesson planning. There is also a general history booklet for each of these sections, and a guide for both students and teachers. The students' guide consists of an excellent range of questions, some suitable for class discussion and others for use as a springboard for further research.

The following mathematicians are included in the kit. Hypatia (c. 370-415) was eventually killed because of her studies. Emilie du Châtelet (1706-1749) had to dress as a man before she was allowed in to the cafe frequented by mathematicians and scientists. Maria Agnesi (1718-1799) "would sleep-walk to her study and back to bed. In the morning she would find the answer to her problem waiting on her desk". The parents of Sophie Germain (1776-1831) forbade her to study and so she had to work at night. When she was later studying, unofficially, at the Ecole Polytechnique she had to adopt a male pen-name in order to get her work noticed. Ada, Countess of Lovelace (1815-1852) was the daughter of Lord Byron. She "understood the limits of computers better than many people in the twentieth century". Sonya Kovalevskaya (1850-1891) left Russia to study in Germany, but still she was not allowed to formally enrol at the university. Grace Chisholm Young (1868-1944), after great success at Cambridge, moved to Germany where she was the first woman to officially gain a doctorate. When Emmy Noether (1882-1935) went to university in Germany she was one of only two women among a thousand students. She was considered to be among the top mathematicians of her time and yet "she only had a job worthy of her talents for the last year and a half of her life". Hanna Neumann (1914-1971)† taught at the Australian National University from 1963 until her death in 1971. She is remembered not only for her contributions to mathematics, but also for her concern for students and her interest in and support for the teaching of mathematics in secondary schools. The scientists included are:

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†See *Function*, Vol.3, Part 1 for a detailed biography of Hanna Neumann.

Laura Bassi (1711-1778), Caroline Herschel (1750-1848), Mary Somerville (1780-1872), Marie Curie (1867-1934), Alice Hamilton (1869-1970), Irene Joliot-Curie (1897-1946), Maria Goeppert Mayer (1906-1972), Rosalind Franklin (1920-1958), Suzanne Cory (b.1942).

Each biography briefly describes not only the achievements of these women, but also many of the challenges, frustrations and prejudices they had to face in the course of their careers. The material aims to encourage female students to study mathematics and science by acknowledging women's contributions in these areas and by the use of such "positive role models".

I found the biographies interesting but felt that the point that women can be successful mathematicians and scientists was rather overstated. A more important criticism is that the author's construction of sentences is at times clumsy and more frequently incorrect.

Nevertheless, the idea of the kit is a good one. As the author points out in her Teacher's Guide, information about such women is not as readily available as information about their male counterparts, and so this material is a valuable reference for both staff and students.

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WOMEN ENGINEERS

We direct readers' attention to an article in *New Scientist* (30.9.82) on an American trend for women to enter the engineering profession.

Prior to 1975, there was a gradual increase in the number of female engineers, but in that year there was a very sudden increase. Whereas, earlier, some 4% of each graduating class were women, by 1980 the figure had jumped to 10% and was rising very rapidly.

The increase has been brought about in part by legislation, partly by the lure of the very high salaries that engineers receive in the U.S., partly by the fact that many engineering schools there seek in a very active way to attract women into their courses and partly by the new awareness that women themselves have achieved.

A woman with a bachelor's degree in engineering can expect a starting salary 14% above that of her counterpart with a qualification in computer science, 20% above her opposite number in mathematics, 34% above that of a woman entering the accounting profession, and 70% above one whose degree is in the area of social science.

A similar picture emerges for men, and this has boosted the total numbers of students entering engineering schools, but the increase has been most spectacular among women. It would seem that the old stereotype of female incompetence in this area is being quickly abandoned. As one brochure put it: "The best man for the job may be a woman".

LETTERS TO THE EDITOR

We ran an item "The Next Term in the Sequence" in our last issue, where we gave the finite sequence

$$1,2,4,8,16,\ldots,$$

and asked for the next term. "Most of us", we wrote, "would unhesitatingly say '32'. Here we show that a perfectly logical answer is 31." We then gave a geometric construction for the sequence

$$1,2,4,16,31,57,\ldots$$

given by $\frac{1}{24}\left(n^4 - 6n^3 + 23n^2 - 18n + 24\right)$. This generated two responses printed here.

A GENERALISED RESULT

I refer to your article "The Next Term of the Sequence" (Volume 6, Part 5) in which you posed the question "What is the next term of the sequence 1,2,4,8,16,..?", and suggested that whereas the "obvious" answer is 32, a perfectly logical" alternative is 31.

As a mathematician, I fear that the answer 32 is not "obvious", and the "perfectly logical" alternative of 31 is in fact only one of an infinite number of "alternative" answers. In fact, for the given sequence, the nth term is

$$\begin{aligned} u_n = {} & 1\frac{(n-2)(n-3)(n-4)(n-5)}{(1-2)(1-3)(1-4)(1-5)} + 2\frac{(n-1)(n-3)(n-4)(n-5)}{(2-1)(2-3)(2-4)(2-5)} \\ & + 4\frac{(n-1)(n-2)(n-4)(n-5)}{(3-1)(3-2)(3-4)(3-5)} + 8\frac{(n-1)(n-2)(n-3)(n-5)}{(4-1)(4-2)(4-3)(4-5)} \\ & + 16\frac{(n-1)(n-2)(n-3)(n-4)}{(5-1)(5-2)(5-3)(5-4)} + (n-1)(n-2)(n-3)(n-4)(n-5)f(n) \end{aligned}$$

where $f(n)$ is any function of n whatsoever.

Simple listing of any finite number of terms without specification of the law of formation is not sufficient to define a sequence, a fact frequently overlooked in elementary treatments of the topic, and by psychologists when they ask "number sequence" questions in their so-called IQ tests.

Incidentally, I once asked the following question on a TOP mathematics examination paper at this College:

"Two students A and B were asked to write down an nth term for the sequence

$$1,16,81,256,\ldots$$

and also to give the fifth term. Student *A* wrote $u_n = n^4$, and gave $u_5 = 625$. Student *B*, however, did not recognize this simple law of formation, and wrote $u_n = 10n^3 - 35n^2 + 50n - 24$, and gave $u_5 = 601$. Which student, if either, was correct, and why?"

I apologize for taking you to task over this matter, but the practice of "proving" theorems by verification in a finite number of instances is an all too commonly encountered phenomenon in mathematics classes, and one which should be stamped out.

J.A. Deakin, Shepparton College of T.A.F.E.

[*We do, by the way, entirely agree with Mr Deakin's last sentence. This was part of the point of the item. Eds.*]

GREENBURY'S PARALLELOGRAM OF REGIONS

With reference to the next term of 1,2,4,8,16,..., *Function* Vol.6 Part 5, the terms of the series listing the number of regions in a circle are readily supplied by *Greenbury's Parallelogram of Regions* as follows.

Write a horizontal series of five ones.

Write a diagonal series of an infinite number of ones.

```
1 1 1 1 1
          1
            1
              1
                1
                  .
                    .
                      .
```

Add the horizontal numbers as you do with Pascal's Triangle.

1	1	1	1	1
2	2	2	2	1
4	4	4	3	1
8	8	7	4	1
16	15	11	5	1
31	26	16	6	1
57	42	22	7	1
99	64	29	8	1
163	93	37	9	1
256	130	46	10	1
386	176	56	11	1
562	.	.	.	.
.	.	.	.	.

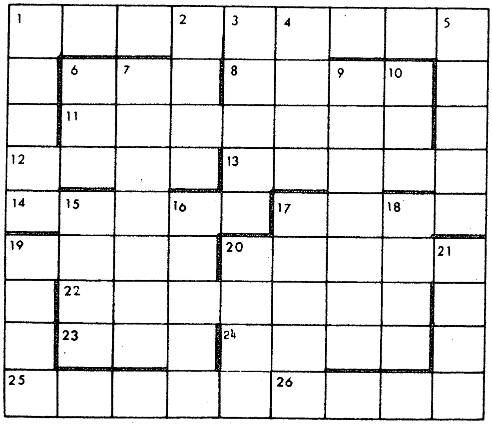
Garnet J. Greenbury,
Taringa, Queensland.

The numbers of regions are underlined.

WORDSUMS PUZZLE

I enclose a copy of a WORDSUMS puzzle I devised two years ago. This form of puzzle was invented by a problemist known as *Proton* for *The Listener*.

In each of the sums below, each different letter represents a different digit. The same letter indicates the same digit throughout the sum, but not throughout the puzzle. The digits are entered in the diagram (e.g. 1A means 1 across, 5D means 5 down, etc.).




```
     1A=1D     3D       20A
A.   GREAT + TITLE = OOGOG

      4A      2D      21D      17A
B.   FURS + SOLO = FLEE + SLED

      6A      8A        13A       11A
C.   (DIE × LOOM) - RAYON = UNROMAN

      12A      20D      19A      22A
D.   (YARN × KEEN) - GLEE = STREAKS

      16D       14A      17D
E.   ASIDE - SOLAR = HELLO

      18D     19D     23A
F.   OILY - OOZZ = ZIT

      6D      24A       25A       9D
G.   (SOA + WADS) × YOWIE = WAYLAID

      15D     26A      5D
H.   MUMS + MALT = STATE

      10D     4D       7D
I.   SUE × TEST = PUNTERS
```

Answer: p.32.

T.N. Halsall, Student,
Pimlico H.S., Townsville,
Queensland.

PROBLEM SECTION

Each issue, *Function* presents a set of problems for solution, together with solutions of earlier problems. We like to have your solutions to these problems. For many student readers, this is the best way to become involved with *Function*. As usual we begin with comments and solutions.

MORE ON PROBLEM 6.1.1.

This problem was to represent the integers 1 to 100 in terms of the digits of 1982, in their correct order and using various specified mathematical operations.

E.g. $23 = 19 + 8/2$

$51 = 1 + (\sqrt{9})! \times 8 + 2.$

We couldn't do, in the terms given: 52,53,93,94. Our counterpart from Newcastle, *School Mathematics Journal*, also carried the problem. They give the obvious

$$53 = -1 + 9 \times (8 - 2)$$

(how did we miss that?), and the beautiful

$$93 = 1 + [((\sqrt{9})!)! \div 8] + 2.$$

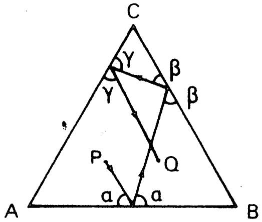
They also give versions for 94, 52, essentially equivalent to our "cheating" solution, although they didn't cheat because they allowed the symbol $\overline{)\quad}$, which we didn't.

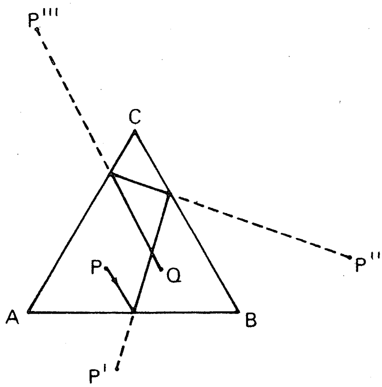
Now, we suppose, someone will ask about 1983.

MORE ON PROBLEM 6.2.5.

Two points P,Q lie inside a triangle ABC. We need the shortest path from P to Q subject to the condition that the path must hit each side of the triangle.

J. Ennis (Year 10, M.C.E.G.S.) partially solved the problem by noting that the shortest path would be that of a beam of light reflected off the sides as it travelled from P to Q. The two figures overleaf show the process.





The path is constructed as in the second diagram, by forming reflections of P (or Q) and then joining the third image P''' to Q and "folding up" the path by back reflection. The difficulty is that there is more than one way to do this. Depending on which order we adopt in reflecting P, we get six possible points P''', giving rise to six possible light paths. (Indeed, at first sight, we might expect twelve, as Q could also be reflected. However, these merely duplicate those already found.)

The problem of determining which of these six paths is the actual minimum can be determined by measurement, but it would be nice to have some way of determining the result in advance. So far, we have not been able to come up with any suggestions that work. Possibly some reader may be able to help us out.

LATE SOLUTIONS TO PROBLEMS 6.4.1, 6.4.2.

J. Ennis solved both these problems, but his letter reached us after we had gone to press.

His solution to Problem 6.4.2 differs from that published last issue. To prove that P_n, the product of n consecutive integers, is divisible by $n!$, he writes:

"If we take n consecutive integers, it follows that one of them must be divisible by n. On this line of reasoning, one of them must be divisible by $(n - 1)$, one by $(n - 2)$, etc. Since $n! = n(n - 1)(n - 2) \ldots \times 2 \times 1$ and P_n is the product of n consecutive integers, each of the factors of $n!$ divides P_n and thus $n! | P_n$."

SOLUTION TO PROBLEM 6.4.3.

This problem concerned a circle rolling inside an ellipse. If the ellipse is a long thin one, the circle can "jam" and so not roll into the ends. We asked for the proof that this occurred if

$$\frac{b^2}{a} \leqslant c \leqslant b$$

where c is the radius of the circle, and a, b are respectively the semi-major and semi-minor axes of the ellipse.

This problem was solved by J. Ennis, then in Year 10, M.C.E.G.S. He writes:

"Obviously c must be less than b, otherwise the circle will not fit inside the ellipse. To determine the lower limit for which the circle jams, we consider the largest circle for which P the point of contact coincides with $(a,0)$, i.e. the largest circle which will fit into one end of the ellipse.

For an infinitesimal distance, the perimeter of the circle then coincides with the ellipse. The x-coordinate of this is $a - \Delta$, where Δ represents a vanishingly small quantity. The equation of the circle is

$$(x - a + c)^2 + y^2 = c^2 \qquad (1)$$

and that of the ellipse is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 . \qquad (2)$$

Replace x by $a - \Delta$ in both equations, giving

$$y^2 = \Delta(2c - \Delta)$$

$$y^2 = b^2\left\{1 - \frac{(a - \Delta)^2}{a^2}\right\}$$

i.e.

$$y^2 = \frac{b^2}{a^2}\,\Delta(2a - \Delta) .$$

Thus

$$\Delta(2c - \Delta) = \frac{b^2}{a^2}\,\Delta(2a - \Delta)$$

or

$$2c - \Delta = \frac{2b^2}{a} - \frac{b^2\Delta}{a^2} .$$

But now take the limit as $\Delta \to 0$, and find

$$c = \frac{b^2}{a} .$$

This is the radius of the largest ellipse that will not jam in the circle, and thus, if the circle is to jam in the ellipse $c \geqslant \frac{b^2}{a}$, the result required."

SOLUTION TO PROBLEM 6.5.1.

This problem concerned a gambling game.

N men each toss a cent. If $N - 1$ cents agree and the Nth does not, the Nth takes all the money. If this does not happen the money jackpots. What, on average, does the eventual winner gain?

The following solution was submitted by David Shaw and the Year 11 students at Geelong West Technical School.

$$\text{Pr(win on first toss)} = \frac{2n}{2^n}$$

$$\text{Winner's expectation on first toss} = \frac{2n}{2^n}.n \qquad (\text{Prob} \times \text{stake})$$

$$\text{Pr(win on second toss)} = \left(1 - \frac{2n}{2^n}\right) . \frac{2n}{2^n} .$$

$$\text{Winner's expectation on second toss} = \left(1 - \frac{2n}{2^n}\right) \frac{2n}{2^n} . 2n .$$

$$\text{Winner's expectation on third toss} = \left(1 - \frac{2n}{2^n}\right)^2 . \frac{2n}{2^n} . 3n .$$

$$\text{Winner's overall expectation} = \frac{2n}{2^n}.n + \left(1 - \frac{2n}{2^n}\right).\frac{2n}{2^n}.2n$$

$$+ \left(1 - \frac{2n}{2^n}\right).\frac{2n}{2^n} . 3n + \ldots = \frac{2n}{2^n}.n\left\{1 + 2\left(1 - \frac{2n}{2^n}\right) + 3\left(1 - \frac{2}{2}\right)^2 + \ldots\right\}$$

$$= \frac{1}{\frac{2n}{2^n}} + \frac{1 - \frac{2n}{2^n}}{\left(\frac{2n}{2^n}\right)^2} \quad \frac{2n^2}{2^n}$$

$$= \frac{\frac{2n}{2^n} + 1 - \frac{2n}{2^n}}{\left(\frac{2n}{2^n}\right)^2} \quad \frac{2n^2}{2^n}$$

$$= \frac{1}{\left(\frac{2n}{2^n}\right)^2} \quad \frac{2n^2}{2^n}$$

$$= \left(\frac{2^{2n}}{4n^2}\right)\frac{2n^2}{2^n} = \frac{2^n}{2} = 2^{n-1}$$

They simulated the game on a computer with the following program.

```
00002  INPUT "NUMBER OF PLAYERS";P
00005  INPUT "NUMBER OF GAMES";N
00007  W=0
00010  RANDOMIZE
00015  FOR J=1 TO N
00020  K=P
00025  C=0
00030  FOR I=1 to P
00040  R=INT(RND*2)
00050  C=C+R
00060  NEXT I
00070  IF C=1 OR C=P-1 THEN 80 ELSE 90
00080  W=W+K\ GOTO 100
00090  K=K+P\ GOTO 25
00100  NEXT J
00110  PRINT "AVERAGE WIN IS";W/J
00999  END
```

We are happy to report that this confirmed the theoretical answer.

SOLUTION TO PROBLEM 6.5.2.

We asked for the smallest positive root of $x = \tan x$. (It is here correct to consider x as being measured in radians.) Mr Shaw and his class also solved this problem.

The smallest positive solution must be in third quadrant. Simple iteration does not converge. Put

$$f(x) = \tan x - x$$

$$f'(x) = \sec^2 x - 1 = \tan^2 x.$$

So, by Newton-Raphson iteration,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{\tan x_n - x_n}{\tan^2 x_n}$$

Taking $x_0 = 4{\cdot}5$ gives a solution $x = 4{\cdot}49340945\ldots$

This elegant solution carefully avoids, in its first sentence, one of the traps into which the unwary may fall. If $0 < x \leqslant \frac{\pi}{2}$ (i.e. in the first quadrant), $\tan x > x$. However, some programs or calculators, due to rounding error, produce a spurious solution near $x = 0$. On an HP35, for example, we get a "solution" $x \approx 0{\cdot}000615$.

Here now are some new problems.

PROBLEM 7.1.1.

The famous British mathematician Augustus De Morgan lived last century. He is once alleged to have said, "I was x years old in the year x^2". When was he born?

PROBLEM 7.1.2.

A triangle has sides of length 13, 14, and 15. What is the length of the altitude drawn to the 14-unit side?

PROBLEM 7.1.3.

A barrel contains n white marbles and m black. Marbles are drawn out one at a time. Prove that the probability that the black marbles are all withdrawn before the white is equal to the probability of obtaining a white marble on the first draw.

PROBLEM 7.1.4.

At a certain Australian casino, there is a game in which three dice are tossed. Before the tossing, you can bet on *one* of the numbers 1, 2, 3, 4, 5, or 6. If the three dice all show your number, you receive \$11, if two show your number you receive \$3 and if one dice shows your number you receive \$2. Of course, you lose your dollar if no dice shows your number.

(i) What is the amount of money you expect to lose per game?

(ii) What is the probability that you would have x losing games in a row before your number came up? ($x = 0,1,2,\ldots$.)

SOLUTION TO WORDSUMS PUZZLE (P.26)

[1] 2	7	9	[2] 1	[3] 3	[4] 2	8	7	[5] 1
7	[6] 9	[7] 1	0	[8] 8	3	[9] 3	[10] 6	5
9	[11] 7	5	4	3	6	2	5	3
[12] 1	2	4	0	[13] 4	2	1	3	5
[14] 3	[15] 7	2	[16] 5	9	[17] 1	4	[18] 5	6
[19] 5	9	3	3	[20] 6	6	2	6	[21] 2
5	[22] 7	8	4	3	2	6	7	4
1	[23] 1	6	8	[24] 3	2	8	9	5
[25] 1	7	3	6	0	[26] 7	3	8	5

MATHEMATICIANS IN DEMAND

It is time, or well past time, for us to encourage bright undergraduates to major in mathematics. We have too long let the dismal academic employment opportunities for Ph.D.'s during the early and middle seventies adversely affect the morale of our whole profession. Anecdotal evidence from many sources indicates that professional opportunities for those with bachelor's, master's, or Ph.D. degrees in mathematics have substantially improved in the past few years.

Recent statistical evidence also supports the view that mathematicians are increasingly in demand and that we should be very upbeat when talking about opportunities for people who successfully study mathematics.

From *Focus* (Newsletter of the Mathematical Association of America, Vol.2, No.4).

[*Australian trends are similar. Currently the employment rate in mathematics graduates is sixth best, behind only (in order): Medicine, Pharmacy, Accountancy, Law, and Engineering. Mathematics also helps, of course, in preparing for these careers. Eds.*]

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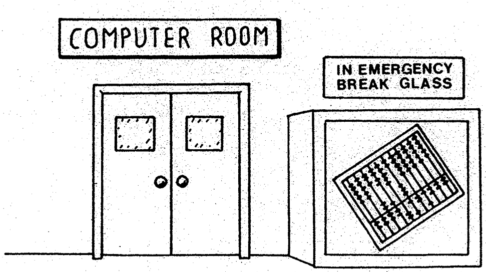
SLOW LEARNER!

"I was shocked to learn that the only ancient languages he could read were Latin, Greek and Hebrew, and that he knows almost nothing of mathematics beyond the elementary levels of the calculus of variations."

Flowers for Algernon, Daniel Keyes, 1960.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞



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