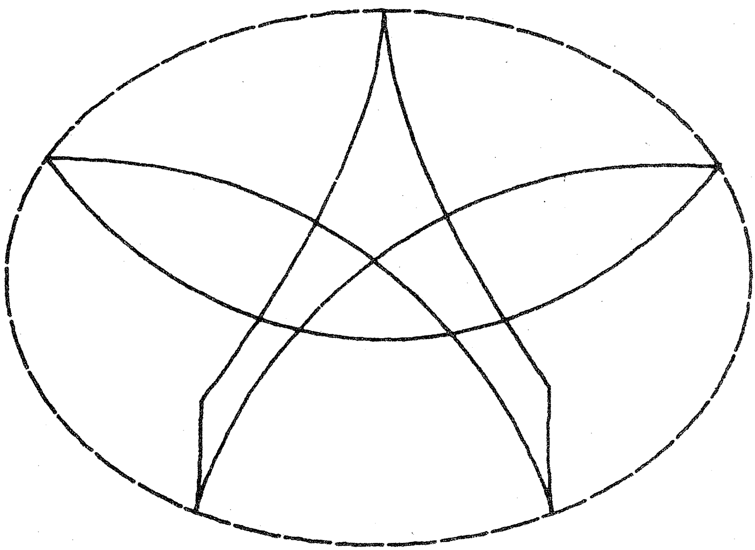
ISSN 0313 - 6825

FUNCTION

Volume 6 Part 4 August 1982



A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Reg. by Aust. Post Publ. No VBH0171

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

The magazine will be published five times a year in February, April, June, August, October. Price for five issues (including postage): \$6.00*; single issues \$1.50. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

*\$4.00 for *bona fide* secondary or tertiary students

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Registered for posting as a periodical - "Category B"

The clock paradox, or twin paradox, of special relativity is one of the more puzzling results of Einstein's theory. One of a pair of twins remains on earth, while the other takes a return trip (at close to the speed of light) to some distant object. When the traveller returns, he is younger than his twin. The result was disputed for many years, but seems to be real and it is not a self-contradiction, as it is still occasionally asserted to be. Dr Sneddon's article analyses the effect very thoroughly. He treats in detail the relation between the clocks used by A (the stay-at-home) and B (the traveller) and shows that no contradiction arises. But wouldn't there be, you might ask, a symmetric relation between A and B, so that B could see himself as staying put, while A travelled? The answer is no. A and B are *not* the only objects in the universe and this fact allows a distinction to be made. B, not A, feels the effect of the various accelerations involved. For a Science Fiction story based on the effect, read Stanisław Lem's *Return from the Stars*.

CONTENTS

The Front Cover. Andrew Mattingly	2
The Clock Paradox of Special Relativity. G.E. Sneddon	4
The Birthday Problem. G.S. Watson and G.A. Watterson	12
A "Papermobile" to Multiply Polynomials. J.-P. Declercq	17
Life in the Round I. M.A.B. Deakin	19
Letter to the Editor	26
Problem Section	27
Miscellanea	29

THE FRONT COVER

Andrew Mattingly, Monash University

The locus of a point P on the perimeter of a wheel, which rolls without slipping, along a flat plane, is known as a *cycloid* (see Figure 1). If, instead, this wheel were to roll inside a

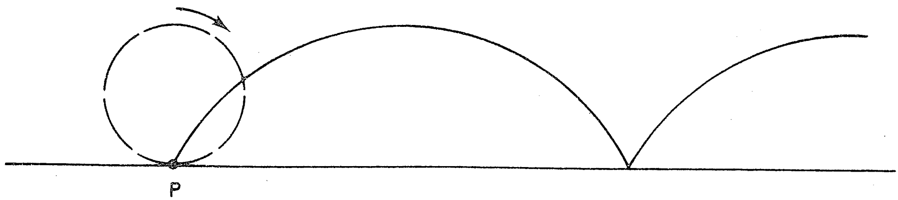


Figure 1

larger circle, the point P would then describe an *epicycloid* (see Figure 2). Extending this further, consider what happens when our wheel is rolled inside an ellipse. If the wheel is of sufficiently small radius, the point P will trace out a curve much like an epicycloid. It is possible, however, that our wheel may jam inside the ellipse (see Figure 3). The curve on the front cover is generated by computing the locus of P, where the wheel is initially jammed on the right-hand side of the ellipse with P in contact with the ellipse. The wheel then rolls along the upper arc of the ellipse until it jams on the left-hand side, whereupon it proceeds along the lower arc, and so on. To generate this figure, the radius of the wheel has been specifically chosen to give a closed, symmetric pattern.

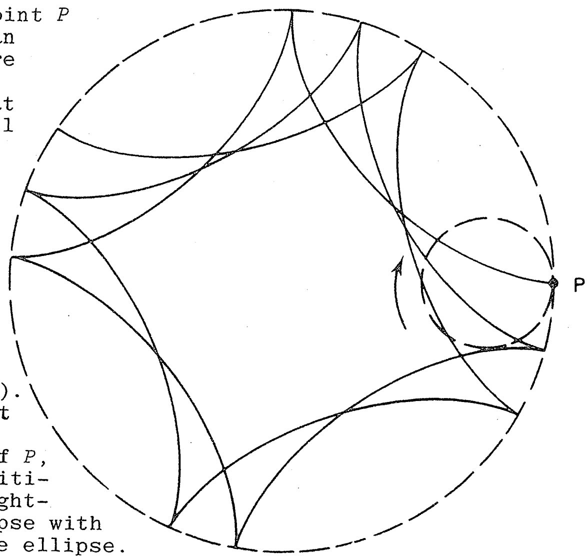


Figure 2

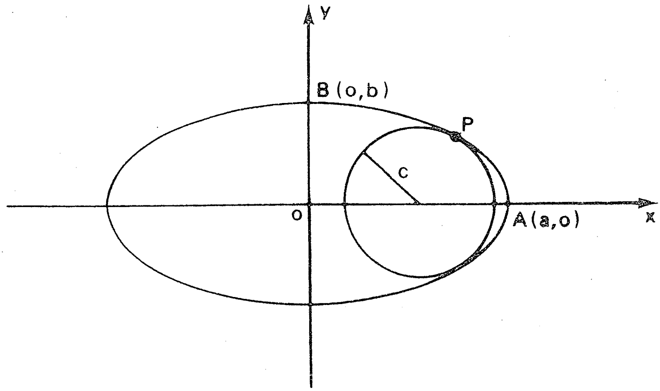


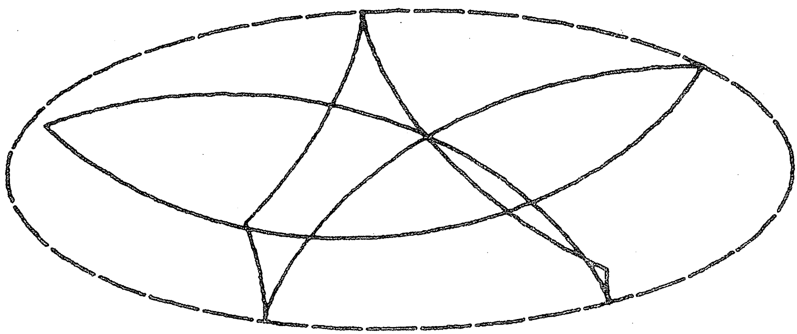
Figure 3

It should be noted that the radius of the wheel is quite crucial in determining the properties of the locus of P. An arbitrary choice of radius (provided the wheel fits inside the ellipse) will, in general, yield a curve that never closes and will eventually fill the entire area that the point P can occupy.

Exercise: The reader may verify that for an ellipse with semi-major and semi-minor axes of lengths a and b respectively, the radius c of a wheel, which will jam in this ellipse, lies in the range:

$$\frac{b^2}{a} \leqslant c \leqslant b.$$

For more patterns generated in this way, see pp. 11, 16, 25, and the inside back cover. A more complicated, non-symmetric pattern appears below.



THE CLOCK PARADOX OF SPECIAL RELATIVITY

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PNG University of Technology

The Special Theory of Relativity

The "clock paradox" is a result arising from Einstein's Special Theory of Relativity. It is surprising and, at first sight, appears to violate common sense. Indeed, some authors have gone so far as to suggest that it invalidates Einstein's theory. This, however, is not the case. Such suggestions depend on misunderstandings of the basic assumptions of the Special Theory. This article will discuss the clock paradox and its resolution. But first it will be useful to review briefly the main results of the theory.

We begin by noting that relativity theory does not distinguish between two separate entities: "space" and "time"; there is rather a single entity, known as "space-time". In other words, it is impossible to define a time coordinate unambiguously. An "event" can be considered to be a point in space-time described by coordinates (x,y,z,t) in some specified reference frame. An "observer" may be regarded as someone capable of making measurements.

It is possible to measure the relative velocity of two observers. However, *there is no such thing as the absolute velocity* of an observer. The velocity of any observer can only be given with respect to another observer. *Absolute acceleration can however be measured* (for example, by measuring the reaction force on the observer). An *inertial observer* is one who is not accelerating. Clearly, two inertial observers must be moving at constant velocity with respect to each other.

Each observer can define a set of coordinates in space-time. The time coordinate will be the "proper time" of the observer, i.e. the time that would be measured by a clock carried by the observer, or any clock that is stationary with respect to the observer.

Consider now two inertial observers, A and B, moving with speed v with respect to each other. Assume that at some time, both observers are at the same point in space. This point can be chosen to have coordinates $(0,0,0,0)$ in A's frame and also in B's frame. Furthermore x-, y- and z-axes of A can be chosen to be parallel to the respective axes of B together with the x-axes parallel to the relative velocity. The conventional diagram used to represent this situation is shown in Figure 1. Observer A is situated on the x-axis at $x = 0$ and observer B is situated on the x'-axis at $x' = 0$.

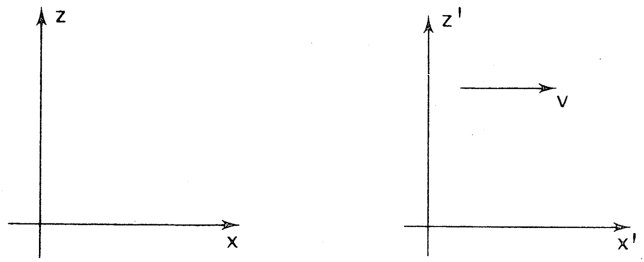

Figure 1

According to Special Relativity, if (x,y,z,t) are the coordinates of an event as measured in A's frame, the coordinates (x',y',z',t') of this same event measured in B's frame are given by the Lorentz Transformations:

$$x' = \frac{1}{\sqrt{1 - v^2}}\, x - \frac{v}{\sqrt{1 - v^2}}\, t \tag{1a}$$

$$y' = y \tag{1b}$$

$$z' = z \tag{1c}$$

$$t' = -\frac{v}{\sqrt{1 - v^2}}\, x + \frac{1}{\sqrt{1 - v^2}}\, t. \tag{1d}$$

The inverse of this transformation is given by

$$x = \frac{1}{\sqrt{1 - v^2}}\, x' + \frac{v}{\sqrt{1 - v^2}}\, t' \tag{2a}$$

$$y = y' \tag{2b}$$

$$z = z' \tag{2c}$$

$$t = \frac{v}{\sqrt{1 - v^2}}\, x' + \frac{1}{\sqrt{1 - v^2}}\, t'. \tag{2d}$$

In these equations, units have been chosen so that c, the speed of light, is equal to 1. This has the effect of simplifying the equations. For instance, if $c \neq 1$, Equation (1a) should be written as

$$x' = \frac{1}{\sqrt{1 - v^2/c^2}}\,x - \frac{v/c}{\sqrt{1 - v^2/c^2}}\,t.$$

This transformation can be compared with the Galilean Transformation used in Newtonian theory:

$$x' = x - vt \qquad (3a)$$

$$y' = y, \quad z' = z \qquad (3b,c)$$

$$t' = t. \qquad (3d)$$

Equation (3d) implies the existence of an absolute time that will be valid for all inertial observers. However, in Special Relativity, equation (1d) implies that no such absolute time can be defined.

Time Dilation

Two of the phenomena of Special Relativity can be derived directly from the Lorentz Transformation. These are the Lorenz contraction and time dilation. The Lorentz contraction refers to the apparent shortening of a rod moving with respect to an observer. Time dilation refers to the apparent slow running of clocks that are moving with respect to an observer. This latter effect can be described as follows.

Consider the points where $x' = 0$. Equation (1a) implies that $x = vt$ (as expected), so Equation (1d) implies that

$$t' = t\sqrt{1 - v^2}. \qquad (4)$$

Since $\sqrt{1 - v^2} \leqslant 1$, the time measured by B is less than the time measured by A for the same event.

It is this effect that is referred to as time dilation. It means that, according to the observer A, B's clock is running slow. The opposite conclusion results from a consideration of the situation at points where $x = 0$. From Equation (2a), $x' = -vt'$, so Equation (2d) implies that

$$t = t'\sqrt{1 - v^2}. \qquad (5)$$

In other words, according to B, A's clock is running slow.

It should be stated that this does not in itself represent a contradiction. It simply means that the two observers A and B each believe the other's clock to be slow. Despite appearances, Equations (4) and (5) are not contradictory. They refer to measurements made at different events. Equation (4) refers to points where $x' = 0$ and Equation (5) to points where $x = 0$. These equations could be rewritten as

$$t_1' = t_1\sqrt{1 - v^2} \qquad (6a)$$

$$t_2 = t_2'\sqrt{1 - v^2} , \qquad (6b)$$

where event 1 is at $x' = 0$ and event 2 is at $x = 0$. There is no contradiction here.

The Clock Paradox

The only way in which a contradiction could occur between Equations (4) and (5) would be the situation in which the two observers meet a second time. That is, if at some later time there is an event with $x = x' = 0$. Only at such an event would $t_1 = t_2$ and $t_1' = t_2'$ in Equations (6). It is this possibility that is the essence of the clock paradox.

It is usually given in the following form.

Consider the situation where observer A remains on earth while observer B undertakes a space journey at speed v. After a time T has elapsed (as measured by observer A on Earth) B turns around and returns to Earth, again with speed v. Once A and B are reunited, it is possible to make a direct comparison of the readings on their respective clocks. On the outward journey, Equation (4) implies that according to A, B's clock is running slow. On the return journey the relative speed is v and Equation (4) implies once again that B's clock is running slow. Both observers will age according to the time measured on their own respective clocks. Therefore, on his return to Earth, B will actually have aged less than A.

This extraordinary conclusion is often emphasised by assuming that A and B are twins. It is possible to arrange matters so that when B returns he will be several years younger than his twin on Earth. It is for this reason that the "paradox" is also called the "Twin Paradox".

Such a possibility is of course far beyond our usual experience. It is not however, a paradox in itself. The apparent contradiction appears when the situation is analysed from B's point of view. Since there is no such thing as absolute motion, B can always be regarded as the observer who is at rest and A can be considered to be the traveller. In that case, an argument similar to that given above shows that, according to observer B, A's clock runs slow on both the outward and return journeys. The argument thus says that, according to B, observer A will not have aged by as much at the completion of the journey and a contradiction arises. Each observer believes that the other observer's clock is slow and so predict that the other observer will not have aged as much as he has at the end of the journey.

This argument must therefore be fallacious.

Resolution of the Clock Paradox

The immediate resolution of this paradox is to note that in fact, the two observers are not equivalent. In order for B to turn round (which is essential if the two clocks are to be brought together for a further comparison) he must undergo an acceleration for a part of the journey. It is possible to detect an absolute acceleration, and so this acceleration will distinguish between the two observers, since A undergoes no such acceleration.

However, this still does not indicate precisely where the preceding argument breaks down. Indeed, the duration of the acceleration could, in principle, be made arbitrarily small and so it might be expected that any effect on measured time resulting from this acceleration could also be made arbitrarily small. In this case, since B is an inertial observer for most of the journey, it might be expected that the argument that results in the paradox would still be valid, in spite of the acceleration.

The situation may be analysed by investigating the time measured by each observer at each stage of the journey. It is convenient to represent these events on a space-time diagram. If x and t axes drawn at right angles are chosen to represent the lines $t = 0$ and $x = 0$ respectively in A's frame, then the journey may be represented as in Figure 2. In this diagram, lines parallel to the x-axis represent t = const, and lines parallel to the t-axis represent x = const. The line E_0E_1 represents the outward journey and E_1E_2 the return journey. Each leg of the journey takes a time T according to A's frame of reference. The period during which B is accelerated is taken to be arbitrarily small, and is represented by event E_1. (Note that the slope of E_0E_1 is $1/\sqrt{1 - v^2}$ which being greater than one, indicates that B is

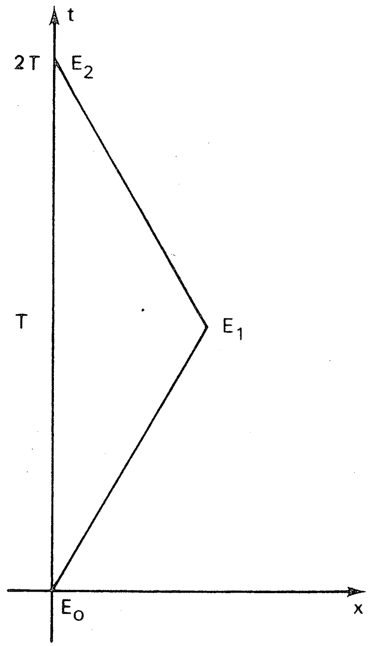

Figure 2

travelling at less than the speed of light, which is one.)

From Equation (1d), the lines of t' = constant are given by the equations:

$$-\frac{v}{\sqrt{1-v^2}}\,x + \frac{1}{\sqrt{1-v^2}}\,t = \text{const} \tag{7}$$

$$\text{i.e.} \quad t = vx + \text{const.}$$

It is helpful to draw the lines t' = const on the space-time diagram (Figure 3a). These lines are not the same as the lines t = const, which indicates that B's time can be expected to be quite different from A's time. In fact, the slope of these lines is v, which is less than one.

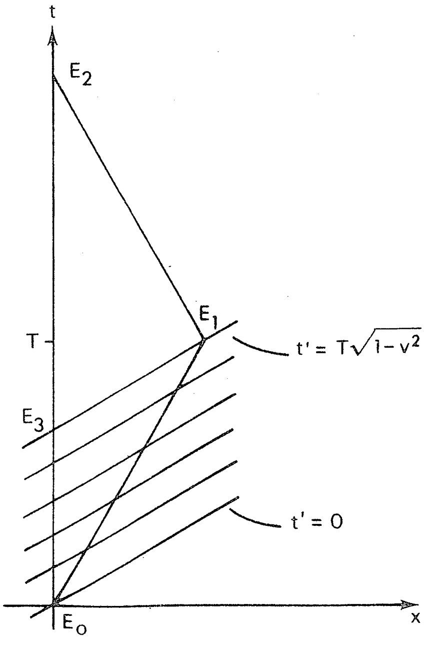

Figure 3(a)

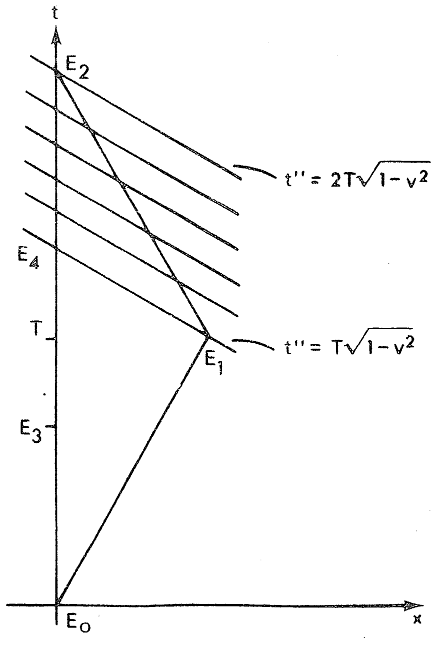

Figure 3(b)

From this diagram, it can be seen that both observers regard the other's clock as being slow. For instance, if the time measured at E_1 is represented by t_1 (if measured by A) or t_1' if measured by B) then

$$t_1 = T \tag{8a}$$

$$t_1' = T\sqrt{1-v^2}\ , \tag{8b}$$

and so B's clock appears to be slower than A's at this point. However, at event E_3 where $t' = T\sqrt{1 - v^2}$ and $x = 0$,

$$t'_3 = T\sqrt{1 - v^2} \ , \tag{9a}$$

and Equation (1d) implies that

$$t_3 = t'_3\sqrt{1 - v^2} = T(1 - v^2) \ , \tag{9b}$$

and so A's clock appears to be slower than B's at this point. These results are not contradictory, since they relate to readings on clocks at different events (i.e. E_1 and E_3).

At event E_1, observer B changes to a reference frame that is once again moving at speed v relative to A's frame, though now in the opposite direction. If quantities measured in the new frame are devoted by a double dash, then the lines $t'' = \text{const}$ are given by

$$t = -vx + \text{const}. \tag{10}$$

The lines $t'' = \text{const}$ appear as in Figure (3b). At event E_1, $t'' = T\sqrt{1 - v^2}$, since this is the value measured by B's clock at this point. At event E_4 (corresponding to $t'' = T\sqrt{1 - v^2}$ and $x = 0$) the time measured by A will be $t_4 = T(1 + v^2)$.

For the return journey, once again, both observers will believe the other's clock to be running slow. A will see B's clock go from $T\sqrt{1 - v^2}$ to $2T\sqrt{1 - v^2}$ while his goes from T to $2T$. B will observe A's clock go from $T(1 + v^2)$ to $2T$ while his own goes from $T\sqrt{1 - v^2}$ to $2T\sqrt{1 - v^2}$.

Thus, throughout the journey, A observes B's clock to be slow. Also throughout the journey, B observes A's clock to be slow except at the point where B accelerates or changes frame. At this point, B changes his refererence frame in such a way that, on the line $x = 0$, the point corresponding to $t' = T\sqrt{1-v^2}$ changes from event E_3 to event E_4. According to B, A's clock moves suddenly from $T(1 - v^2)$ to $T(1 + v^2)$ and observer A will age accordingly. It is this sudden change in A's clock (for no change in B's clock) that enables B to predict that when the observers meet at E_2, he will not have aged as much as A. Since this is the same prediction that A will make, there is no contradiction.

Thus, the paradox may be resolved by consideration of what happens near event E_1, which is the only region in which B is not an inertial observer. It should also be pointed out that the "jump" in A's time is not a real effect. It arises purely because B has redefined his coordinate system. Note that B will not actually see observer A suddenly age by $2Tv^2$. Any such observation would have to rely on light signals passing between the two observers. An analysis of this can be made. For example, Appendix B of Resnick's *Introduction to Relativity* (Wiley, 1968) discusses the matter and shows that the sudden jump in A's age is not actually observed by B. However the overall conclusion remains unchanged. That is, at event E_2 when the observers again meet, A's clock shows that time $2T$ has elapsed, while B has measured a time of $2T\sqrt{1 - v^2}$ only. Observer B will return to find he is younger than observer A.

A direct experimental verification of these conclusions was undertaken in 1971 by two researchers, Hafele and Keating, who used extremely accurate clocks travelling around the world in high speed aircraft. To the limits of accuracy of their equipment, they verified that the effect does occur and that the difference is as predicted.

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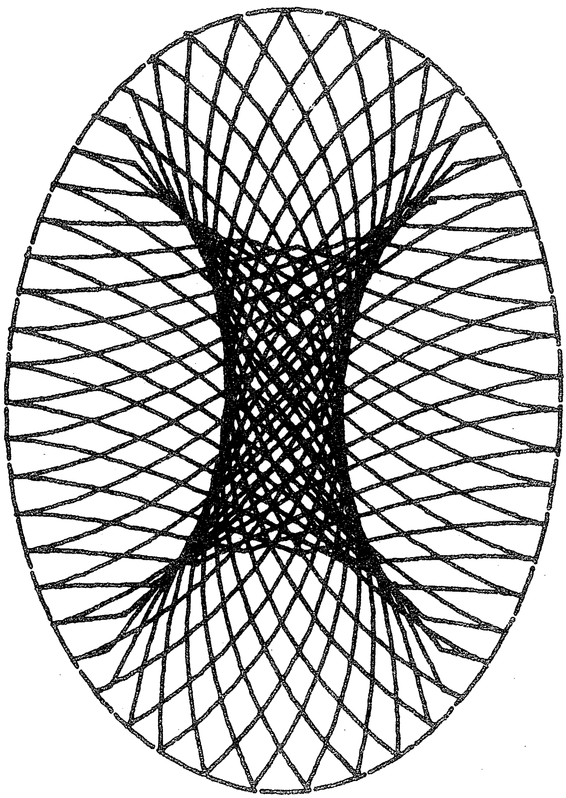

THE BIRTHDAY PROBLEM

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What is the probability that, in a set of r people, at least two people have birthdays on the same day of the year?

This "birthday problem" is an intriguing one, because most people think that you would need a lot of people, say 100 or so, before there was much chance of two of them having birthdays on the same day. However, it turns out that far fewer people are needed, as we shall see.

We will assume that each person is equally likely to have any one day as any other day (out of the 365) as his/her birthday. This may not be strictly correct. If, for instance, summer birthdays were more likely than winter ones, then there would be a *greater* chance of two people having the same birthday than the answer we will find. Also, we are ignoring the leap-year complication. Moreover we will assume that the group of people we deal with is randomly chosen from the population as far as birthdays go. It could be the members of a mathematics clan, two opposing football teams, two opposing cricket teams, etc. The argument would be wildly wrong if we had an Astrological Society of Sagittarians, or of Leos, etc.

Imagine the r people numbered off: $1, 2, 3, \ldots, r$. Let us then make a list of birthdates in that order. The first person can have any of 365 days as his birthday. So can the second, the third, etc. The total number of possible birthday lists is

$$365 \times 365 \times \ldots \times 365 = 365^r.$$

Our list is equally likely to be any one of these 365^r lists. Let us now calculate the number of lists in which *no* two people have the same birthday. Again, the first person can have any of 365 days for his birthday. But now the second person can have only 364 possible days for his birthday, excluding the first person's birthday. The third person can have 363 possible days, the fourth person 362 possible days, etc. Thus the number of lists with r different birthdays is

$$365 \times 364 \times 363 \times \ldots \times (365 - r + 1).$$

The probability that all r birthdays are *different* is then

$$Q = \frac{365 \times 364 \times 363 \times \ldots \times (365 - r + 1)}{365 \times 365 \times 365 \times \ldots\ldots\ldots \times 365}$$

$$= \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{(365 - r + 1)}{365}$$

$$= 1 \times \left(1 - \frac{1}{365}\right) \times \left(1 - \frac{2}{365}\right) \times \ldots \times \left(1 - \frac{r-1}{365}\right) ; \quad (1)$$

and the probability that at least two people have birthdays on the same day is

$$P = 1 - Q = 1 - \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \ldots \left(1 - \frac{r-1}{365}\right) . \quad (2)$$

This is the answer we seek.

If you have a calculator or computer, it is easy for you to calculate P for various values of r, by multiplying the terms together. But there is a simple approximation which yields answers correct to 5 decimal places without multiplying so many terms together. The approximation, which we shall derive later, is

$$P \approx 1 - 1{\cdot}649286257\left(\frac{365}{365{\cdot}5 - r}\right)^{365{\cdot}5-r} e^{-r} \quad (3)$$

where $e \approx 2{\cdot}71828\ 18285$.

In Table 1, we show the exact and approximate values for P to 6 decimal places, calculated from (2) and (3) respectively.

TABLE 1

r	P exact	P approximate
1	0	0
2	0·002740	0·002740
3	0·008204	0·008205
4	0·016356	0·016357
5	0·027136	0·027137
⋮	⋮	⋮
10	0·116948	0·116951
⋮	⋮	⋮
20	0·411438	0·411442
21	0·443688	0·443692
22	0·475695	0·475699
23	0·507297	0·507301
⋮	⋮	⋮
40	0·891232	0·891233
41	0·903152	0·903153
⋮	⋮	⋮
57	0·990122	0·990123
⋮	⋮	⋮

Notice that with only 23 people (about two cricket teams), you can be more than 50% sure that at least two will have the same day of the year as their birthday; with 41 people you can be

90% sure and with 57 people you can be 99% sure. As we are ignoring leap-years, with 366 people you are 100% sure, because there are not 366 different days available then.

To arrive at our somewhat mysterious approximation in equation (3) to the exact expression in equation (2), we first save ourselves a bit of writing by letting n stand for the number of days in the year. Of course $n = 365$ in our above discussion, but perhaps we should have taken $n = 366$ to be on the safe side, or, if we were on some other planet, then n may take a different value again. Equation (1) says

$$Q = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{r-1}{n}\right) .$$

Taking the natural logarithm of both sides yields

$$\log_e Q = \log_e\left(1 - \frac{1}{n}\right) + \log_e\left(1 - \frac{2}{n}\right) + \ldots + \log_e\left(1 - \frac{r-1}{n}\right) \quad (4)$$

because the logarithm of a product is the sum of the individual logarithms.

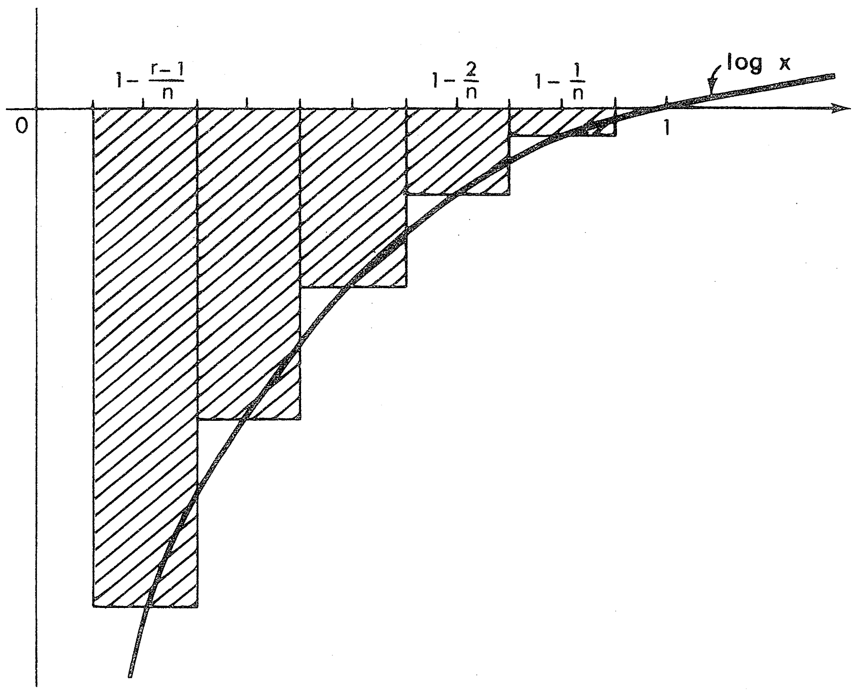

Figure 1

In Figure 1 we have represented the logarithms on the right of (4) by the heights (or, more correctly, by the depths, as logarithms of numbers less than 1 are negative) of certain rectangles. The bases of the rectangles are each $\frac{1}{n}$ long, so that the shaded area represents $\frac{1}{n}$ times the right hand side of (4). We have to treat the area as being negative, so we call it the "signed area":

$$\log_e Q = n \times \text{signed area}.$$

But the shaded area can also be approximated by the area above the smooth curve for $\log_e x$ between $x = 1 - \frac{r-1}{n} - \frac{1}{2n} = a$, say, and $x = 1 - \frac{1}{n} + \frac{1}{2n} = b$. Hence

$$\log_e Q \approx n \int_a^b \log_e x \, dx.$$

You can easily check, by differentiating $x \log_e x - x$, that its derivative is $\log_e x$, so that $x \log_e x - x$ is an antiderivative of $\log_e x$, and so

$$\log_e Q \approx n \Big[x \log_e x - x \Big]_a^b$$

$$= n[b \log_e b - b - a \log_e a + a] .$$

Thus

$$Q \approx e^{n[b \log_e b - b - a \log_e a + a]}$$

$$= b^{nb} a^{-na} e^{n(a-b)} ,$$

because $e^{\log_e b} = b$, etc. Substituting the values for a and b yields

$$Q \approx \left(1 - \frac{1}{2n}\right)^{(n-\frac{1}{2})} \left(1 - \frac{r - \frac{1}{2}}{n}\right)^{-(n-r+\frac{1}{2})} e^{(1-r)}$$

It is then a simple matter to substitute $n = 365$ and evaluate the terms which do not involve r. This leads to the approximation (3) for $P = 1 - Q$. We have not seen this approximation in print before, but it may be known to others.

The birthday problem is just one of many "Coincidence" problems, some of which are just fun and some of which are scientifically important. Formulate for yourself the problem of r people entering a lift in the ground floor of a tall building with n floors. The coincidence might be "no two people get out at the same floor". Think of the situations when it would be *incorrect* to assume that formula (2) is applicable. How tall does the building have to be before the approximation (3) would be any good?

If you get this straight you are beginning to see how to *apply* mathematics to real world situations.

First: you must be clearheaded about the real situation.

Second: you must make a slightly simplified mathematical model which captures the essence of the situation.

Third: you must solve the mathematical problem.

Fourth: you must often approximate this solution.

Fifth: you must examine these solutions in the light of the real problem.

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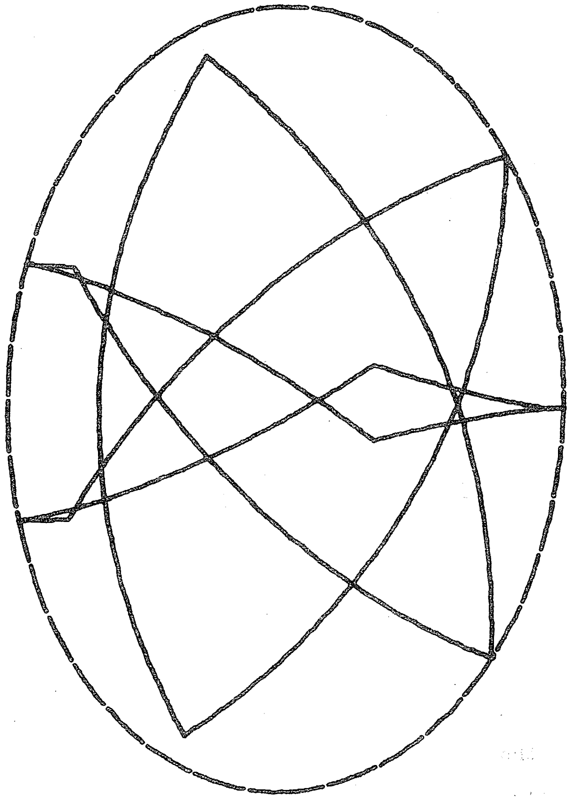

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.... AND SIX DIMENSIONS TO PUT THEM IN!

"[Uncle Scrooge McDuck] owns 3 cubic acres of cash ..."

Newsweek 5/7/'82 (p.45).

A "PAPERMOBILE" TO MULTIPLY POLYNOMIALS†

Jean-Pierre Declercq,
Comines, Belgium

If I asked you to multiply $P(x)$ by $Q(x)$, where

$$P(x) = 5x^{10} - 2x^9 + 8x^8 - 6x^5 + x^4 + x^3 - 2x + 1,$$

and

$$Q(x) = 6x^7 + 2x^4 - 9x^2 + 2x + 8,$$

you could certainly tell me that the result would be a 17th degree polynomial with 18 terms. If you reflected a little longer you could say that we must find

$$30x^{17} + \ldots + 8.$$

How can we find the other 16 coefficients? Of course, you could write the whole multiplication out, but this leads to some very onerous calculations. However, we can organise the computation so that the work becomes a mere mechanical process. Here is how I proceed.

(a) On a sheet of paper, I enter, at regular intervals, the 11 coefficients of the polynomial $P(x)$ in decreasing order of the powers of x. This gives

x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x^1	x^0
5	-2	8	0	0	-6	1	1	0	-2	1

Note that I put in *all* coefficients, inserting zeros where necessary.

(b) On a small strip of paper, I next enter the coefficients of the polynomial $Q(x)$ in the same way except that here the powers of x increase. So I get this

---

†This article is a translation from the French. It first appeared in the Belgian journal *Math-Jeunes* and is reproduced here under an exchange agreement between *Function* and *Math-Jeunes*, and with the kind permission of Professor Declercq.

8	2	-9	0	2	0	0	6
x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7

My machine is now ready for use.

Step 1. To find the coefficient of x^{17}, I place the strip at the left of the sheet of paper so that the right-hand coefficient on the strip lies just under the left-hand coefficient on the sheet, like this

$$\begin{array}{cccc} x^{10} & x^9 & x^8 & \dots \\ 5 & -2 & 8 & \end{array}$$

0	0	6
x^5	x^6	x^7

Then I multiply the adjacent numbers to get $5 \times 6 = 30$. This gives the coefficient of x^{17}.

Step 2. Next I slide the strip one space to the right and work out the sum of products of adjacent coefficients. We have

$$\begin{array}{ccccccl} & 5 & -2 & 8 & \dots & & \text{on the sheet} \\ \dots & 0 & 0 & 6 & & & \text{on the strip} \end{array}$$

We calculate $5 \times 0 + (-2) \times 6 = -12$ = the coefficient of x^{16}.

Step 3. I continue in this way. At each step, I calculate the sum of products of adjacent coefficients, until the extreme left of the strip lies underneath the extreme right-hand digit on the sheet. This final calculation gives the coefficient of x^0 and the product polynomial is determined.

At one stage, I will have

$$\begin{array}{cccccccccccl} 5 & -2 & 8 & 0 & 0 & -6 & 1 & 1 & 0 & -2 & 1 & \text{(sheet)} \\ & & & 8 & 2 & -9 & 0 & 2 & 0 & 0 & 6 & \text{(strip)} \end{array}$$

The computation goes: $0 + 0 + 54 + 0 + 2 + 0 + 0 + 6 = 62$. Can you tell me the degree of the term in question? I bet on 7. Am I right? Have you spotted my method? Try to complete the calculation yourself. My answer is on the inside back cover.

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LIFE IN THE ROUND I

M.A.B. Deakin, Monash University

Much of our world of experience conforms to euclidean plane geometry. With this geometry, the following property (called the parallel postulate) is true of points and (straight) lines:

> *If ℓ is a line and P is a point not lying on ℓ, then there is exactly one line passing through P and parallel to ℓ.*

But there are other useful geometries which do not obey this postulate - for example, geometry on the surface of a sphere, which is described in this article. The practical uses of spherical geometry rest on the fact that the earth we live on is (to an excellent approximation) a sphere, and the sky surrounding us is an apparent sphere (of arbitrarily large radius). As you read this article, you may find it helpful to cut up an orange and peel the pieces to verify some of the claims being made.

On the surface of the sphere, we have a two-dimensional space that is curved. No straight lines can be drawn upon it, so our first task is to find some analogue of the straight line. We may approach this experimentally by taking a string and stretching it between two points on a globe or a soccer ball. Such a string adopts the shortest route between the two points.

We may also notice that the string takes up the form of a circular arc. Any straight cut through a spherical surface produces a circle (test this by slicing an orange). Such circles may be small (radii only just exceeding zero, even) or large. But there is a maximum size, attained by circles whose radius equals the radius of the sphere itself. Such circles are called *great circles*. (All others are called *small circles*.)

On the earth's surface, meridians (lines of longitude) form great circles and the equator is also a great circle. Lines of latitude (other than the equator) are small circles.

Figure 1 shows this. N and S are the north and south poles respectively. NXS is the Greenwich meridian and P is an arbitrary point (drawn by convention in the northern hemisphere) whose meridian meets the equator at Y. O is the centre of the earth. Then $\measuredangle XOY$ (known as ϕ) is the longitude and $\measuredangle POY$ (known as λ) is the latitude of P. The radius of the circle of latitude through P is $R \cos \lambda$, where R is the radius of the earth.

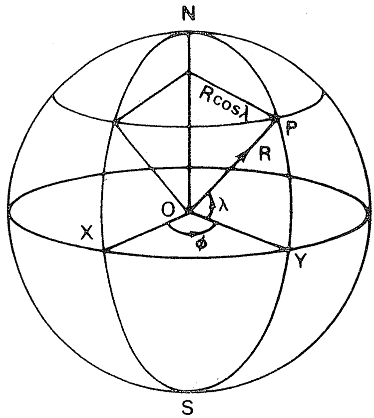

Figure 1

Now return to our string stretched between two points on a sphere. It lies along a circular arc. Let the radius of this be r, where $r \leqslant R$ (R now standing for the radius of whatever sphere is under consideration).

Now the larger the radius of a circle, the straighter an arc will appear. This suggests that if we try to find the shortest distance between two points on a spherical surface, we use the *straightest* path available, i.e. the one that curves least. This is the great circular arc joining the two points. Great circular arcs do in fact provide the shortest distances between pairs of points on a sphere. This property underlies the practice of great circle navigation (*Function*, Vol.4, Part 4). For such arcs $r = R$.

In our spherical geometry, therefore, we use great circles in place of the euclidean straight lines.

Notice first of all that Euclid's parallel postulate now does not hold. Any two great circles, unless they happen to coincide) meet in precisely two points (again you can check this by cutting an orange). So our geometry is non-euclidean.

Second, between any two points on a sphere, there are two great circular arcs. The smaller of these measures the distance between the points. See Figure 2. Here Q,P are joined by the great circular arc drawn, but also by the continuation of this "around the back" of the sphere. By convention, the smaller of these two distances is the distance QP. (When the points are diametrically opposite, infinitely many great circle arcs join them, but the distance is the same on all of these.)

Third, while the length of the great circle arc QP could be measured in terms of ordinary units of length, it is convenient not to do this, but to use the angle QOP as the "length" of the arc QP. In any case, these two measures are closely related for $QP = R\measuredangle QOP$, where $\measuredangle QOP$ is measured in *radians*. (Remember $\frac{\pi}{2}$ radians make one right angle.)

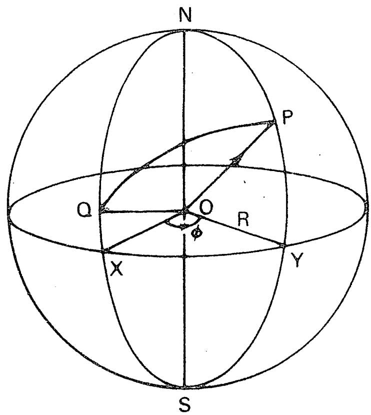

Figure 2

We need to measure also the angles at which great circular arcs meet. This may be done in two ways. The great circular arcs QN, PN meet at an angle QNP which (in the case of the earth) could be measured locally by an observer standing at N (the north pole) to whom the arcs would be seen as straight lines (to a very good approximation). Something similar could be done with a piece of orange peel, by pressing the relevant "corner" to a protractor.

The other way is to take this angle as being measured at the centre of the sphere as $\measuredangle XOY$. The two methods give the same result. Can you see why?

We begin now to consider the geometry of the sphere. First, those aspects that are unfamiliar; next those that are more acceptable to us Flatlanders; finally, the beauty of symmetry achieved through the power of Mathematics.

Two lines in a plane either meet exactly once, or are parallel. In neither case do they enclose an area. Two great circles, by contrast, necessarily meet twice to divide the sphere into eight regions called *lunes*. The region $NQXSYPN$ (Figure 2) is a lune (actually one of two so described - the other is the remainder of the sphere's surface, which, by convention, we neglect).

The area of the lune may be found in terms of the angle at N, i.e. $\measuredangle N$ or $\measuredangle XOY$. (Again, there are actually two such angles; by convention, the smaller is implied.)

The total area of the spherical surface is $4\pi R^2$. (We do not prove this here - it is normally demonstrated using calculus.) The proportion of this contained in the lune formed at N is merely $(\measuredangle XOY)/2\pi$ or $N/2\pi$.

Thus: *area of lune* $= \left(\frac{N}{2\pi}\right)4\pi R^2 = 2R^2N$.

We may proceed now from lunes (two-sided figures) to the more interesting *spherical triangles* (three-sided figures). These have some properties in common with plane triangles, but in other ways, they behave differently. In Figure 2, the points N, P, Q and the great circle arcs joining them describe a spherical triangle - actually several, but again we can use a convention to restrict this number. Our convention is that each side is to be *less than a semicircle*. This now limits our triangle to the "obvious" one of Figure 2. (Can you, using orange peel or otherwise, find the others? There are fifteen possibilities, although some look rather strange.)

We will first find the sum of the angles in the spherical triangle NPQ (which is perfectly typical - N's being at the top is pure convention on a perfect sphere). Call the angles N, P, Q. N has an antipodal point (a diametrically opposite point) S and P, Q have antipodal points P', Q' (not diagrammed) respectively.

But the spherical triangle NPQ forms part of the lune $NPSQ$ and also part of the lunes $PNP'Q$ and $QPQ'N$ and so:

$$\begin{aligned} \text{Area of } \Delta NPQ + \text{Area of } \Delta SPQ &= 2R^2 N \\ \text{Area of } \Delta NPQ + \text{Area of } \Delta NP'Q &= 2R^2 P \\ \text{Area of } \Delta NPQ + \text{Area of } \Delta NPQ' &= 2R^2 Q \end{aligned} \tag{1}$$

But now we have

$$\begin{aligned} &\text{Area of } \Delta NPQ + \text{Area of } \Delta SPQ + \text{Area of } \Delta NP'Q + \text{Area of } \Delta NPQ' \\ &\quad = \text{Area of one hemisphere (check with orange peel)} \\ &\quad = 2\pi R^2. \end{aligned} \tag{2}$$

Thus, if we now add up Equations (1) and use this relation, we find:

$$2 \times \text{Area of } \Delta NPQ + 2\pi R^2 = 2R^2(N + P + Q),$$

or

$$\text{Area of } \Delta NPQ = R^2(N + P + Q - \pi). \tag{3}$$

As the left-hand side of this expression is clearly positive, the three angles of the spherical triangle thus add up to more than π, i.e. 180°. The quantity in parentheses is known as δ, the *spherical excess*. It is always positive and it increases in size as the triangle gets larger.

We might perhaps have expected the angles of the triangle not to add up to π, as the euclidean proof that they do so in the plane case uses the parallel postulate, which does not apply in spherical geometry.

The connection between the area and the spherical excess of a spherical triangle has a long and not entirely settled history. Historians agree that the first correct proof to be published was that of the Italian mathematician Cavalieri (1598? - 1647) in his book *Directorium generale uraniometricum* (Bologna, 1632). Prior to this, however, it had appeared in print in the work *Invention Nouvelle en Algèbre* (Amsterdam, 1629) by the Dutch mathematician Girard (1595? - 1632). Girard, however, was not satisfied with his proof of the result.

Even before this, as we now know, the result was stated by the English mathematician Thomas Harriot (c.1560 - 1621) in 1603; Harriot did not publish his result. It is not clear whether he had a proof. Some historians regard it as likely that the even earlier mathematician Regiomantanus (Italian 1436 - c.1476) may have known the result and regard it as possible that the Polish scientist Witelo (c.1230/5 - 1275) may have discovered it. This, however, is largely speculation.

The theorem is often referred to as *Girard's Theorem*, which seems a little unfair to Cavalieri and Harriot.

A consequence of Equation (3) is that two spherical triangles, from the same sphere but of different size, cannot be similar - i.e. they cannot be scale models of one another. For this to happen we would need the angles in the first triangle to equal their corresponding angles in the second and so the areas of the two triangles would be equal.

In fact, in this case, the triangles are not only equal in area, but they are essentially equal in all respects - i.e. corresponding sides are also equal.

In euclidean geometry if two triangles are equal in all respects, they are also congruent - that is to say they can be superposed. E.g. in Figure 3, $AB = DE$, $BC = EF$, $CA = FD$ and the angles A, B, C are equal, respectively to the angles D, E, F. The two triangles also have equal areas.

We have drawn the case in which the triangles are mirror images - to superpose them, we need to turn one or the other of them over.

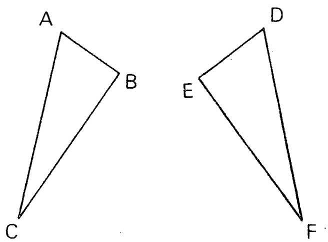

Figure 3

In the case of spherical triangles, corresponding sides and angles (and hence areas) may all be equal and yet the triangles may not superpose because they "bulge" in opposite directions. If you checked Equation (2) with orange peel, you will already have encountered this phenomenon. Such triangles are termed *anti-congruent* or *symmetrically equal*. They are mirror images of each other.

A triangle and its antipodal triangle (the triangle made up of its antipodal points) are anticongruent (e.g. $\Delta SP'Q'$ is anticongruent to ΔNPQ). In everyday experience, we could regard our hands as anti-congruent; likewise our feet. That feet are not superposable may be (rather painfully) demonstrated by trying to squash your right foot into your left shoe.

In euclidean geometry, two triangles are congruent if:

(1) The three sides of the first respectively equal the three sides of the second;

(2) Two sides of one and the angle between them respectively equal two sides of the other and the angle between them;

(3) Two angles of one and the side between them respectively equal two angles of the other and the side between them.

Actually, because the angles of a plane triangle add up to π, we may deduce from (3) the stronger:

(3') Two angles and a side of one respectively equal two angles and the corresponding side of the other

as a condition sufficient to ensure congruence.

Triangles in the plane need not be congruent if

(4) The three angles of one respectively equal the three angles of the other,

although this does ensure that the triangles are similar.

When we consider geometry on the surface of a sphere, matters are simpler and more elegant. We have the

THEOREM: *Two triangles, drawn on the same sphere are either congruent or anti-congruent if any one of Conditions (1), (2), (3), (4) holds.*

(Note that Condition (3') is *not* listed.)

On the sphere we may note an elegant symmetry. If, in Condition (1) we interchange the words "side" and "angle", we obtain Condition (4). These conditions are said to be the *duals* of one another. Conditions (2) and (3) are also dual. In spherical geometry and trigonometry, the truth of a theorem about triangles always ensures also the truth of its dual. (We do not, however, here consider the dual of the concept of area.)

We may derive another interesting result in this way. Clearly, any two sides of a spherical triangle are greater in sum than the third side, since that side is the shortest possible distance between its end-points. But it now follows that any two angles of a spherical triangle always exceed in sum the third angle. This theorem is not true in plane geometry (consider, for example, obtuse-angled triangles).

In a sequel to this paper, to appear in our next issue, I shall consider the equations of spherical trigonometry and their applications.

Geometries without parallel lines were first investigated systematically by *Riemann* (1826 - 1886). That spherical geometry is such a geometry and that it can exist in a (three-dimensional) euclidean context shows that the parallel postulate cannot be proved - i.e. that a consistent riemannian geometry can exist. Other non-euclidean geometries were intoduced somewhat before this by Bolyai, Gauss and Lobachevsky (see *Function*, Vol.3, Part 2) but these have a different character again.

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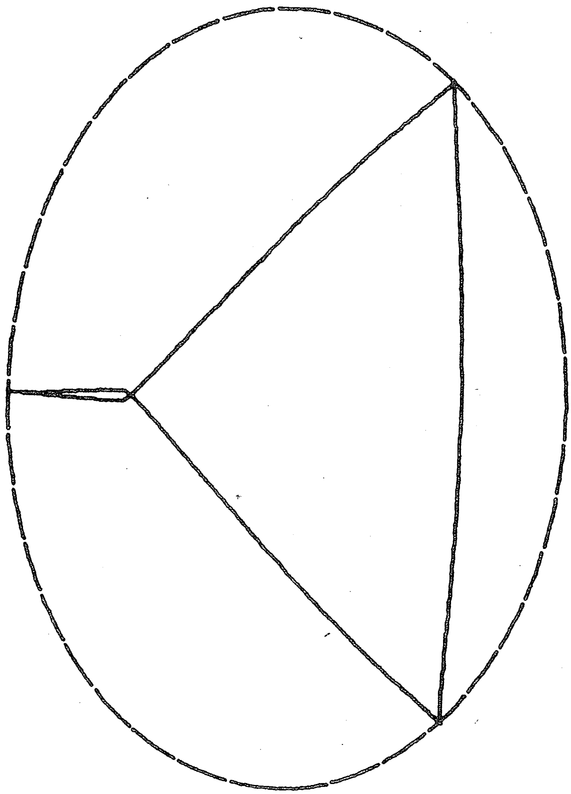

LETTER TO THE EDITOR

WITH A POCKET CALCULATOR WHO NEEDS A NORMAL AREA TABLE?

If your pocket calculator has the functions e^x, x, $1/x$ and also π, it is easy to use an approximation for the standard normal distribution function which is

$$P(-\infty < Z < Z^*) \approx \left(e^{-\sqrt{\frac{8}{\pi}}Z^*} + 1\right)^{-1};\ Z^* \geqslant 0.$$

The basic idea comes from Tocher (*The Art of Simulation*, p.32) but his account is decidedly garbled. He attributes it to H. Kahn, but I haven't tried to follow this up.

The approximation is quite good. It gives, for example,

$P(-\infty < Z < 0.3) = 0.6174$ (actually 0.6179)

$P(-\infty < Z < 1.0) = 0.8314$ (actually 0.8413)

$P(-\infty < Z < 1.5) = 0.9163$ (actually 0.9332)

$P(-\infty < Z < 2.0) = 0.9605$ (actually 0.9772)

$P(-\infty < Z < 3.0) = 0.9917$ (actually 0.99865).

The accuracy of Kahn's approximation can be considerably improved by adding a correction term c where

$c = 0,$ for $0 \leqslant Z^* < 0.5,$

$c = \frac{Z^{*2}}{100},$ for $0.5 \leqslant Z^* < 1.0,$

$c = \frac{Z^*}{100},$ for $1.0 \leqslant Z^* < 2.0,$

$c = \frac{3.7 - Z^*}{100},$ for $2.0 \leqslant Z^* < 3.0.$

With the added Preston correction term we get:

$P(-\infty < Z < 0.3) = 0.6174$ (actually 0.6179)

$P(-\infty < Z < 1.0) = 0.8414$ (actually 0.8413)

$P(-\infty < Z < 1.5) = 0.9313$ (actually 0.9332)

$P(-\infty < Z < 2.0) = 0.9775$ (actually 0.9772)

$P(-\infty < Z < 3.0) = 0.9987$ (actually 0.99865).

Esme Preston,
Monash University.

PROBLEM SECTION

We have had good solutions to some of our outstanding problems, but we would still like to see more. Attention to the solution of problems is very much a part of mathematical learning.

Here are some of the solutions we have received.

SOLUTION TO PROBLEM 6.2.1

This problem, from the Australian Mathematical Olympiad, had A tossing $n + 1$ fair coins while B tossed n. We asked for the probability that A throws more heads than B.

Any attempt to calculate the result directly leads to difficult summation problems involving the binomial coefficients. However, Evan Thorley (Year 12, Scotch College) found a neat and efficient solution.

Let p be the probability that A throws more heads than B.

Let q be the probability that A throws more tails than B.

Clearly $p = q$, as all the coins are fair.

But note further that the two events described above are mutually exclusive - if one occurs, the other cannot. Furthermore, one of them must occur. Thus $p + q = 1$.

It thus follows that $p = \frac{1}{2}$.

[Note that this reasoning would break down if A *tossed* $n + 2$ *coins for then it would be possible for* A *to throw more heads than* B *and also more tails. This case is much more complicated, and the required probability is a function of* n, *which does, however, tend to* $\frac{1}{2}$ *as* $n \to \infty$.]

SOLUTION TO PROBLEM 6.2.3

This problem, also from the Australian Mathematical Olympiad, was solved by J. Ennis (Year 10, M.C.E.G.S.). It read:

Let $p_1 = 2$ and if $n \geqslant 2$ define p_n to be the largest prime divisor of $p_1p_2 \ldots p_{n-1} + 1$. Prove that $p_n \neq 5$ for any value of n.

If $p_n = 5$, then $P_n = p_1p_2 \ldots p_{n-1} + 1$ is divisible by 5, and possibly by 2 and/or 3, but has no other prime divisors.

However $p_1 = 2$, so P_n is odd and thus 2 does not divide it. Furthermore $p_2 = 3$ (as is readily calculated), and so 3 cannot divide P_n, by a similar argument. Thus P_n is a power of 5.

Now all p_n (apart from p_1) must be odd primes, so $p_1p_2 \cdots p_{n-1} \equiv 2 \pmod 4$, and thus $P_n \equiv 3 \pmod 4$. But $5 \equiv 1 \pmod 4$ and $5^k \equiv 1 \pmod 4$ also. Thus we reach a contradiction and 5 is not a member of the sequence.

SOLUTION TO PROBLEM 6.2.5

Three cabinets each contain two drawers. In each drawer a gold or a silver coin has been placed and the following information is supplied: *One cabinet contains two gold coins, another two silver, and the third one of each.*

A drawer is opened at random and is found to contain a gold coin. The other drawer in the same cabinet is then opened.

What is the probability that it too contains a gold coin?

J. Ennis (Year 10, M.C.E.G.S.) also solved this problem. He writes:

"If a drawer is opened at random, and is found to contain a gold coin, then there are three possibilities: (i) the drawer is in the cabinet containing the silver and the gold, and we have found the coin which I shall call G_1; (ii) the drawer is one in the cabinet containing two gold coins, and yields the coin G_2, say; (iii) the drawer is the other in the same cabinet, the coin G_3, say.

But if the coin found is G_2 or G_3, then the second drawer opened will contain a gold coin. If the coin found is G_1 it will not. Thus the probability that the second drawer contains a gold coin is $\frac{2}{3}$."

[*It is surprising how often such problems are solved incorrectly. A well-known fallacious "solution" goes thus: the coin found is from either the first cabinet or the third; hence the probability is* $\frac{1}{2}$. *Unfortunately for this argument, the probability of its coming from the first cabinet is twice the probability of its coming from the third. When due allowance is made for this, the correct answer,* 2/3, *again emerges. For more on the care needed in such problems,* *see p.*29.]

Try your hand at these problems and send in your solutions for publication in *Function*.

PROBLEM 6.4.1

On a piece of paper are N statements. Statement n (where $1 \leqslant n \leqslant N$) reads: "There are exactly n incorrect statements on this page." Which statement(s), if any, are true? What if the word "incorrect" in each statement were replaced by the word "correct"?

PROBLEM 6.4.2

Prove that if P_n is the product of n consecutive integers, then $n!$ divides P_n.

PROBLEM 6.4.3

See the problem set in the article describing our cover diagram.

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BOYS - WHAT CHANCE?

The Watterson household was divided against itself! "She-who-must-be-obeyed" had read the following problem in a school textbook and couldn't understand why that textbook said the answer was $\frac{1}{2}$ when a university textbook said the answer was $\frac{1}{3}$. "He-who-does-what-he-is-told" was instructed to sort out the mess *immediately*; the H.S.C. class would want to know the correct answer in the morning.

Problem: "A man visits a couple who have two children. One of the children, a boy, comes into the room. Find the probability p that the other is also a boy."

School textbook answer (1): $p = \frac{1}{2}$, because the other child is equally likely to be a boy or a girl.

University textbook answer (2): The sample space for the sex of two children is

$$S = \{bb, bg, gb, gg\}$$

with probability $\frac{1}{4}$ for each point. (Here the sequence of each point corresponds to the sequence of births.) But given that at least one child is a boy, the reduced sample space consists of three equally likely elements $\{bb, bg, gb\}$; hence $p = 1/3$, because only one of these sample points has two boys.

"He-who-does-what-he-is-told" had a severe case of divided loyalties. On the one hand, he thought the university textbook ought to be correct, but on the other hand he was a friend of the (university-employed!) author of the school textbook. He finally decided that his friend's answer (1) was correct, and answer (2) was incorrect. Simplicity wins out over complexity!

Why is answer (2) incorrect? Let us take the sample space S as defined above, and let B be the event "that at least one child is a boy", or, in set-notation

$$B = \{bb, bg, gb\}.$$

Let A be the event that the family has two boys:

$$A = \{bb\}.$$

Then without a doubt,

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P\{bb\}}{P(\{bb,bg,gb\})} \\ &= \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} \\ &= \frac{1}{3}\ . \end{aligned}$$

But this is just an even more complicated way of getting answer (2). Where lies the mistake?

In my view, the mistake is in equating the verbal sentence "One of the children, a boy, comes into the room" with the event B defined on the sample space S. A sample space has to be the collection of all possible outcomes, and an event has to be a set of sample points. B is definitely an event in S; but I do not believe the sentence quoted above is an event in S, and so is not the same as B. For instance, consider the sample point "bg". If this occurs, does "One of the children, a boy, comes into the room" occur? Not necessarily; even if a child did come into the room it could have been the girl. A similar doubt occurs if we consider the sample point "gb". If this occurs, B does; but the verbal statement may, or may not, be true. There is some extra randomness not allowed for in the sample space S.

What we need to do is to define a sample space, and its associated probabilities, which adequately reflect the randomness in the whole problem. We could do that in various ways. The simplest way is along the lines of answer (1). Knowing that one child is a boy, we could set up a sample space $\{b,g\}$ for the *other* child and allot probabilities $\frac{1}{2},\frac{1}{2}$ to the two sample points (irrespective of whether the other child was the older or the younger). Alternatively, we could try to extend the method of answer (2) to include the choice of which child (older or younger) came into the room. The sample space would then have 8 sample points

$$\{bbo, bby, bgo, bgy, gbo, gby, ggo, ggy\}$$

where "o" = older child seen, "y" = younger child seen. If we let p = probability that the older child is seen, and $1 - p$ = probability that the younger child is seen, we could allot probabilities to the respective sample points as follows:

$$\tfrac{1}{4}p, \tfrac{1}{4}(1-p), \tfrac{1}{4}p, \tfrac{1}{4}(1-p), \tfrac{1}{4}p, \tfrac{1}{4}(1-p), \tfrac{1}{4}p, \tfrac{1}{4}(1-p).$$

The event corresponding to "one of the children, a boy, comes into the room" is

$$C = \{bbo, bby, bgo, gby\}$$

with $P(C) = \frac{1}{4}p + \frac{1}{4}(1-p) + \frac{1}{4}p + \frac{1}{4}(1-p) = \frac{1}{2}$. So the required probability is

$$P(A|C) = P(\{bbo,bby\}|C) = \frac{P(\{bbo,bby\})}{P(C)}$$
$$= \frac{\frac{1}{4}p + \frac{1}{4}(1 - p)}{\frac{1}{2}}$$
$$= \tfrac{1}{2}.$$

Notice that this agrees with answer (1), whatever the value of p.

Hence, on balance, I come down on the side of answer (1). But what more could be said? What problem does answer (2) actually answer? As calculated above, $P(A|B)$ is 1/3, but the practical question is how would you ever know B = "at least one boy" to justify answer (2) without looking at a child who happened to be a boy, forcing you to use answer (1). One way would be if you knew the school principal was only visiting families who had at least one boy and you saw him entering the house. Another way would be if a biochemist was given a sample of cells from each child but pooled together. She could establish that at least one child was a boy by chromosome testing without, perhaps, knowing whether both were boys. (Boys have X and Y chromosomes, girls have only X chromosomes.)

Another remark on this problem is that even the answers we have given make important assumptions, namely that the sexes of children are equally likely boys or girls (not exactly true in practice) that the sexes of different children are independent, and that the child coming into the room was not biased to do so by their sex, i.e. a girl would come into the room and be seen as often, but no more often, than a boy in the long-term (if a boy and a girl were both in the family).

Finally, the author of this paper is guilty of making the same mistake as was made in answer (2). In a *Function* article, Volume 1, Part 5, p.23, I discussed the probability that in a community more than one person might have the same characteristics as a criminal was known to have. I treated this probability as a conditional probability of more than one person, given at least one person, had the characteristics. I now think it would have been better to use the answer (1) method. Too late, too late!

G.A.W.

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G.P.'S IN TRANSYLVANIA

Dracula, Bram Stoker's famous vampire (*To Kill a Corpse*, 1897), though outwardly dead, was able to survive by rising at the time of the full moon and gorging on the blood of the living. These rapidly died and in turn became vampires. In Bram Stoker's story, the Dutch expert van Helsing finally rid the world of Dracula, after many difficulties.

If vampires in truth existed, we would be flat out to get rid of them. Let P be the living population and let V be the population of vampires. Then V people are, at any given time, being preyed upon by vampires.

The $P - V$ "normals" increase according to Malthus' law (say) and produce ΔP normals, but the V people already undergoing predation are killed off. In time Δt, we have

$$\Delta P = (P - V)r\Delta t - VR\Delta t \ , \tag{1}$$

where r, R are positive constants. The $VR\Delta t$ individuals killed by vampires swell the vampire population so that

$$\Delta V = VR\Delta t \ . \tag{2}$$

Let $\Delta t \to 0$ and write Equations (1), (2) as differential equations

$$\frac{dP}{dt} = rP - (R + r)V \tag{3}$$

$$\frac{dV}{dt} = RV. \tag{4}$$

Equation (4) has the solution

$$V = V_0 e^{Rt} \ , \tag{5}$$

where V_0 is the initial number of vampires, so that the number of vampires increases in geometric progression. Substitute from Equation (5) into Equation (3) to find

$$\frac{dP}{dt} - rP = -(R + r)V_0 e^{Rt} \ , \tag{6}$$

a differential equation whose solution is

$$P = P_0 e^{rt} - \left(\frac{R + r}{R - r}\right) V_0 e^{Rt}. \tag{7}$$

These equations will be valid as long as $P > V$; if $P \leqslant V$, the world population is rapidly extinguished. Supposing it took a vampire one year to kill its victim, this gives $R = \log_e 2 \approx 0{\cdot}693$ in units of years^{-1}. In the same units, r is known to be approximately $0{\cdot}027$. We thus need to solve the equation $P = V$ or (taking $P_0 = 4 \times 10^9$ and $V_0 = 1$)

$$4 \times 10^9 e^{0{\cdot}027^t} = 2{\cdot}08 e^{0{\cdot}693^t} \tag{8}$$

giving

$$t \approx 32.$$

Thus a single vampire could wipe out the world's entire population within 33 years. This hardly gives even a small army of van Helsings time to eradicate the plague once it became established. (Consider that there would probably be 256 or 512 vampires before the outbreak was diagnosed and this would already have used up 8 or 9 valuable years - then there would be problems of legality, organisation, etc. Of course, none of these things would apply if the world were *really* threatened with annihilation, would they now?)

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The ABC News 7.7.82 told of a Perth woman who was charged with "pretending to tell fortunes" - i.e. predict the future. Her defence was that she foretold future events more accurately than meteorologists and economic forecasters whose activities are (presumably) not illegal. However the magistrate, after initially reserving his decision, found her guilty.

At the time of his reserving his decision, she could have put him in a spot by announcing: "You will find me guilty", as he would have had to let her off to disprove her predictive powers!

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"If you ask six friends to name the commonest bird in Britain, the odds are that nine out of ten would say the sparrow."

Weekend, reprinted in *The Best of Shrdlu*, Denys Parsons, 1981

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The polynomial product (pp.17-18) is

$$30x^{17} - 12x^{16} + 48x^{15} + 10x^{14} - 4x^{13} - 65x^{12} + 34x^{11} - 30x^{10} - 12x^{9} + 54x^{8} + 62x^{7} - 21^{6} - 61x^{5} + 12^{4} + 26x^{3} - 13x^{2} - 14x + 8.$$

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