ISSN 0313 - 6825

FUNCTION

Volume 5 Part 1

February 1981

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A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Function,
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Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the addresses shown above.

The magazine will be published five times a year in February, April, June, August, October. Price for five issues (including postage): $4.50; single issues $1.00. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

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Registered for posting as a periodical - "Category B"

We welcome with this issue new readers and old friends alike. Once again we hope you enjoy and learn from the articles, letters, tidbits and problems. It is the problems that often give the entry by which our readers can play a more active part in *Function*. Send us solutions or even problems of your own or that you would like to see solved. Or you may care to send us letters or articles.

In any case, we wish you happy *Function*ing in 1981.

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THE FRONT COVER – HEADS AND TAILS WITH PI

C.H.J. Johnson, CSIRO.

The games of chance that everybody knows arise in the external world, quite outside the abstract world of mathematics. For example, games based on the throwing of a pair of dice or on the tossing of a coin can hardly be said to have a mathematical origin but their interpretation in mathematical terms must by necessity refer to idealised models with "perfectly made dice" and "prefectly fair coins". But there is no need to go outside mathematics to get material for games of chance for it is there in front of us in the very heart of that most ordered of all mathematics source material, the real numbers themselves, an area where we might expect that no element of chance might possibly occur. If we examine the decimal expansion of "almost" any real number in the interval (0,1), say, we find that over a long enough sequence in the expansion each of the digits 0 through 9 occurs with its "proper" frequency of 1/10. Remarkable? If, for example, you take the first two thousand digits in the decimal expansion of the fractional part of π as shown on the cover of this issue of *Function*, and determine their distribution you will obtain the results given in the table below.

digit	0	1	2	3	4	5	6	7
frequency	0.091	0.106	0.103	0.095	0.097	0.103	0.100	0.099
digit	8	9						
frequency	0.101	0.106						

Examination of these results shows them to be eminently consistent with the statement that each of the digits 0 through 9 occurs with frequency 1/10. That is, it appears that the digits are uniformly distributed. (Of course the digits of π are not arbitrary and can be computed.)

Numbers for which every allowable digit in their decimal expansion occurs with its "proper" frequency were called normal numbers by the French mathematician Emile Borel (1871 - 1956). Of course, not every number is normal; in particular rationals like 1/2 and 355/113, which gives a five-figure approximation to π, are not. It can be shown that the non-normal numbers in the interval (0,1) can be set in one-to-one correspondence with the positive integers, that is, they can be set in a countable sequence. On the other hand, it can be shown the totality of numbers in (0,1) cannot be set in such a sequence - that is, they form an uncountable set. It follows from this that there must be an uncountable number of normal numbers in (0,1). It is in terms of this uncountability that we must interpret the word "almost" when we say that almost all numbers are normal. Let us now examine this concept of a normal number and determine what we mean by the phrase "digits appearing with their proper frequency".

If x is a number between 0 and 1 we may represent it in binary notation. For example the number $0{\cdot}65625$ may be converted to binary form as

$$1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} + 0 \times 2^{-4} + 1 \times 2^{-5}.$$

The coefficients (1,0,1,0,1) in this or any other such binary number are written as $\varepsilon_k(x)$, where x is the number represented and k is the relevant power of 2^{-1}. In the example above $\varepsilon_1(0{\cdot}65625) = 1$, $\varepsilon_2(0{\cdot}65625) = 0$, etc. For $k > 5$ in this example $\varepsilon_k(0{\cdot}65625)$ is always zero.

It is customary to work with the so-called *Rademacher functions* defined by the equation

$$r_k(x) = 1 - 2\varepsilon_k(x).$$

In our example ($x = 0{\cdot}65625$), we find $r_1(x) = r_3(x) = r_5(x) = -1$, with all other $r_k(x)$ being +1. Since x is determined by its binary expansion, that is to say by the functions $\varepsilon_k(x)$, it is also determined by the values of its Rademacher functions.

Suppose now that we choose an x at random between 0 and 1. The probability that $r_k(x)$ is +1 is $\frac{1}{2}$ and the probability that $r_k(x)$ is -1 is also $\frac{1}{2}$. This holds for all values of k, and is relatively easy to prove, for example by graphing successive $r_k(x)$. The choice of each successive $r_k(x)$ is exactly like the toss of a fair coin, with say "heads" corresponding to a value +1 and "tails" to a value -1.

Now as such a coin, tossed sufficiently often, will tend to turn up heads or tails with equal frequency, so the average of the first n Rademacher functions will tend to zero for a randomly chosen x, as there will tend to be the same number of +1 and -1 values. This means that almost all values of x lead to this result, or to get back to the $\varepsilon_k(x)$, almost all values of x have (if we follow the expansion far enough) equal numbers of 0's and 1's in their binary expansions.

A similar result applies with only a little more work if we use base 10 instead of base 2. This is the result that holds for examples such as π.

What I have demonstrated here is that almost all numbers are normal; it is quite a different matter to prove that a particular number like $0{\cdot}123456789101112131415\ldots$ is normal (it is).

CURVE–STITCHING AND ENVELOPES

M.J.C. Baker, R.A.A.F. Academy

Introduction.

In his article in *Function* (Vol.4, Part 3) P. Greetham gave an interesting account of the origins of curve-stitching and showed some of the attractive figures that may be produced. The present article goes on to describe how to find the equation of one stitched curve.

Let us begin by ruling two axes OX, OY at right angles on a piece of card, and making holes along the axes at the points with integer co-ordinates (using a suitable scale, say 5mm to the unit). The first stitch joins any hole on OX (it doesn't matter which) to any hole on OY. For the next stitch we move one hole towards 0 on the X-axis and one hole away from O on the Y-axis. Suppose for instance that we started with a line from $A(7,0)$ on OX to $B(0,3)$ on OY, then our next line would go from (6,0) to (0,4), the next from (5,0) to (0,5), and so on. (Going backwards we should get the line from (8,0) to (0,2), and so on.) When we reach the line from (0,0) to (0,10) we may go further by using holes on the negative X-axis.

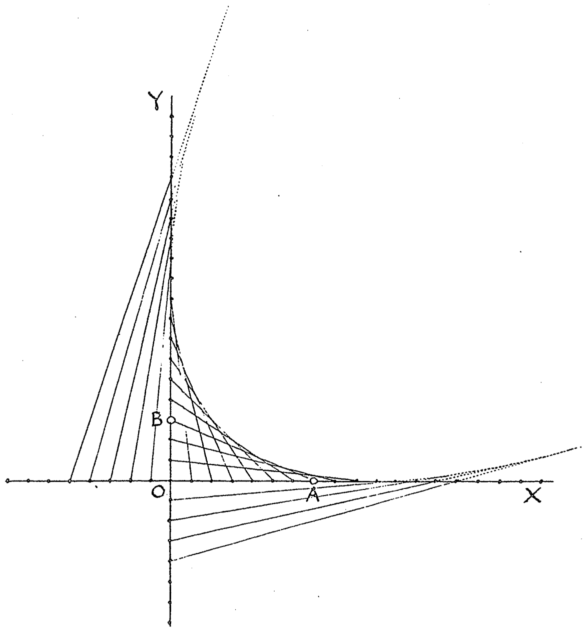

Figure 1

Figure 1 shows the resulting pattern. The threads outline a curve. In mathematical language one says that the family of lines *envelopes* a curve. The curve is called the *envelope* of the family of lines; and each line of the family is a *tangent* to the envelope. In this case we shall prove that the envelope is a parabola. (It will be seen in Figure 1 that the stitches using points on the negative axes need to be extended in order to reach the envelope.)

We are concerned with a *family* of lines, and shall need to have a way of describing mathematically not only a line but a whole family of lines. For this we first need to be clear about the meaning of the word '*parameter*'. The next section is designed to make this plain.

Variables, constants, and parameters

$2x + y = 5$ is an *equation*. Appearing in it are the letters x, y. They are *variables*. The variables can be given *values* by substituting numbers in their place. For instance we might substitute 3 for x and $2\frac{1}{2}$ for y. In this case the equation becomes a statement "Two times three plus two-and-a-half is equal to five" which we see is false. Values that *satisfy* the equation - that is make it into a statement which is true - are called *solutions*. Thus $x = 2$, $y = 1$ is a solution; so is $x = 0$, $y = 5$.

If we mark on a diagram with co-ordinate axes all the solutions by means of dots, the set of dots forms the graph of the equation. The graph of the above equation is a straight line. The co-ordinates of any point on the line satisfy the equation.

The 2 that appears in the equation is called the *coefficient* of x. The coefficient of y is 1. The 5 may also be called a coefficient.

The graph of $y = 3x$ is a straight line through the origin. We may equally well say that $y = 3x$ *represents* a straight line through the origin. So does $y = -2x$. So does $y = mx$. In the last equation m stands for a coefficient of x which has not been specified. m is called a *constant*. The coefficient -2 is also a constant of course (it doesn't change!); but very often we wish to talk in general terms and not to tie ourselves down to particular numbers, and then we use letters for constants.

Now let us turn to our original curve-stitching example: $\frac{x}{7} + \frac{y}{3} = 1$. (It is a useful trick to remember: the line joining $(a,0)$ and $(0,b)$ is $\frac{x}{a} + \frac{y}{b} = 1$.) The next line was $\frac{x}{6} + \frac{y}{4} = 1$. We can get all our lines (almost) by putting a suitable number n under the x, and $10 - n$ under the y, to give $\frac{x}{n} + \frac{y}{10 - n} = 1$. (The only exceptions are the axes themselves: $y = 0$ and $x = 0$; also we may not use 0 or 10 as values for n.)

Thus we see that the equation $\frac{x}{n} + \frac{y}{10 - n} = 1$ represents a whole family of lines. Any of them may be singled out by giving n the appropriate value: $n = 7$ gives us the first of our lines. For that line n has a constant value, 7. If we change the value of n we change to a different line. n is called a *parameter*, which is as it were a 'variable constant'. Parameters occur in connection with families of lines. For any one line of the family the parameter is constant; but we may change the parameter to single out a different member of the family.

Suppose finally that we did not wish to specify 10 in particular as the sum of OA and OB but only that this sum was a fixed but unspecified number c. We should then get the equation $\frac{x}{n} + \frac{y}{c - n} = 1$. Here x,y are the variables, c and 1 are the constants, and n is a parameter.

Calculation of the envelope

Returning to our original family of lines $\frac{x}{n} + \frac{y}{10 - n} = 1$, let us tackle the problem of finding the equation of their envelope. Suppose we choose some value for x, 2 for example, and consider the line $x = 2$. (See Figure 2.) This cuts our first line $\frac{x}{7} + \frac{y}{3} = 1$ where $y = 3(1 - \frac{2}{7})$, that is at the point $(2,2\frac{1}{7})$. Similarly it cuts the general line of the family $\frac{x}{n} + \frac{y}{10 - n} = 1$, where $y = (10 - n)(1 - \frac{2}{n})$. Now to find where $x = 2$ cuts the

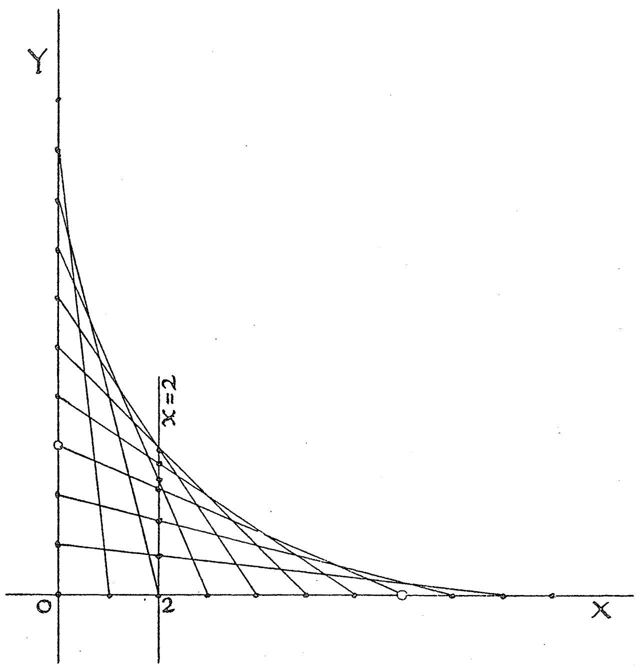

Figure 2

envelope we need to find the highest point where it cuts one member of the family. So we would like to know the maximum value of $y = (10 - n)(1 - \frac{2}{n})$ as we change n (that is as we let $x = 2$ cut the different members of the family). By calculus, the maximum may be found to occur when $n = \sqrt{20}$, i.e. the line of the family that cuts $x = 2$ highest up is the one for which $n = \sqrt{20}$ (we should have to make two new holes in our piece of card). The value of y is approximately 3·06. So (2,3·06) is a point on the envelope.

More generally if we choose x_0 as a value for x, and find the highest point of intersection of $x = x_0$ and a line of the family, we shall have to find the maximum of $y = (10 - n)(1 - \frac{x_0}{n})$. This occurs when $n = \sqrt{(10x_0)}$. We have identified the right line of the family. To get y_0, the maximum value for y on $x = x_0$, we substitute this value of n in $y = (10 - n)(1 - \frac{x_0}{n})$.

$$\begin{aligned} \text{This gives } y_0 &= (10 - \sqrt{(10x_0)})(1 - \frac{x_0}{\sqrt{(10x_0)}}) \\ &= (10 - \sqrt{(10x_0)})(1 - \frac{\sqrt{x_0}}{\sqrt{10}}) \\ &= 10 - 2\sqrt{(10x_0)} + x_0. \end{aligned}$$

So if $P(x_0,y_0)$ is on the envelope its co-ordinates satisfy the above equation.

We may now drop the subscripts and say that the equation of the envelope is $y = 10 - 2\sqrt{(10x)} + x$.

The envelope is a parabola

The last equation may be converted as follows:

$$x - y + 10 = 2\sqrt{(10x)},$$

$$x^2 - 2xy + y^2 + 20x - 20y + 100 = 40x,$$

$$x^2 - 2xy + y^2 - 20x - 20y + 100 = 0. \quad \ldots(*)$$

Since this is a second degree equation it must represent a conic. By the symmetry of the original family of lines we know that the line $x = y$ must be an axis of the conic. This suggests a rotation of axes through 45° (see the box overleaf).

Rotation of axes through an angle θ

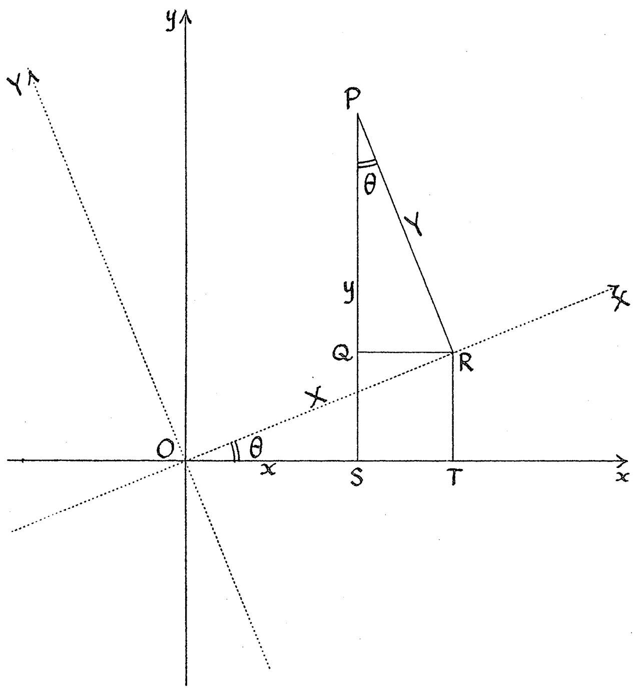

Figure 3

If P has co-ordinates (x,y) in the old set of axes and (X,Y) in the new set, then

$$x = OS = OT - ST = OT - QR = OR\cos\theta - PR\sin\theta = X\cos\theta - Y\sin\theta,$$
$$y = PS = QS + PQ = RT + PQ = OR\sin\theta + PR\cos\theta = X\sin\theta + Y\cos\theta.$$

To change axes by a rotation of 45° we use the relations $x = \dfrac{X - Y}{\sqrt{2}}$, $y = \dfrac{X + Y}{\sqrt{2}}$, as $\sin 45° = \cos 45° = 1/\sqrt{2}$. Equation (*) is thus converted into the following equation representing the conic in the new axes:

$$\frac{(X-Y)^2}{2} - (X^2 - Y^2) + \frac{(X+Y)^2}{2} - \frac{20}{\sqrt{2}}(X - Y) - \frac{20}{\sqrt{2}}(X + Y) + 100 = 0.$$

This reduces to $2Y^2 - \dfrac{40}{\sqrt{2}}X + 100 = 0$, or $Y^2 = 4.\dfrac{5}{\sqrt{2}}X - 50$, which is a standard equation for a parabola. We have finished the problem.

QUADRATIC EQUATIONS

John N. Crossley, Monash University†

Quadratic equations such as

$$x^2 + 10x = 39 \tag{1}$$

are very familiar to us and so is a method of solving them. The solutions of

$$ax^2 + bx + c = 0 \tag{2}$$

are given by

$$x = \tfrac{1}{2}(-b \pm \sqrt{b^2 - 4ac})/a. \tag{3}$$

How were quadratic equations first solved? and when? and have things changed since those times?

In many books on the history of mathematics you will find it said that the Babylonians (who lived in what is now south-east Iraq) solved quadratic equations in about 1700 B.C. This is only partly true. They certainly solved problems which we would write as quadratic equations and they also used a calculation corresponding to that in equation (3). However, they only dealt with numerical coefficients as in equation (1) and had no means of writing down anything corresponding to equation (2) which has letters for coefficients. In addition they could not write down equations like (2) because they had no zero. They also did not have negative numbers. So an equation such as (1) corresponds to a problem they could solve while an equation like

$$x^2 + 10x - 39 = 0 \tag{4}$$

just had no equivalent for the Babylonians even though (1) and (4) both appear to us as the same equation (or at least equivalent equations).

These quadratic equations are generally presented in geometric terms, so the answers always have to be actual lengths (or areas etc.). Because of this there are never any answers other than positive numbers to any problem considered.

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† A version of talks given at a Monash School Mathematics lecture on 28 March, 1980, and to the Southeast Asian Mathematical Society, 19 June, 1980. This article also appears in *Matimya's Matematika 4* (1980) (Philippines).

When we turn to the Greeks who were doing mathematics from about 600 B.C. we find them less interested in practical problems and more concerned with pure, ideal questions. Their main contributions were in geometry, the most famous work being Euclid's *Elements* which was written about 300 B.C. There the questions posed usually require a construction using a ruler and compass. As an example consider the following problem taken from another book of Euclid's, the *Data*.We are told that two lengths add up to a certain amount (let us call this b) and when used as two of the sides of a rectangle they enclose a certain area (let us call it c). In modern terms we think of the two lengths as x_1 and x_2 where

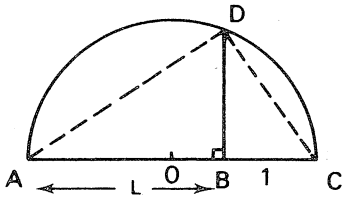
$$x_1 + x_2 = b \quad \text{and} \quad x_1 x_2 = c.$$

From our knowledge of quadratic equations we would say that we were trying to find the roots of a quadratic where the sum of the roots is b and the product is c. So we wish to solve the quadratic

$$x^2 - bx + c = 0. \qquad (5)$$

But what Euclid does is to give a construction which involves only drawing lines and right angles starting from a line of length b and a rectangle of area c. The method does not depend on which rectangle we choose, only on its area. (And you will remember how to construct a right angle just using a compass and ruler.) To get the answers one then just has to measure a couple of lines in the figure constructed.

Here is another example. We want to construct a length x such that x^2 is a given length L. So we draw a line AC of length $L + 1$ and construct a semi-circle on that line as diameter.

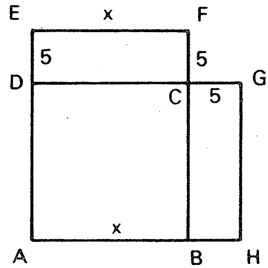


Now erect the perpendicular at B and let it meet the circle in D. Then BD is the required length. (This example is from Euclid, Book II. The proof that BD is the correct solution is found by considering the similar triangles ABD and DBC - make sure the order is right!)

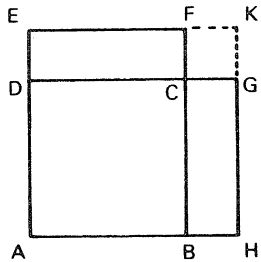
After the Romans had taken over Alexandria (in northern Egypt) where Euclid worked, there was little dramatic development in mathematics and it is not until 830 A.D. that we find a novel treatment of quadratic equations.

In that year al-Khwarizmi, an Arab mathematician whose name gives us the word 'algorithm', wrote a book about *al-jabr* and *al-muqabala*, that is to say, algebra. In it he gave the rule we have in (2) but he also gave geometric proofs of solutions. In particular he solved the equation (1). One of his versions is roughly as follows.

We are given a square of side x and an area 10 units by x units. Divide this extra area into two pieces each 5 by x and put them on two adjacent sides of the square. We then have the figure below:

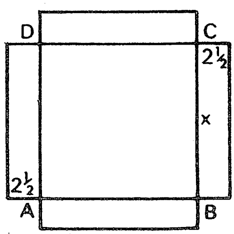


The total area is then $x^2 + 10x$ which we are told (by equation (1)) is 39. Now add on the top corner to give a square $CFKG$ of size 5 by 5, that is, 25 units and then the area of the big square



$AEKH$ is then $x^2 + 10x + 25$ or, using the 39 instead of $x^2 + 10x$, $39 + 25$, that is, 64. So $AEKH$ is a square of area 64. Its side AE is therefore $\sqrt{64}$, that is, 8. But AE is made up of $AD = x$ and $DE = 5$. So $x + 5 = 8$ and $x = 3$. Thus the equation is solved.

He also solved the equation another way. Instead of adding 5 by x rectangles on two sides, he added $2\frac{1}{2}$ by x rectangles on all four sides to get the figure below.



Now we have to fill in the corners. Perhaps you would like to finish off solving the equation.

Strangely enough it was a very long time before people started accepting negative roots of quadratics in the West, though as early as the seventh century A.D. Hindu mathematicians had considered them. In fact before negative roots became accepted complex numbers were used. The question of what happens when $b^2 - 4ac$ is negative had been sidestepped by Euclid, who had specifically imposed a condition equivalent to $b^2 \geqslant 4ac$. It was not until cubic equations were being solved algebraically that complex numbers came into mathematics. Once they had entered mathematics they were soon considered as roots of quadratic equations. With them I think most people believe that we now have the complete solution of quadratic equations. But in fact there are endless questions still to be asked. Some are simple and have already been asked, others may suggest themselves to you. Indeed, that is the way mathematical research progresses. Anyhow, here are a couple for the equation $ax^2 + bx + c = 0$.

What conditions must a, b, c satisfy if x is to be a whole number?

When are both roots (i) positive, (ii) negative?

Finally, to widen the picture, can you draw a three-dimensional picture to solve the cubic? (Try $x^3 = a - 3uvx$ and consider the cube with side u where $u = x + v$.)

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MESSAGES FROM THE CALCULATOR

Complete each calculation then turn your calculator upside down to read the answer.

1. Q: What did the society lady call the hobo?

   A: $1938 \times 25 \times 4 \div 24 =$

2. Q: What did the doctor tell Robert's mother?

   A: $(555^2 + 520803) \times 93 + 70804 =$

3. Q: What did the casino boss think as he watched William winning at the game tables in Las Vegas?

   A: $7 \times 5 \times 100 + 7{\cdot}7718 =$

4. Q: When the ghost frightened the little girl, what did she say?

   A: $0{\cdot}07 \times 0{\cdot}111 \times 5 + 0{\cdot}00123 =$

5. Q: How is maths this year?

   A: $2^4 \times 3 \times 5 \times 83 \times 277 =$

FAST ADDITION FOR COMPUTER ARITHMETIC

Ron Sacks-Davis, Monash University

When representing integers, positional notation using radix (i.e. base) ten is defined by the rule

$$a_m \ldots a_2 a_1 a_0 \quad \text{means} \quad a_m 10^m + \ldots + a_2 10^2 + a_1 10^1 + a_0,$$

where the conventional decimal system is obtained when the coefficients $a_0, a_1, \ldots, a_m$ are integers lying between 0,9 inclusive. For example, 230 means $2 \times 10^2 + 3 \times 10^1 + 0$.

Other number systems have also been studied. We consider one called a *signed-digit* number system where the coefficients may take negative values as well as positive. In this system, the coefficients are restricted to be integers in the range -6 to 6 rather than 0 to 9. Note that 13 digit values are now allowed in contrast to the 10 of the conventional number-system. The six negative digits are indicated by placing a bar over the digit.

Example 1: $3\bar{4}2$ means $3 \times 10^2 - 4 \times 10^1 + 2$ or 262.

Example 2: $\bar{1}01$ means $-1 \times 10^2 + 0 \times 10^1 + 1$ or -99.

Note that there may be more than one signed-digit representation for a particular number (in Example 1, both $3\bar{4}2$ and 262 satisfy the requirements of the system). For this reason, signed-digit number systems are sometimes referred to as *redundant* number systems.

Let us look at some of the properties of signed digit numbers. If A is a signed digit number with value $a_m 10^m + \ldots + a_2 10^2 + a_1 10^1 + a_0$ ($a_m \neq 0$), then since

$$|a_{m-1}|10^{m-1} + \ldots + |a_2|10^2 + |a_1|10^1 + |a_0| < |a_m|10^m,$$

a number of important consequences follow.

(1) The sign of A is the same as that of a_m.

(2) $A = 0$ if and only if $a_0 = a_1 = a_2 = \ldots = a_m = 0$.

(3) If A is represented in signed-digit notation as $a_m \ldots a_2 a_1 a_0$, then $-A$ can be represented as $\bar{a}_m \ldots \bar{a}_2 \bar{a}_1 \bar{a}_0$ (where, if necessary, we use the convention $\bar{\bar{a}} = a$).

However, the property of signed-digit number representation that makes it so attractive to designers of computer arithmetic units is the following.

(4) When adding two signed-digit numbers, we may add the digits in corresponding positions in totally parallel fashion.

Thus, the carry propagation associated with the addition of numbers in the conventional representation is eliminated if a signed-digit representation is used.

The addition process is as follows. Given two signed-digit integers X, Y, where

$$X = x_n 10^n + \ldots + x_2 10^2 + x_1 10^1 + x_0$$

$$Y = y_n 10^n + \ldots + y_2 10^2 + y_1 10^1 + y_0,$$

addition is performed in two stages. First, from each of the terms $x_i + y_i$, $i = 0, 1, \ldots, n$, a transfer (or carry) digit t_{i+1} and an interim sum digit w_i are formed, satisfying

$$x_i + y_i = 10t_{i+1} + w_i.$$

The values of t_{i+1} and w_i are given by this table.

$x_i + y_i$	As at right with all signs reversed.	0	1	2	3	4	5	6	7	8	9	10	11	12
t_{i+1}		0	0	0	0	0	0	1	1	1	1	1	1	1
w_i		0	1	2	3	4	5	−4	−3	−2	−1	0	1	2

In the second stage, the digits, s_i, of the sum are formed:

$$s_i = w_i + t_i.$$

It may be observed from the addition table that $|w_i| \leqslant 5$ and $|t_i| \leqslant 1$, so that s_i may be formed without any further carry propagation.

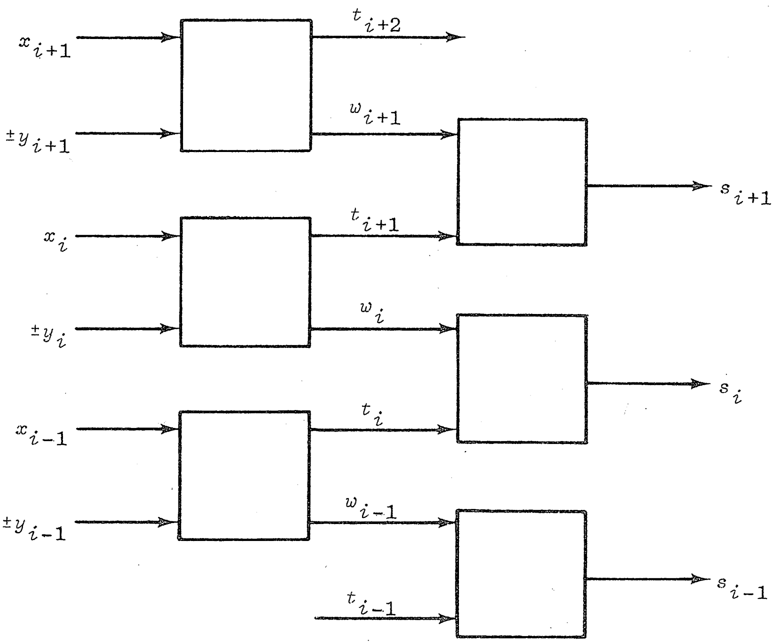
The formation of each of the pairs of digits t_{i+1} and w_i in the first stage of the addition process and the formation of each of the sum digits in the second may be performed in perfectly parallel fashion (i.e. simultaneously). Thus addition in signed-digit arithmetic is achieved in constant time (independent of the number of digits in X, Y). Contrast this with addition using the conventional representation of numbers. There, due to carry propagation, addition time is roughly proportional to n, where n is the number of digits in X,Y.

As an example of addition, take $X = 24\bar{3}5\bar{1}$ (or 23749) and $Y = 11\bar{6}16$ (or 10416). The calculation then proceeds as follows

x_i	:	2	4	$\bar{3}$	5	$\bar{1}$
y_i	:	1	1	$\bar{6}$	1	6
w_i	:	3	5	1	$\bar{4}$	5
t_i	:	0	$\bar{1}$	1	0	0
s_i	:	3	4	2	$\bar{4}$	5

The sum is represented in conventional notation as 34165.

Below is a block diagram of a totally parallel adder.



A section of a totally parallel adder.

Subtraction in signed-digit arithmetic is performed by complementing (changing the sign of) each digit in the number to be subtracted and then adding. To subtract 10416 from 23749, we proceed as follows.

$$
\begin{array}{lccccc}
x_i : & 2 & 4 & \bar{3} & 5 & \bar{1} \\
\bar{y}_i : & \bar{1} & \bar{1} & 6 & \bar{1} & \bar{6} \\
\hline
w_i : & 1 & 3 & 3 & 4 & 3 \\
t_i : & 0 & 0 & 0 & \bar{1} & 0 \\
\hline
d_i : & 1 & 3 & 3 & 3 & 3
\end{array}
$$

The digits d_i are those of the difference which, written conventionally or in signed-digit notation, is 13333.

Variants of the above technique are used to increase the performance of many computer arithmetic units. Usually binary arithmetic is used but the ideas are similar. Note that the larger range of allowable digit values in a signed-digit number system compared to a conventional number system implies that in general the signed-digit number systems are not as economic of storage as conventional number systems. As a consequence, signed digit addition is commonly used for the addition of intermediate quantities such as the partial products generated during multiplication, the initial arguments and the final product are represented by a conventional number system.

As an exercise, try to develop an algorithm for adding decimal signed-digit numbers in three stages (rather than two) if the a's are restricted to lie in the range -5...5 rather than -6...6.

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"The universe is infinite but bounded, and therefore a beam of light, in whatever direction it may travel, will after billions of centuries return - if powerful enough - to the point of its departure; and it is no different with a rumor, that flies about from star to star and makes the rounds of every planet. One day Trurl heard distant reports of two mighty constructor-benefactors, so wise and so accomplished that they had no equal; with this news he ran to Klapaucius, who explained to him that these were not mysterious rivals, but only themselves, for their fame had circumnavigated space."

The Cyberiad, Stanislaw Lem, 1974.

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"Half Past Five is Twenty to One."

Brian Martin, 3UZ race preview,
Twenty Five to Two, 7.10.80.

"Come in more like a Quarter to Six."

Disgruntled punter, Five to Two.

I ALWAYS LIE WHEN I WRITE ARTICLES†

Vicki Schofield, University of Newcastle

Around the end of the last century, a number of paradoxes were "discovered", which seemed to threaten the foundations of logic and mathematics. Paradoxes occur when the "conceptual apparatus of science is more or less radically revised" [1, p.493]. The study of these paradoxes has led to a greater insight into mathematics and reasoning, and often intuitive ideas have been rejected in favour of concepts which at first seem unbelievable.

Zeno of Elea - the follower of Greek philosophy - propounded four famous paradoxes on motion, two of which are:

I. *The Dichotomy* - There is no motion; because that which is moved must first arrive at the middle before it arrives at the end. It must traverse the half of the half before it reaches the middle and so on ad infinitum. (i.e. How is it possible to reach an infinite number of positions in a finite time?)

II. *The Achilles* - The slower when running will never be overtaken by the quicker, for that which is pursuing must first reach the point from which that which is fleeing started, so that the slower must necessarily be some distance ahead. (i.e. If the quicker must occupy an infinite number of positions in order to overtake the slower, how will he be able to do this in a finite time?)

To explain this paradox of infinity, Cantor, in 1882, proposed a new concept for "counting". If the elements of 2 sets can be put into one-to-one correspondence then the 2 sets have equal cardinality. A set is then said to be infinite if it can be put into 1-1 correspondence with one of its parts. If this is the criterion of equality, then it can be seen that a finite length can contain an infinite number of positions.

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† This article first appeared in *School Mathematics Journal* No.14 (July, 1980) published by the Newcastle Mathematical Association. It is reproduced with their permission.

However, Cantor's work produced its own paradox, which arises when the set m of all sets is considered. Its cardinal number $NC(m)$ (the number of elements) is the largest which can exist. However the set $B(m)$ of all subsets of m, according to a theorem in set theory, has its cardinal number $NC(B(m))$ larger than the cardinal number $NC(m)$ of m.

This paradox derives its origin from the possibility of constructing the set m of all sets. It can be seen [1, p.496] that the set m cannot be constructed using the various systems of axioms of set theory available.

The Liar Paradox has been known since the sixth century B.C. when Epimenides the Cretan said "Cretans always lie". Suppose he is lying. Then what he says cannot be true, and as he says that he is lying he is thus speaking the truth.

Suppose he speaks the truth. Then what he says must be true, and he is thus lying. Therefore contradictions arise whatever alternative is chosen. It is shown [1, p.510] that in a formal system with an adequate definition of truth and falsehood, the statement concerning the liar doesn't exist. Thus, for sentences contained in ordinary language, such a definition cannot be given.

The Grelling Paradox: If and only if an adjective can be applied to itself, it is called "autological"; "heterological" if and only if it can't. The words "English" and "polysyllabic" are autological while "French", "monosyllabic", "red" are heterological. If "heterological" is to be considered heterological, it can be applied to itself and it is thus autological, and vice versa.

In 1926, F.P. Ramsey observed that paradoxes of logic could be divided into two classes. Group A consists of contradictions which, "were there no provision made against them, would occur in a logical or mathematical system itself. They involve only logical or mathematical terms such as class and number, and show that something must be wrong with our logic or mathematics. But the contradictions of Group B are not purely logical and cannot be stated in logical terms alone, for they contain some reference to thought, language, or symbolism, which are not formal but are empirical terms" [1, p.503].

Group B are referred to an "semantical paradoxes" and include the Liar paradox, as well as those of Grelling, and Berry. If we formalise logic and mathematics, the semantical paradoxes don't enter into this system and no revision of the formal system can be of any use in getting rid of them.

The Berry Paradox is this: Suppose we are given a set A containing every word occurring in this essay; the number of words contained in this set will be finite. We consider the set P (also finite) of sentences which contain at most 50 words, all of which come from set A. Let Q be the set of sentences in P which define a natural number (Q is finite). Consider the set R of the natural numbers which are defined by a sentence in Q. The set R will be finite, consequently there are natural numbers which are not in R; the first (according to the usual arrangement of natural numbers) is to be called the Berry number.

Consider the sentence: "The Berry number is the first number which cannot be defined by means of a sentence containing at most fifty words, all of them taken from set *A*."

This sentence is a correct definition of the Berry number; contains less than fifty words (all of which are taken from set *A*) and is therefore contained in *P*. As it constitutes the definition of a natural number, it is also in *Q*. The Berry number is therefore in *R*. However, by its definition - the Berry number cannot be in *R*. This leads to a formal contradiction.

The Paradox of Denotation shows the confusion which can occur with usage of symbols.

The statements $\log 343 > 2$

and $343 = 7^3$,

imply $\log 7^3 > 2$.

This inference conforms to accepted standards of logic, while the following does not.

343 contains three figures

$343 = 7^3$

7^3 contains three figures.

A further example of a paradox of denotation is this:

"Do you know your father?"

"Yes."

"But if you were shown a masked man and were asked if you knew that man, wouldn't you say that you didn't know him?"

"Yes."

"Now that man happens to be your father. So, as you don't know him, you don't know your father."

If a set may be a class of classes, the elements of a set may themselves be sets. Bertrand Russell (in 1903) brought to notice the class of all those classes that are not members of themselves, and asked whether the set of sets so described was a member of itself. If the answer is no - then it is a class that doesn't contain itself, and thus a contradiction. If the answer is yes, the class is a member of the "all" and is therefore a set that doesn't contain itself, which is again a contradiction.

Another version Russell produced was: In a certain village the village barber shaves all those men and only those men who don't shave themselves. Who shaves the barber? If he is one of those men who do not shave themselves, he must be shaved by the barber (himself), and thus he does shave himself. If he is someone who shaves himself, then he cannot be shaved by the barber, which contradicts the fact just stated.

Much more is gained by examining the apparent inconsistencies in mathematics, than by merely disregarding them. By the critical evaluation of current knowledge and understanding, further progress can be made into mathematics.

References

1. E. Beth, *The foundations of mathematics*, North Holland Publishing Company, Amsterdam.

2. Edna E. Kramer, *The nature and growth of modern mathematics*, Hawthorn Books Inc. New York.

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The World Almanac and Book of Facts, 1980 contains on p.154 some information on the speeds of animals. Some 36 critters are listed and opposite them their speeds. The cheetah travels, we are told, at 70 m.p.h. (i.e. about 110 k.p.h.) and is the fastest. The garden snail manages only 0·03 m.p.h. (i.e. about 0·05 k.p.h.). It must have been quite invigorating collecting all this data. The book comments:

"Most of these measurements are for maximum speeds over approximate quarter-mile distances. Exceptions are the lion and the elephant, whose speeds were clocked in the act of charging; the whippet, which was timed over a 200-yard course; the cheetah over a 100-yard distance; man for a 15-yard segment of a 100-yard run (of 13·6 seconds); and the black mamba, six-lined race runner, spider, giant tortoise, three-toed sloth, and garden snail, which were measured over various small distances."

The animals listed include the chicken (9 m.p.h., or 14 k.p.h.). We wonder who timed a chicken over a quarter-mile and in what circumstances.

* * * *

"... in regard to young people at an age when memory is tenacious, imagination vivid and invention quick. At this age they may profitably occupy themselves with languages and plane geometry, without thereby subduing that acerbity of minds still bound to the body which may be called the barbarism of the intellect. But if they pass on while yet in this immature stage to the highly subtle studies of metaphysical criticism or algebra, they become overfine for life in their way of thinking, and are rendered incapable of any great work."

New Science, Giambattista Vico, 1744.

THE SURPRISE PARTY

Aidan Sudbury, Monash University

In a previous issue of *Function* (Volume 3 Part 3) we included an excerpt from a play which was built around various logical paradoxes. It was a version of the notorious Prediction Paradox which (perhaps) has had a good deal more serious thought expended on it in philosophical journals than it has merited. I will restate it in a form more suitable for logical exploration than that given in the play. The persons involved are a hotel manager and a guest, Professor Fist.

The manager tells Fist that he will be given a party at 8 p.m. one day next week, and it will be a surprise in the sense that on the day before the party, he will not know the date of its occurrence. Fist reasons as follows:

1. If the party were to be held on the last day of the week (Saturday), then the night before he would be in a position to predict its occurrence on the following day.

2. However he will not, apparently, ever be in such a position.

Hence:

3. The party cannot be held on Saturday.

4. However, since Saturday is excluded, Friday is not possible either. For, having excluded Saturday he would be in a position on Thursday evening to predict a Friday party, contrary to the manager's promise.

Hence:

5. The party will occur before Friday ... etc.

He thus eliminates all available days, but, suppose, in spite of this, the manager were to give him the party on Wednesday, then, as promised, Fist could not know this on Tuesday. How could the manager achieve the apparently impossible?

What follows is the explanation given by Professor Crispin Wright and myself in the *Australian Journal of Philosophy* of May 1977. I should point out that there is rarely universal agreement as to the solution of paradoxes and this is no exception. In fact the norm is universal disagreement, and the explanation of why this should be so would probably be the most important philosophical solution of all.

Consider a "one-day" version of the paradox. Suppose someone said to you 'I am going to give you a party to-morrow, but you won't know the date of it beforehand', surely, the only appropriate reaction would be bewilderment, for there is nothing that can be sensibly deduced from the statement, and because of this, he can do exactly as he said. You can't know the date of the party if you can't believe the only statement that might have given you the information.

Let us now look at how we eliminated the Saturday as a possibility in the "one-week" version of the paradox. We imagined ourselves at the end of Friday wondering whether we could or could not deduce what would happen the next day. Our situation by then, however, would be exactly the same as those who had been given the one-day version of the paradox, and we have seen that in that case it is possible for the person promising the party to do exactly as he said. Thus we cannot actually eliminate the Saturday as a possibility, and this is how the manager kept his promise to Fist.

In the play he did indeed give Fist the party at the last possible moment. 'I left it so late you had no reason to believe I'd give you the party, and then I could do what I said'.

[*We invite reader's comments on this paradox and solution. Dr Sudbury's play* Language Takes a Holiday *is being produced for radio by the ABC. We are not informed exactly when it will be broadcast, but readers may like to look out for it. Eds.*]

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TO FIND SQUARE ROOTS WITHOUT A $\sqrt{\ }$ KEY

1. Have a guess.
2. Divide the number by the guess.
3. Average the answer and the guess.
4. This number becomes a new "guess".
5. Go back to stage 2, and repeat the steps.

Stop when your answer is close enough.

e.g. $\sqrt{24}$

1. guess 4.
2. $24 \div 4 = 6$.
3. $\dfrac{4 + 6}{2} = 5$.
4. new guess 5.
2. $24 \div 5 = 4{\cdot}8$.
3. $\dfrac{5 + 4{\cdot}8}{2} = 4{\cdot}9$.
4. new guess $4{\cdot}9$.

Continue until the desired accuracy is obtained.

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In *Function*, Vol.2 Part 4, we published a letter from Dr Scott of the department of Botany, Monash University. Dr Scott wrote that a cutting, descended from the apple tree which helped Newton formulate his theory of gravity, is now established in the grounds of Monash University. Mr G. Smith has written the following somewhat sceptical letter. The Editors still hope that the Monash tree is, indeed, descended from Newton's.

LETTER TO THE EDITORS

from G.C. Smith, Monash University

NEWTON'S APPLE TREE - AND ITS SUPPOSED DESCENDANTS

Not far from where I sit writing this note there is an apple tree which is said to be a descendant of the tree from which the fall of an apple inspired Newton to conceive the theory of gravitation.

Isaac Newton told William Stukeley the story about the apple in 1726. The vital apple fell in the autumn of 1665 or 1666. We therefore can be sure that the garden of Newton's family estate at Woolsthorpe contained at least one apple tree in the 1660's. For ease of reference let us call the tree that bore the inspiring apple N.

In 1732 the Woolsthorpe estate was bought by Edmund Turnor; it is still owned by the Turnor family. Edmund Turnor claimed that an apple tree then living was N; the tree that he found on buying the property I shall refer as T (for Turnor). The question that immediately arises is how certain could Turnor be that T was the same as N? Over sixty years had elapsed since the fall of the apple, and it is therefore not unlikely that N would have died in this period; even if N had survived, there might have been other apple trees at Woolsthorpe, and then the question arises as to the identification of which particular tree was the one that bore the apple that fell. Apple trees may survive for more than 60 years but relatively few people do - so who could have told Turnor 'that was the very tree...'? As far as I know, there is no evidence of the apple story being told before 1723; so apart from Newton, there is no reason why anyone should have any knowledge that apples or apple trees were of interest to make it likely that the story should be passed on. Perhaps Turnor's identification of T with N was just a guess.

In 1820 (or in 1814 according to one source) T finally fell in a gale. It had had to be propped up for many years. If we agree that T was N this is hardly surprising, as the tree would be in excess of 120 years old! I am told that apple trees *can* live to this age - but owners generally do not aim at prolonging the life of fruit trees to such an advanced age. As Turnor was an amateur local historian I think there can be no doubt that the tree that fell in 1820 was *T*; Edmund Turnor's family valued the association of the Woolsthorpe property with Newton.

So much for the original tree (or trees); now for the descendants.

At some time - no-one seems to know just when - scions from *T* were grafted on to a stock at Lord Brownlow's estate at Belton which is a few km from Woolsthorpe. I should explain here that scions are pieces of twig bearing the fruit buds and hence transmitting the genetic material of the original tree; a stock is the tree which receives the graft. Let us call this tree and its descendants *B*. There were trees descending from the original graft at Belton at least until 1939, for in either 1939 or 1940 scions from a tree at Belton were grafted onto a tree at the Fruit Research Station at East Malling. And scions from the East Malling tree have been taken to graft upon stocks at a number of places in the United Kingdom, U.S.A. and Australia. A picture of one may be found in Colin A. Ronan's *Sir Isaac Newton* (International Profiles, 1969). The trees bear a variety of apple known as Flower of Kent; the apples are said to be pear-shaped.

But are these trees really descended from the original tree *N*? The evidence that they are descended from some apple tree at Woolsthorpe seems to me to be fairly strong. However I am not convinced that we can be even reasonably sure that they come from the tree that had the honour of inspiring Newton. As I have pointed out above the period that elapsed between 1666 and the 1720's (when the apple story came into the open) is a long one and probably too long for anyone to have been able to transmit the evidence.

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MATHEMATICS great, Sir Isaac Newton may have suffered mercury and lead poisoning as a result of his interest in alchemy, according to a British study.

Recent tests on his hair showed high levels of mercury and lead.

The metal poisoning is thought to have occurred from 1678 to 1692 when Sir Isaac performed hundreds of experiments in alchemy - trying to make gold from base metals.

He frequently slept in his laboratory while experiments were in progress, breathing in toxic fumes.

The Sun, 5.5.80.

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PROBLEM SECTION

Each issue of *Function* contains a number of problems of varying levels of difficulty. You are invited to submit problems, solutions to problems, comments or partial solutions.

Here we begin by printing some comments and solutions.

COMMENT ON PROBLEM 4.2.4.

This problem was, in paraphrase, the following:

Baggage trains used at airports, railway stations, etc. have a small tractor which pulls a train of 4-wheeled trailers, each connected to the one in front. The back axle of each trailer is fixed, and the front axle pivots, being steered by the towing bar connecting the trailer to the one in front. The wheelbase has length b, the towing bar length a, and the rear connection length c. How should the dimensions a,b,c be proportioned so as to make the train follow as nearly as possible the path taken by the tractor?

In the previous issue of *Function* (Vol.4, Part 5), we gave the solution $a^2 + b^2 = c^2$, which is the correct result to allow tracking round and round a circle.

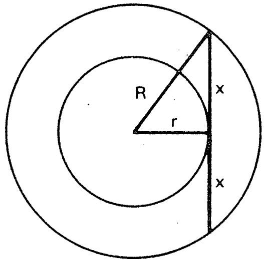
Research carried out at Flinders Street railway station, however, fails to confirm the result. We have there approximately $a = b = 120$ cm, $c = 45$ cm. Moreover, a little thought shows a reason for the discrepancy. If we take account of the width w, we find (approximately) $w = 150$ cm. Two trucks can pivot in a tight circle in which the first is at right angles to the second. The condition for this is $a = w/2 + c$, i.e. $a > c$, which cannot happen if $a^2 + b^2 = c^2$. As reasonable width is a prerequisite for a usable trolley, the published solution is unfeasible. We also note that the axles collide if (at very least) $b < \frac{w}{2}$, so that b is also subject to limitation.

Our solution is thus impractical, although not, strictly speaking, incorrect. The Flinders Street trolleys will *not* track in a circle, nor is there much point in their doing so. They will turn corners of 90° or more. On a 90° turn they track reasonably exactly. On a 180° turn they don't, as you can observe. Nor is there any reason in this circumstance whey they should.

The actual design problem is very difficult. Professor Tom Morley of Illinois, an engineer turned mathematician, remarked that the problem was one of those mathematically ill-posed questions engineers have to solve routinely.

SOLUTION TO PROBLEM 4.3.3.

The problem was to paint the ring-shaped (or annular) region in the target shown.



(i) Supposing that it is sufficient information to know the length, $2x$, of the chord in the diagram, what is the *area* to be painted? (No complicated calculations!)

(ii) Prove that $2x$ *is* sufficient information to find the shaded area.

(i) If the other dimensions are irrelevant, then if we shrink the inner circle down to zero radius, we just have to find the area of a full circle with x as radius, i.e. πx^2,

(ii) That the circles' radii are irrelevant follows by letting the two circles have radii r, R $(r < R)$. Then the required area is

$$\pi(R^2 - r^2) = \pi x^2.$$

SOLUTION TO PROBLEM 4.4.1.

There are seven persons and seven committees. Each committee is to have three persons. Can you share the committees out to the people so that each person is on the same number of committees? Does this problem have any connection with the Seven-Point Geometry article in this issue?

The Seven-Point Geometry in *Function*, Vol.4, Part 4 certainly helps. Looking at the figures on p.20 of that article, we see that the seven lines each contain 3 points and each point is on three lines. Thus if our people are called points and our committees are called lines, we can read off the following committees: $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$, $\{2,4,6\}$, $\{2,5,7\}$, $\{3,4,7\}$, $\{3,5,6\}$.

SOLUTION TO PROBLEM 4.4.2.

Determine the continued fractions for $\sqrt{2}$, $\sqrt{3}$, $\sqrt{11}$, $\sqrt{23}$, and find which numbers equal the continued fractions

$$\langle 1,3,\overline{1,3}\rangle, \quad \langle 1,1,2,2,1,1,2,2\rangle.$$

The article on Continued Fractions in *Function*, Vol.4, Part 4 tells you how to solve this question. The answers are:

$$\sqrt{2} = \langle 1,2,2,2,\ldots\rangle, \qquad \sqrt{3} = \langle 1,1,2,1,2,1,\ldots\rangle$$

$$\sqrt{11} = \langle 3,3,6,3,6,3,\ldots\rangle, \qquad \sqrt{23} = \langle 4,1,3,1,8,\overline{1,3,1,8}\rangle$$

$$\langle 1,3,\overline{1,3}\rangle = \tfrac{1}{2}\left(1 + \sqrt{\tfrac{7}{3}}\right) \approx 1{\cdot}2638,$$

$$\approx \langle 1,1,2,2,\overline{1,1,2,2}\rangle = \frac{9 + \sqrt{221}}{14} \approx 1{\cdot}7047.$$

SOLUTION TO PROBLEM 4.4.3.

In a Chinese game, six dice are tossed. Among various possible outcomes, "two pairs" (e.g. the dice might fall 2,1,5,2,5,3) are rated more highly than "one pair" (e.g. 2,1,5,6,5,3). What are the probabilities for getting "two pairs", "one pair"? Do you think the relative ratings are sensible?

The probability of two pairs is $\dfrac{\binom{6}{2}\binom{4}{2}6!}{6^6 \times 2! \times 2!} = \dfrac{25}{72}$.

The probability of one pair is $\dfrac{\binom{6}{1}\binom{5}{4}6!}{6^6 \times 2!} = \dfrac{25}{108}$.

Thus two pairs are *more common* than one pair! [For instance, in the first answer there are $\binom{6}{2}$ ways of choosing the two numbers to be pairs, $\binom{4}{2}$ ways of choosing the two remaining numbers to be singletons, $6!/(2! \times 2!)$ ways of rearranging the 6 numbers so obtained. There are 6^6 total possible outcomes, counting order as important. Similarly for the second answer.]

We now give some more problems for you to work on. First we restate an old one.

PROBLEM 3.3.5 RESTATED.

Consider the set $\{2^n$, where $0 \leqslant n \leqslant N\}$ - i.e. the first $N + 1$ powers of 2. Let $p_N(a)$ be the proportion of numbers in this set whose first digit is a. Find $\lim\limits_{N\to\infty} p_N(a)$. Is the first digit of 2^n more likely to be 7 or 8?

This is an interesting problem which no reader has yet attempted. We would be interested to see computational results, which lead to an interesting pattern.

PROBLEM 5.1.1.

This one was passed on to us from a Russian problem book. Three poor woodcutters, stranded in the bitter winter seek shelter in an abandoned cottage. "I", said the first, "have 5 logs of wood to help keep us warm". "And I", said the second, "have 3". "Alas", said the third, "I have no wood, but I have 8 kopeks to repay you for allowing me to share your fire".

How should the 8 kopeks be distributed between the first two woodcutters?

PROBLEM 5.1.2.

This problem has generated some controversy, but we print it anyway.

A boy, a girl and a dog go for a walk down the road, setting out together. The boy walks at a brisk 8 kph, while the girl strolls at a leisurely 5 kph. The dog frisks backwards and forwards between them at 16 kph. After one hour, where is the dog, and in what direction is it facing?

PROBLEM 5.1.3.

Suppose a debt of \$1000 incurs simple interest of 10% p.a. The borrower can repay a maximum of \$25 per month. How long will it take to pay the debt off? This problem is easily reduced to a simple equation, but here is how some accountants do it.

First estimate is $\frac{1000}{25} = 40$ months.

Interest on \$1000 for 40 months = $\$333\frac{1}{3}$.

New principal = $\$1333\frac{1}{3}$.

Second estimate is $\frac{1333\frac{1}{3}}{25}$, which rounds up to 54 months.

Interest on \$1000 for 54 months = \$450.

New principal = \$1450.

Third estimate is $\frac{1450}{25} = 58$ months.

Interest on \$1000 for 58 months = $\$483\frac{1}{3}$.

New principal = $\$1483\frac{1}{3}$.

Fourth estimate is $\frac{1483\frac{1}{3}}{25}$, which rounds up to 60 months.

Interest on \$1000 for 60 months = \$500.

New principal = \$1500.

Fifth estimate is $\frac{1500}{25} = 60$ months.

Thus the answer to the problem is 60 months.

Does this method always work, and if so, why?

PROBLEM 5.1.4. (Submitted by Garnet J. Greenbury, Brisbane.)

Prove that the sum of the squares of any five consecutive integers is always divisible by five.

A THERMODYNAMIC PROOF OF AN INEQUALITY

Suppose a, b are two positive numbers. Their average or, more correctly, *arithmetic mean* is $\frac{1}{2}(a + b)$. However, for some purposes, the average is computed differently, as the *geometric mean* $\sqrt{ab}$. Let A be the arithmetic mean and G the geometric. A relatively elementary theorem states that $A \geqslant G$.

The simplest proof runs as follows. We consider the expression $(\sqrt{a} - \sqrt{b})^2$. This, being a square, cannot be negative. So

$$(\sqrt{a} - \sqrt{b})^2 \geqslant 0,$$

or, if we expand

$$a + b \geqslant 2\sqrt{ab}.$$

In other words, $A \geqslant G$, as required.

Recently (1978), Dr P.T. Landsberg of the University of Southampton devised an alternative and startling proof. It uses thermodynamic rather than mathematical concepts, and so may be viewed as an application of Physics to Mathematics (rather than the reverse - more commonly seen).

Landsberg takes two bodies of equal mass, but at different temperatures. He supposes that these two are brought into contact, but isolated from the outside world. (We might, for instance, put equal masses of water and ice into a calorimeter or thermos flask.) Let the temperatures at the start of the experiment be a, b.

After a period of time, the two bodies exchange heat and both reach an equilibrium temperature. Now, by the first law of thermodynamics, energy is conserved and, as energy here is proportional to the temperature, the initial energy is proportional to $a + b$ and the final energy to $2c$ (say) where c is the equilibrium temperature. Hence $a + b = 2c$ or $c = A$.

But now we may use the second law of thermodynamics, which states that the entropy must increase. (Entropy measures the degradation in the quality or usefulness of energy.) Here entropy is proportional to the logarithm of temperature. The initial entropy is thus $\log a + \log b$ or $\log ab$. The final entropy is $2 \log c$ or $\log c^2$.

So $c^2 \geqslant ab$, or $A \geqslant G$.

This "proof" is readily extended to more general cases, such as

$$\frac{a + b + c}{3} \geqslant \sqrt[3]{abc}$$

and so on.

COMPUTERS CAN PLAY CHESS

[*We have had several requests for an article on computer chess and hope soon to supply one. Meanwhile, here is some background on the history and present state of the art. Eds.*]

As early as 1769, one Baron von Kempelen claimed to have constructed a chess-playing automaton. In this, however, he lied - a human was cunningly concealed inside the works of the "machine". The "machine" claimed many famous opponents, including Napoleon who lost to it very quickly.

Charles Babbage, who suggested many of the principles of computing in the course of work on his "analytical engine" (see *Function*, Vol.3, Part 1), discussed the topic of computer chess, as rather later did Norbert Wiener in his book *Cybernetics*. Between 1937 and 1945, one Konrad Zuse began work on a program and made very considerable progress.

It was, however, Claude Shannon whose work was most influential. He began to address the problem in 1948 and laid the groundwork for almost all the subsequent advances. A programme was begun by Alan Turing in 1951, and though never brought to perfection, was the first to (after a fashion) work.

By 1957, the Los Alamos laboratory in New Mexico had produced a programme that played a simplified version of the game (there were no bishops). Nonetheless it took on average 12 minutes per move and was programmed by a "brute force" method consisting of an exhaustive search two moves (or more accurately 4 plies - two moves and two replies) deep.

In 1958, the first genuine programme appeared at Massachusetts Institute of Technology. This also implemented some of the programming short-cuts that have entered modern programmes. This led to further theoretical advances.

Meanwhile Soviet programmers had not been inactive. Dr Mikhail Botvinnik, on and off through this period the world champion, and an electrical engineer, interested himself in the problem and made his expertise available to the programmers. In the west, a number of masters, most notably the ex-world champion and grandmaster, Dr Max Euwe (a mathematician) involved themselves in the endeavour.

Further developments were made through the sixties by Alan Kotok and Richard Greenblatt. In 1966, two programmes, one essentially by Kotok, the other developed by the Soviet investigators, played a match. The Soviet programme won two games and drew the remaining two. This gave the search for improved programmes much publicity. It was felt that Greenblatt's programme, Mac Hack VI, had it been ready in time, might have done rather better.

Two years later, the British master Levy bet that no programme could beat him within the next ten years. The stake was £10,000, the deadline 31 August, 1978. Botvinnik's comment was "I'm sorry for his money".

This spurred a flurry of activity including the institution of tournaments between programmes. The first of these to be held at the world level occurred in 1974. It took place in Stockholm and was won by the Soviet programme KAISSA. The first of a number of related U.S. programmes (CHESS 4.0) was runner-up. CHESS 4.6 reversed this three years later.

Levy and CHESS 4.6 that year played with a win to Levy and a win (but in a game not played to tournament rules) to CHESS 4.6. Later CHESS 4.6 went on to beat Michael Stean, then Britain's number two player, and a grandmaster.

In the following year, Levy won his bet by beating CHESS 4.7 with a score of $3\frac{1}{2}$ - $1\frac{1}{2}$. Last year, the world champion, Anatoly Karpov played 25 computers simultaneously. Only one managed to achieve a winning position against him and even this one subsequently lost.

Currently the best programmes (BELLE and CHESS 4.9) play at a very high level. Levels of play are assessed by an elaborate rating system. On this system, BELLE rates at somewhat above 2000, which puts it ahead of all but about 10-15 of Australian players, but a long way behind Karpov who rates at nearly 2700.

A related development has been the production of small single-purpose home computers. Several are commercially available in Australia and these play at a level of about 1000. (The earlier models were much worse.) A couple of years ago, one lost all its games when entered in the Tasmanian junior championship. Any regular club player can beat them with relative ease.

Recently a former Australian champion, Fred Flatow, reviewed one for the publication *Chess in Australia* (known affectionately to its devotees as *C.I.A.*). He noted that it played endgames very badly but had good tactical "skill" in some situations. To beat such a programme, exchange material and win the endgame. (The current versions are still incapable of mating with a queen advantage against a lone king.) This method of play is boring, but invariably successful. For an interesting game, play an opening that is unfamiliar or inferior and make tactical complications.

The topic may seem to be a trivial one, but it is related to the development of what is termed "artificial intelligence". Furthermore, the financial rewards are not inconsiderable - there is still money to be made in the entertainment industry.

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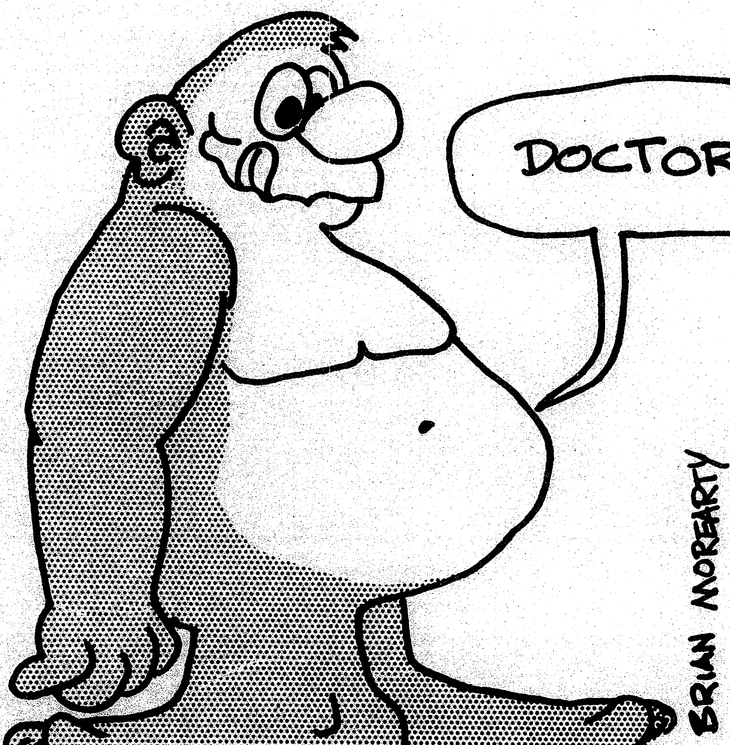
THE LITERAL MEANING

Mathematicians, like lawyers, interpret statements literally and take care to phrase them in such a way as to avoid misunderstandings. Overleaf Mike and Brian Morearty (Year 10, Mt Tamilmas H.S., California) offer their literal interpretations of two familiar instructions.

KEEP OUT OF REACH OF CHILDREN



IF SWALLOWED SEEK MEDICAL ADVICE