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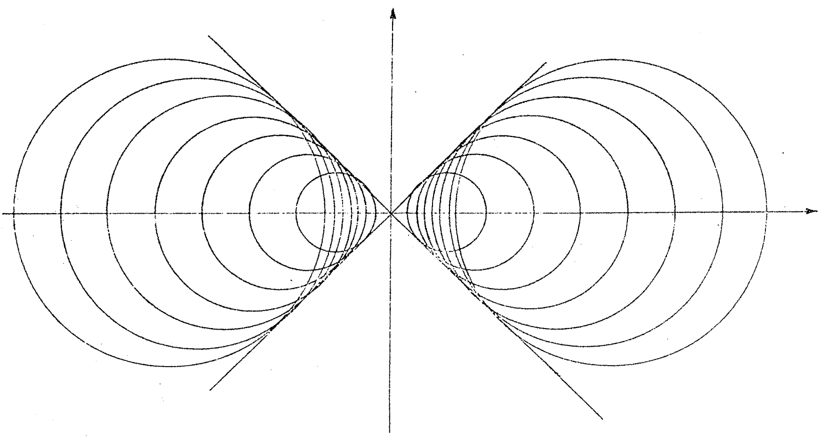

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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The first two articles in this issue were originally presented as part of the Monash Schools' Mathematics Lecture Series. John Stillwell spoke on 4th May, 1979, and Malcolm Clark on 11th April, 1980. We realise that country students cannot attend the lectures, and we hope that, by reproducing the articles in Function, we are making amends to some extent.

Another article by a Monash mathematician, Hans Lausch, indicates that Australian aborigines have strict rules about family relationships. These rules are of quite considerable mathematical interest.

THE FRONT COVER

Phil Greetham kindly provided our Front Cover illustration for this issue, and for the two previous issues. He is head of the mathematics department at Boronia Technical School, and is also currently enrolled as a mathematics and computer science student at Caulfield Institute of Technology. He originally trained as an applied chemist, and then taught chemistry, until he suddenly found a strong interest in mathematics! His article on curve stitching in this issue explains the front cover diagram, and several other such figures.

CONTENTS

WHY MATHEMATICS IS DIFFICULT

John Stillwell
Monash University

At one time or another most of us have had a teacher who tried to persuade us that mathematics was easy. We were told that mathematics has only a few basic concepts, a few basic facts, and everything follows by a few principles of logic. Some have gone so far as to say that the subject is suited only to feeble or lazy intellects, incapable of storing the large amounts of information required for subjects like history, languages or chemistry.

So why do we all find mathematics difficult?

The answer, I believe, lies in a curious one-sidedness about mathematical knowledge. Solutions of problems are hard to find, but once found they are often easy to check. This is what makes it possible for a teacher or a textbook writer, who knows the solution, to make a problem look easy. I can illustrate this by a simple mathematical question: what are the factors of *99208417133957* ?

To all of you this is unknown and difficult, and you could only find the factors by systematic search, perhaps taking weeks. But to the poser of the problem (me) it is known that

$$9918851 \times 10002007 = 99208417133957$$

(because I picked these numbers out of a table of primes and multiplied them) and I can claim that this fact follows by multiplication, which any numerate child can do! The point is, the problem is easy <u>only</u> when the answer is known, because no one knows an easy way to factorize large numbers. At present, factorization is an art which depends on clever guessing as much as brute computation.

The one-sidedness of the factorization problem may only be apparent; perhaps it will be overcome by new discoveries which make quick factoring possible. However, research in logic has shown a genuine, unavoidable one-sidedness in certain mathematical problems, and evidence is accumulating that problems as simple-looking as factorization may indeed be difficult, in the sense that the only way to solve them in reasonable time is by clever guessing.

The next section deals with a form of one-sidedness that is known to exist in mathematics.

<u>Recursively enumerable sets</u>

Most sets of whole numbers that we meet in mathematics can be generated by mechanical rules. Here are four examples, in order of increasing complexity.

(1) $\{positive\ integers\} = \{1, 2, 3, 4, 5, 6, \ldots\}$

Rule: start with 1 and keep adding 1.

(2) $\{even\ numbers\} = \{2, 4, 6, 8, 10, \ldots\}$

Rule: start with 2 and keep adding 2.

(3) $\{prime\ numbers\} = \{2, 3, 5, 7, 11, 13, 17, 19, \ldots\}$

Rule: for each number x in turn from $2, 3, 4, \ldots$ try dividing x by all smaller numbers, and if none divide it, put x in the set.

(4) $\{x \mid y^2 - yx - x^2 = 1$ for some $y\}$

Rule? Here we have to be careful. We cannot say: for each number x from $1, 2, 3, 4, \ldots$ try $y = 1, 2, 3, 4, \ldots$ and see whether $y^2 - yx - x^2 = 1$; because this will never get us past $x = 2$. We will unsuccessfully try $y = 1, 2, 3, 4, \ldots$ without getting $y^2 - 2y - 4 = 1$. To avoid getting stuck on any value of x we must try only finitely many y for a given x before going back to smaller x values that are not yet settled. One method is to divide the computation into stages, at stage s trying all values $y \leqslant s$ for each $x \leqslant s$. Then if $x \leqslant y$ and $y^2 - yx - x^2 = 1$ we will discover this fact at stage y, and then put x in the set.

The set in example (4) differs from the previous ones in that we don't get its members in increasing order, and hence it is not clear how to find the x's not in the set. Actually, enough is known about the equation $y^2 - yx - x^2 = 1$ to enable us to do this, but only by another method. The importance of the method given is that it can be used for any algebraic equation which we may know nothing about, even one with a series of variables $y_1, \ldots, y_m$ in place of y. We make this our last and most general example of set generation.

(5) $\{x \mid p(x, y_1, \ldots y_m) = 0$ *for some positive integers* $y_1, \ldots y_m\}$
where p is a polynomial with integer coefficients.

Rule: at stage s try all values $x, y_1, \ldots, y_m \leqslant s$ and see whether $p(x, y_1, \ldots, y_m) = 0$. If so, put x in the set.

A set which can be generated by a mechanical rule is called recursively enumerable. This idea can be made precise in several equivalent ways. One way is to take a general purpose computer and let the rules for set generation be programs. A list of all programs can itself be generated, as $P_1, P_2, P_3, \ldots$ say. This gives us the ability to find any member of any recursively enumerable set: if S_n is the set generated by P_n, and $x \in S_n$, then we shall find x by simply running P_n long enough. However, this knowledge of the S_n is one-sided - we do not know how to find their <u>non</u>-members. In the next section we shall prove that there is no mechanical way to do this, even for one specific recursively enumerable set.

From now on we abbreviate "recursively enumerable" to "r.e.".

<u>The diagonal argument</u>

We shall now use a classic argument of George Cantor (1845-1918), known as the "diagonal argument". Cantor points out that for any list of sets $S_1, S_2, S_3, \ldots$ we can define a set D not on the list by

$$D = \{n \mid n \notin S_n\}.$$

Then D differs from each S_n - if $n \in S_n$ then $n \notin D$, if $n \notin S_n$ then $n \in D$ - and hence is not on the list. (The construction of D is called "diagonal" because if we make an infinite table with rows $S_1, S_2, S_3, \ldots$ and columns $1, 2, 3 \ldots$, putting a mark in the (i,j) position if $i \in S_j$, then to build D we work just on the diagonal of the table, for example,

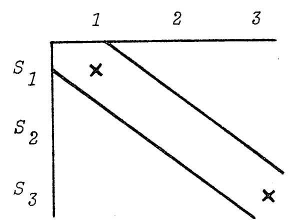


Figure 1.

Figure 1 shows $1 \in S_1$, $3 \in S_3$, and we make D different from these sets at these positions.)

Thus when $S_1, S_2, S_3, \ldots$ are the r.e. sets, D is a specific non-r.e. set, and hence there is no mechanical rule for listing its members. Now for the surprise: D is the complement of an r.e. set! The complement of D is

$$D' = \{n \mid n \in S_n\}$$

and we can list the members of D' in stages: at stage s run each of the programs $P_1, \ldots, P_s$ for s steps, and if P_n is found to list n, put n in D'.

This is conclusive proof that there are "one-sided" sets: the members of D' can be listed mechanically, but not the members of its complement. It follows that there is no mechanical rule to decide, for given n, whether $n \in D'$. If there were a rule to do this, applying it to $1, 2, 3, \ldots$ in turn and keeping the numbers for which the answer was "no" would be a mechanical listing of D.

Various forms of this result were found by the logicians Gödel, Church and Turing in the 1930's, but they had little impact on mathematics at the time, because the set D seemed to be of no mathematical interest. The picture has changed as more mathematical definitions of r.e. sets became available, especially since 1970 when the Russian mathematician Matiasevich found a definition in terms of equations.

Definition of r.e. sets by polynomial equations

Our example (5) showed that if $p(x,y_1,\ldots,y_m)$ is a polynomial with integer coefficients then

$$\{x \mid p(x,y_1,\ldots,y_m) = 0 \text{ for some positive integers } y_1,\ldots,y_m\}$$

is an r.e. set. Matiasevich's theorem is that <u>any r.e. set can be defined in this way</u>. This amazing result shows that recursive enumerability is really as elementary an idea as polynomial equations, yet at the same time it explains why some equations are hard to solve. In particular, there is a polynomial p' such that

$$D' = \{x \mid p'(x,y_1,\ldots,y_m) = 0 \text{ for some positive integers } y_1,\ldots y_m\}$$

and hence the non-solutions x of $p'(x,y_1,\ldots,y_m) = 0$ are just the elements of the non-r.e. set D. It follows that there is no mechanical rule to decide which values x are solutions of the equation and which are not since such a rule would enable us to generate D.

If you are wondering why I don't write down the polynomial p' which defines D', it is because the simplest one known has degree 4 and 153 variables, and its discoverers didn't think it was pretty enough to publish.

Feasibility

The fact that some quite natural sets are not r.e. shows that there are many things we can never hope to know. This can be shrugged off with remarks like "who wants to solve $4th$ degree equations anyway?", but worse is to come. Many of the things we thought we could do are actually not feasible to carry out. Processes which are easy to describe can take so long to execute that the universe will collapse (or maybe, repeat itself) before we get an answer. The factorization problem mentioned at the beginning of this article gives an example. Nothing could be simpler than dividing a number n by $2, 3, 4, \ldots, n-1$ to see whether n has factors. It is also easy to write down a number of, say, *50* digits. But then there are about 10^{50} numbers less than n and there is no way it is feasible to divide by them all, even using the fastest computers.

In the past decade a new discipline of complexity theory has developed, which attempts to measure the complexity of mathematical problems. We assume that a "problem" consists of infinitely many instances or "questions" so that a method of solution is required, rather than, say, looking up answers in tables. Then a problem is intrinsically complex if any method of solution takes time which is long in comparison with the length of questions.

Only a few problems seem to be intrinsically simple. For example, addition. The time taken to add two n digit numbers is proportional to n if one uses the usual method: write the numbers one above the other and scan from right to left, doing the carries from memory of a fixed table of addition facts ("*7* plus *9* gives *6* carry *1*" etc.). Since each carry is absorbed immediately, the scan can be made at constant speed (provided one's memory contains all addition facts for $0, 1, \ldots, 9$!) and hence total time is proportional to n. This is fast, since we produce the answer virtually as soon as we have read the question. Multiplication takes a little longer with the usual method, of order n^2 for n digit numbers a, b, since the products of a by all the n digits of b have to be worked out. Time n^2 can also be regarded as feasible for any n digit numbers we would bother to write down.

In general, if each instance of a problem can be solved in time bounded by a polynomial function of the "length" of the instance then we say we have a polynomial time solution and that the problem belongs to class P. Length is usually measured by the number of symbols, thus in the case of addition and multiplication it is the number of digits in the numbers, and we have just shown that addition can be done in time proportional to length, multiplication in time proportional to $(\text{length})^2$. Thus we have polynomial time solutions to these problems, and addition and multiplication are in class P.

In contrast, just listing all numbers less than an n digit number takes about 10^n steps, or exponential time. Exponential functions like 10^n, or even 2^n, grow much faster than any polynomial, becoming astronomical in size even for small values like $n = 50$, hence exponential time solutions are not regarded as feasible. Complexity theory has been able to show, by diagonal arguments, that certain (solvable) problems in logic can be solved only in exponential time. We do not know yet whether this is true of the factorization problem. It does seem reasonable to expect, however, that complexity theory will eventually find some natural mathematical problems which require exponential time for their solution. In the next section some candidates for this position are discussed.

NP-complete problems

The apparent one-sidedness of the factorization problem can be expressed in terms of complexity; a systematic search of all potential factors takes exponential time, but a correct guess takes only polynomial time to check (since division of numbers with $\leqslant n$ digits takes around n^2 steps). Thus the time taken to solve the problem drops from exponential to polynomial if only we allow the "non-deterministic" step of a correct guess. There are many other problems like this; here are two of them.

(1) Knapsack problem. Given a set $Y = \{y_1, \ldots, y_m\}$ of numbers, and a number x, is there a subset of Y whose members sum to x? (Think of x as the capacity of a knapsack, and $y_1, \ldots, y_m$ as sizes of certain objects. We want to know whether objects can be selected so as to exactly fill the sack.) There are 2^m subsets of Y (why?), so trying them all takes exponential time; but if we correctly guess the right numbers among $y_1, \ldots, y_m$ we can quickly add them and check that the total is x.

(2) Travelling salesman problem. A salesman wants to visit n cities $C_1, \ldots, C_n$ using the shortest possible route. Given the distance between each pair of cities and a number x, can we decide whether there is a route of total length $\leqslant x$ which visits all cities? In general the number of routes grows exponentially with n, but again the problem is easy if a correct guess is made first. We have only to add the lengths of the steps between successive cities on the route guessed, and check that it comes to $\leqslant x$.

The interesting feature of "good guessing" in these problems is that it is one-sided. If there is a solution, then it can be quickly confirmed by a correct guess. But if there is no solution, guessing doesn't seem to help. For example, in the event that we have an instance of the knapsack problem with no solution, there seems to be no alternative but to check all 2^m subsets of $\{y_1, \ldots, y_m\}$ and show that none of them sum to x.

Problems whose solutions can be obtained in polynomial time assuming correct guesses are called *NP* problems. (*N* for non-deterministic, *P* for polynomial). It is conjectured that some *NP* problems can be solved in polynomial time <u>only</u> non-deterministically, i.e. by making correct guesses. This is called the "$NP \neq P$ conjecture". Strong candidates in support of $NP \neq P$ are a class of problems, including knapsack and travelling salesman, called <u>*NP*-complete</u>. It has been proved that if any one of these problems can be solved in polynomial time without guessing, then so can all *NP* problems. The *NP*-complete class includes hundreds of well-known problems about equations, paths in networks, map colouring, scheduling and time-tabling. Since many people have worked on these problems without success, the $NP \neq P$ question has become one of the major unsolved problems in mathematics.

If it turns out to be true that *NP* ≠ *P*, this will not only help explain why mathematics is difficult, it will also confirm that imagination and guessing are essential in solving difficult problems.

STONEHENGE AND ANCIENT EGYPT: THE MATHEMATICS OF RADIOCARBON DATING

Malcolm Clark
Monash University

How old is old? For all of us, the passage of time is inexorable, and yet intangible. It is little wonder that events which occurred long before recorded history have held a peculiar fascination. Consider, for example, Stonehenge, the huge and enigmatic monument of great sarsen stones on the Salisbury Plain in southern Britain. Who built it? Why, how and *when* was it built?

The dating of events in man's past is the central problem facing archaeologists and prehistorians. Prior to 1950, almost all archaeological dating was done on the basis of the similarity, or otherwise, of items (such as bronze spearheads, fragments of pottery, etc.) found at various archaeological sites. Radiocarbon dating, developed in 1950 by W.F. Libby, provided for the first time a method of dating which was independent of archaeological assumptions, apparently sufficiently accurate, and widely applicable.

Radiocarbon, or carbon-14, is produced in the upper atmosphere by the action of cosmic-ray-produced neutrons on N^{14} atoms. This resulting radioactive isotope of carbon behaves chemically like the other non-radioactive isotopes, C^{12} and C^{13}, and so in particular, combines with oxygen to form carbon dioxide. After relatively rapid mixing in the atmosphere, this (radioactive) carbon dioxide becomes absorbed by plants and animals. While the plant or animal remains alive, any carbon-14 which decays away is presumed to be replaced by "fresh" carbon-14 from the atmosphere. Under such equilibrium conditions, the concentration of C^{14} at any particular time in the past is assumed to have been the same in all living organisms (and in the atmosphere).

However, once an organism dies, its radiocarbon is no longer replaced but decays exponentially, at a known rate. Suppose now that we have a sample from an archaeological site, say a piece of charcoal. This sample was once a living organism (and so would have absorbed carbon-14), but it died when burnt say x years ago. In this context, the exponential decay of carbon-14 may be represented by the equation

$$A_m = A(x)e^{-\lambda x} , \qquad (1)$$

where $A(x)$ denotes the concentration of carbon-14 in the sample when it died x years ago, and A_m the concentration in the sample *now*. The parameter λ is related to T, the half-life of carbon-14, by the equation $\lambda = (\ln 2)/T$. For any sample, its carbon-14 concentration at the end of *any* interval of T years is exactly half its concentration at the beginning of that time-interval. For carbon-14, $T = 5730$ years, so that $\lambda \simeq 0.000121$.

For most archaeological samples, both x, its "true" age, and $A(x)$, its initial carbon-14 concentration, are unknown. In order to proceed, Libby made the crucial assumption that the global concentration of carbon-14 in the biosphere has remained constant over at least the last 60,000 years (corresponding to about 10 half-lives). In other words, $A(x) \equiv A_0$, the concentration of carbon-14 in living matter *now*. Under this assumption, the radiocarbon age, y, of our sample is found by solving the equation

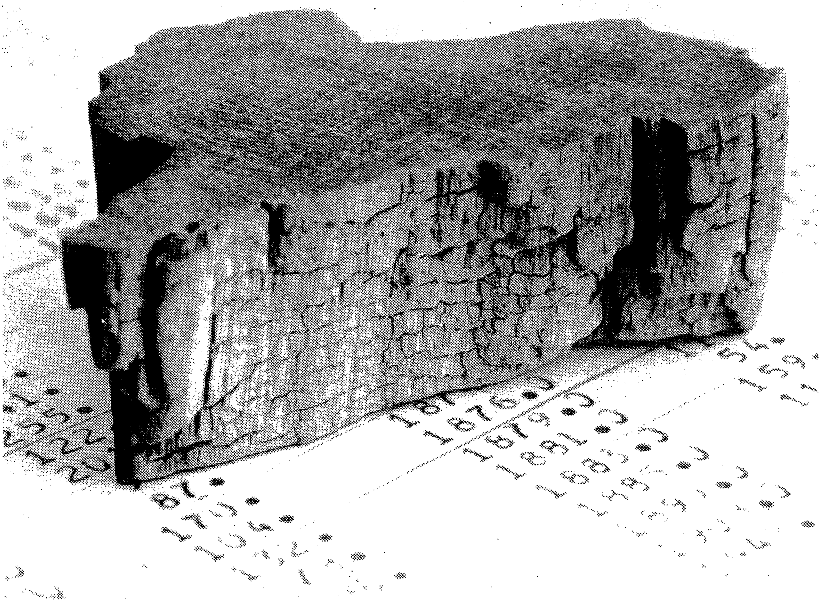
$$A_m = A_0 e^{-\lambda y} , \qquad (2)$$

giving

$$y = -\frac{1}{\lambda} \ln \left[\frac{A_m}{A_0}\right] \qquad (3)$$

Libby was well aware that the accuracy of radiocarbon dating rested heavily on the validity of the latter assumption. Clearly, the only way to check such assumptions was to obtain radiocarbon dates from a series of samples whose age is known independently by other means. This was done, in the first instance, using objects from certain archaeological sites in Egypt and the Middle East, such as pieces of wood, flax or linen from inside the tombs of the Pharoahs. The ages of such samples could be determined to within about 100 years from the Egyptian historical calendar. The agreement between these historical dates and their corresponding radiocarbon dates was good, but not exact. Perfect agreement was not expected since, as we shall see later, radiocarbon dates contain unavoidable random errors of measurement, and for these samples, the historical dates were not known exactly either.

Tree-ring dating provided the opportunity for a more precise and extensive comparison, especially when applied to samples of bristlecone pine. These trees, growing at an altitude of 3000 metres in California, live for up to 4,600 years. Samples of wood from any living bristlecone pine tree can be dated directly, with virtually no error, simply by counting annual growth rings, working backwards from the outermost ring. Furthermore, it has been possible to date fragments of bristlecone pine which are even older than the oldest living tree, by looking at patterns in the ring-widths. By matching ring-width patterns from successively older trees, Ferguson (of the University of Arizona) has built up a tree-ring chronology of bristlecone pine extending back 8200 years, with at most 2 years error.



This is a sample of bristlecone pine, about 6000 years old. The sample represents about 200 years of growth, with about 3 rings per mm.

Over 1000 tree-ring-dated samples of bristlecone pine have now been radiocarbon-dated as well. These results show that for samples up to 3000 years old, the radiocarbon dates and tree-ring dates agree quite well. But beyond 3000 years ago, the radiocarbon dates diverge progressively from the corresponding tree-ring dates, the former being systematically too young (i.e. more recent) by up to 700 years. (See Figure 1.)

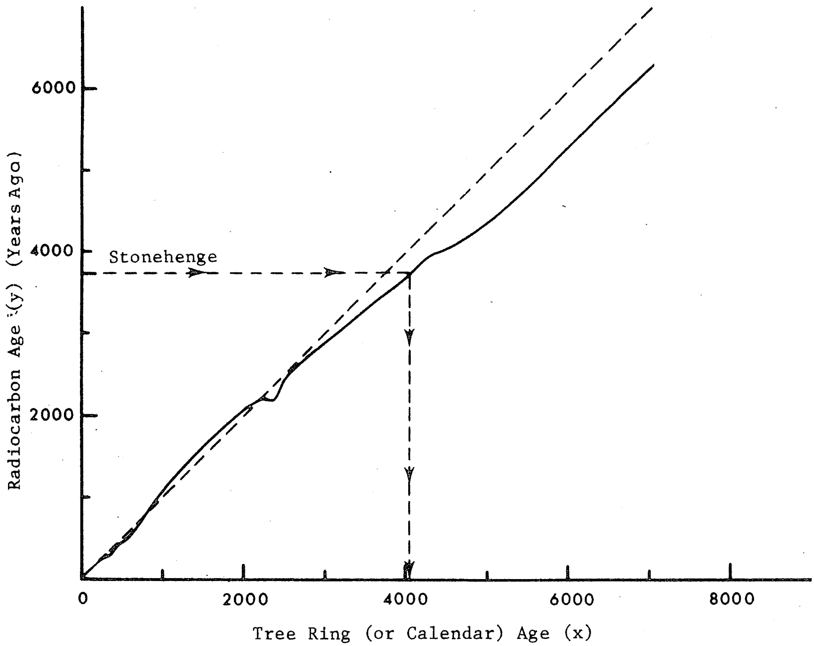


Figure 1.

Since the tree-ring dates are correct to within 2 years or so, what has gone wrong? If we substitute equation (1) in equation (3) and simplify, we obtain the equation

$$y = x - \frac{1}{\lambda}\,\ell n\left[\frac{A(x)}{A_0}\right] \tag{4}$$

defining the theoretical relationship between radiocarbon dates y, tree-ring dates or calendar ages x, and past levels of carbon-14 in the atmosphere, $A(x)$. The measurements on the bristlecone pine samples demonstrate unequivocally that, for $x > 3000$, $y < x$. This implies, from (4), that

$$\ell n\left(\frac{A(x)}{A_0}\right) > 0, \text{ or equivalently, } A(x) > A_0.$$

(See Figure 2.) Thus, contrary to Libby's original assumption, the atmospheric concentration of carbon-14 has not remained constant. The relatively high concentration from 8000 to 3000 years ago could have been caused by changes in the earth's magnetic field, the earth's climate, or solar activity. Indeed, the bristlecone pine data is now of equal interest to geophysicists as to archaeologists.

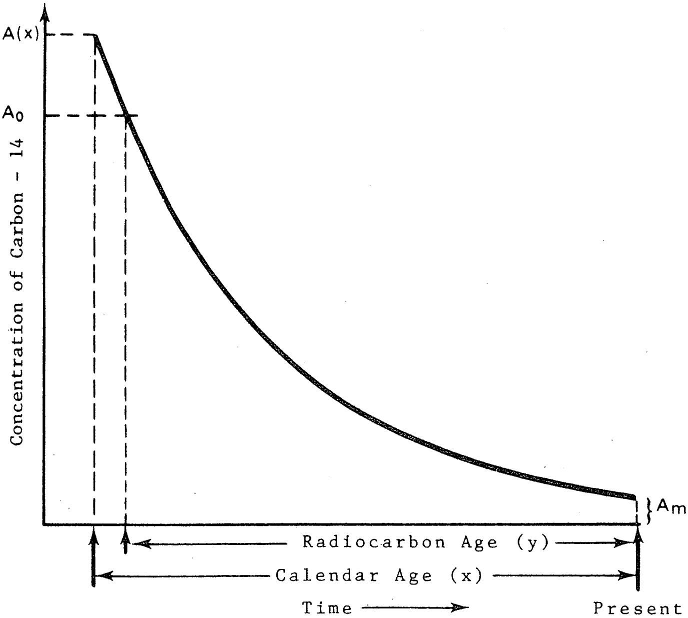


Figure 2.

In practice, to obtain the radiocarbon date of a sample, its current carbon-14 concentration, A_m, is measured indirectly, by counting the number of C^{14} atoms which disintegrate over a given time interval. Each disintegration results in the emission of a beta-particle. It turns out that the number, $N(t)$, of particles emitted per gram of carbon in an interval of t minutes, is a random variable having a Poisson distribution with parameter $A_m t$. In particular, the mean value of $N(t)$ is $A_m t$, which is also the variance of $N(t)$. Now the natural estimate of A_m is

$$\overline{N} = \frac{N(t)}{t} = \text{average rate of emission of } \beta\text{-particles.}$$

It then follows that $\overline{N}$, our estimate of A_m, is also a random variable, with mean value A_m and variance given by

$$\frac{1}{t^2} \times \text{Variance of } N(t) = \frac{1}{t^2} \cdot A_m t = \frac{A_m}{t} .$$

Thus, in principle, the variance of $\overline{N}$ can be made arbitrarily small simply by counting for a sufficiently long time. In practice, radiocarbon laboratories cannot afford to devote a very long time to any particular sample. Because of the random nature of radioactive disintegrations, all radiocarbon dates are subject to unavoidable random errors of measurement, with a resulting standard deviation of between 40 and 120 years for most samples.

To the archaeologist, it does not matter why radiocarbon dates are not always correct. All that he needs is a correction curve or table giving the empirical relationship between radiocarbon dates and tree-ring or calendar dates. The derivation of such a curve is essentially a statistical problem, since one must take account of the likely measurement errors in the radiocarbon dates, as specified by their standard deviations. Several alternative correction curves have been derived from the bristlecone pine data; these curves all show the same general trend but differ in their fine structure.

There have been numerous theories and speculations concerning the origin and purpose of Stonehenge. Some archaeologists have suggested that it was built by Bronze-Age people of the so-called Wessex culture, whose artifacts bear striking similarities to those found at Mycenae (in Greece). It has even been suggested that Stonehenge was built by craftsmen and builders from Mycenae, in which case Stonehenge would have been built around 1500 B.C., or 3480 years ago. This Mycenean connection appeared to be confirmed with the discovery in 1953 of stone carvings at Stonehenge similar to those at Mycenae. In 1959, the first radiocarbon sample from Stonehenge (an antler pick) yielded an uncorrected radiocarbon age of 3700 years, with a standard deviation of 150 years, and therefore not inconsistent with the hypothesis of Mycenean influence. However, the corrected radiocarbon age, after correction against the bristlecone pine data (see Figure 1), is 4000 years (2020 B.C.), some 500 years before Mycenae. Mycenean influence in the building of Stonehenge is clearly impossible.

Stonehenge is just one example of the archaeological implications of the correction curve for radiocarbon dates. In general, the revised radiocarbon dates show that the Neolithic and Bronze Age cultures in western and northern Europe are consistently older than the corresponding Aegean and Near Eastern cultures from which, according to the traditional "diffusion theory" of archaeology, they were assumed to be derived. Prehistorians can no longer regard the Near East as the ultimate source of European civilization.

Further reading:

1. C. Renfrew, "Ancient Europe is older than we thought". *National Geographic*, **152**, 615-623, November 1977.

2. C. Renfrew, *Before Civilisation: The Radiocarbon Revolution and Prehistoric Europe*, Penguin Books, 1976.

CURVE STITCHING AND SEW ON

P. Greetham
Boronia Technical School

Since its birth, Function magazine has been decorated a number of times with curve-stitching designs. (See Vol. 1 Pt. 5, Vol. 2, Parts 1, 2, 4, Vol 3 Pt. 3.) The beauty of many of them is undeniable. The purpose of this article is to consider them a little more closely.

Curve stitching seems to have been the creation of Mary Everest Boole (1832-1916). The life and work of Mary Boole have been paled by the brilliance of her better-known husband, George. She was, however, a fascinating and intelligent person and Bell[1] is perhaps a little unkind when he claims that George was "subconsciously striving for the social respectability that he once thought a knowledge of Greek could confer", when he married her. With ideas often beyond her time, she wrote on topics including early learning experience, mathematics education, love, the occult and vegetarianism. For a list of her books, see reference 2.

In contrast to her husband's humble birth, Mary's family was well-connected. One uncle was Sir George Everest the geographer, who had a mountain named after him. He was the first to survey Mt. Everest. Her father was a friend of Herschel and Babbage, founders of the Analytical Society which had a profound effect upon the development of mathematics in England. An aunt was professor of classics at Cork University, the university in which George was later to hold the chair of mathematics. Mary herself held a position at Queen's College in London.

Mary's interest in curve-stitching is related to her ideas on child learning and development. Whilst being a believer that young children should not be crammed with concepts and work too advanced for them lest it fall on "infertile soil", she advocated that they should have creative, thought-provoking, and interesting experiences so that later in life when ideas are more directly presented the seeds of understanding should have taken root from the childhood "play" activities. It was against this background that curve-stitching was developed.

At the risk of too-lengthy a quotation, she writes of her own early experience of this art: "In my young days cards of different shapes were sold in pairs, in fancy shops, for making needle-books and pin-cushions. The cards were intended to be painted on; and there was a row of holes round the edge by which twin cards were to be sewn together. As I could not paint, it got somehow suggested to me that I might decorate the cards by lacing silk threads across the blank spaces by means of holes. When I was tired of so lacing that the threads crossed in the centre and covered the whole card, it occurred to me to vary the amusement by passing the thread from each hole to one not exactly opposite it, thus leaving a space in the middle. I can now feel the delight with which I discovered that the little blank space so left in the middle of the card was bounded by a symmetrical curve made up of tiny bits of each of my straight silk lines, and that I could modify it by altering the distance of the down-stitch from the up-stitch immediately preceding." Early examples of the cards developed by Mary were called "Boole Curve-Sewing Cards". Mary introduced the subject to her friend Edith Somervell who consequently wrote a book entitled "A Rhythmic Approach to Mathematics", published in 1906, which became a popular text on what was to become known as curve-stitching. The growth in popularity over the past decade or so has often left curve-stitching as an art-form with little mathematics, so now let us tip the scales towards mathematics a little.

Basically, curve stitching consists of forming straight line segments by sewing thread between pairs of holes. If the holes are suitably located, the line segments are tangents to a smooth curve, and they outline the shape of that curve. We discuss several examples below.

1. The Parabola.

In Figure 1, S is a given point, the "focus", and the "directrix" is a given line.

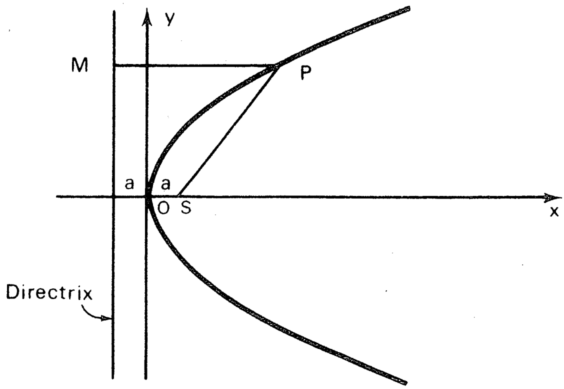


Figure 1.

The set of all points, P, such that $PS = PM$, forms a parabola whose cartesian equation, choosing coordinates as in Figure 1, is

$$y^2 = 4ax .$$

It is easily checked that any line, other than the y-axis, which is tangent to the parabola must have an equation of the form

$$y = mx + \frac{a}{m} \qquad (1)$$

Conversely, if we draw lines whose equation is (1), varying m from line to line, we will get a set of tangents to the parabola, and the shape of the parabola will be discernible from its tangents. In Figure 2, equally spaced

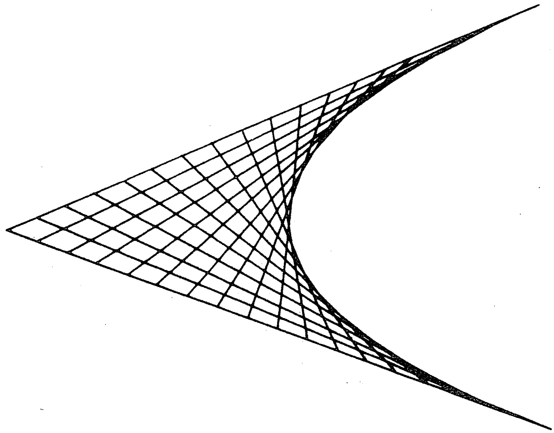


Figure 2.

points along two intersecting lines have been joined. It is an interesting exercise to show that, by suitably choosing an origin and axes, the stitched lines all have equations of the form (1), which explains why the lines appear to bound a parabolic region. The tangent lines are said to "envelop" the parabola.

2. The Cardioid.

Just as a parabolic envelope is based upon the intersection of two straight lines, the cardioid is based on the circle.

Several approaches may be taken. The cardioid may be constructed by drawing a base circle, dividing it into any number of equal arcs, and joining point *1* to *2*, *2* to *4*, etc., more generally x to $2x$, as shown in Figure 3.

Another construction is shown in Figure 4. Again a base circle is constructed and divided into an arbitrary number of arcs. A tangent is drawn to each point, and a perpendicular to each tangent drawn to a point *0* on the circle. The locus of the intersections of the perpendiculars with the tangents is a cardioid. If we use this last construction, we may derive the equation of the cardioid as follows. We refer to Figure 5.

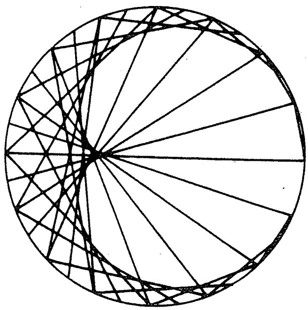


Figure 3.

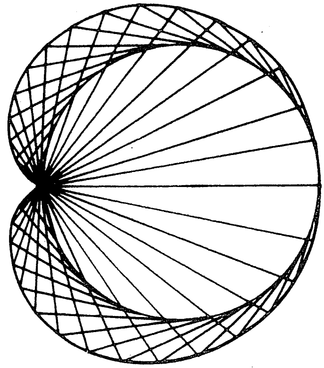


Figure 4.

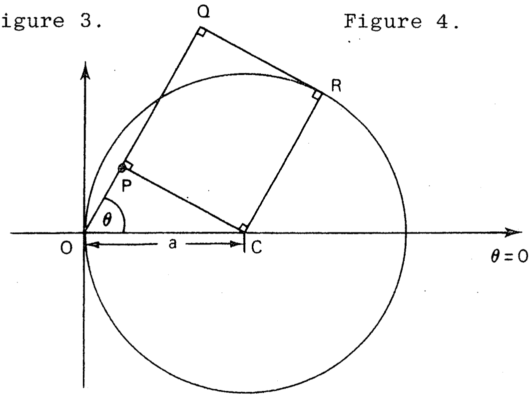


Figure 5.

Let Q be a point on the cardioid, R be a point on the base circle, centre C, radius a, θ be the angle QOC, and $OQ = r$. Then $OQ = PQ + OP$. Now $OP = a\cos\theta$ and $PQ = CR = a$, so that

$$OQ = r = a(1 + \cos\theta),$$

which is the polar equation of a cardioid.

3. The Astroid.

The astroid (or four-cusp hypocycloid), see Figure 6, is sometimes described as the envelope of a line segment of fixed length which slides between two perpendicular lines.

For example, think of a ladder as it slides down a wall,

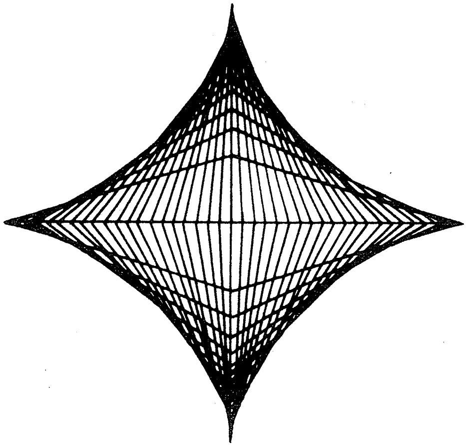


Figure 6.

We shall discuss the cartesian equation of the astroid by using the notation of Figure 7. We shall consider only the first quadrant part of the curve. Let RQ be a line

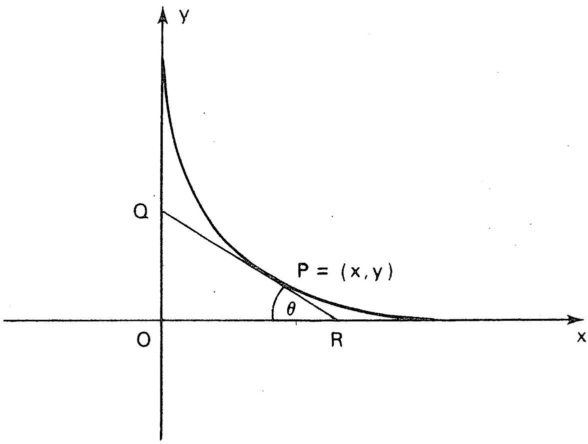


Figure 7.

segment of unit length. Its equation is

$$y = -(\tan \theta)x + \sin \theta. \tag{2}$$

Consider now the curve given parametrically by

$$x = \cos^3\theta, \qquad y = \sin^3\theta, \tag{3}$$

so that its cartesian equation is

$$x^{2/3} + y^{2/3} = 1 . \tag{4}$$

The curve has a derivative got by differentiating (3),

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} ,$$

and hence the tangent to the curve at the point $(\cos^3\theta, \sin^3\theta)$ has equation

$$y - \sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - \cos^3\theta) ,$$

which may be simplified to agree with (2). Hence the tangent lines (2) envelop the curve with equation (4), an astroid.

4. Straight Lines.

In the above situations, the curves have been generated by a set of straight lines. Although it would be more difficult to stitch, it is possible for a pair of straight lines to be generated by a family of circles. In the figure on the front cover the set of circles

$$(x - a)^2 + y^2 = \frac{1}{2}a^2$$

generates the straight lines $y = \pm x$ as a varies as a parameter.

Of course the preceding examples are only a few of the myriad of designs possible, but they are further limited in that they are in two dimensions only. From its inception, curve stitching in more dimensions has been considered.

Quoting again from Mary Boole: "The use of the single sewing cards is to provide children in the kindergarten with the means of finding out the exact nature of the relation between one dimension and two.

There is another set of sewing cards which is made by laying two cards side by side on the table and pasting a tape over the crack between them. This tape forms a hinge. You can lay one card flat and stand the other edgeways upright, and lace patterns between them from one to the other.

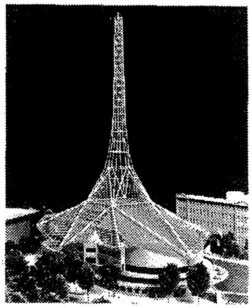
The use of this part of the method is to provide girls [Mary taught in an all-girls school] in the higher forms with a means of learning the relation between two dimensions and three.

There is another set of models, the use of which is to provide people who have left school with a means of learning the relation between three dimensions and four.

The use of the books which are signed George Boole or Mary Everest Boole is to provide reasonable people, who have learned the logic of algebra conscientiously, with a means of teaching themselves the relations between n dimensions and $n + 1$ dimensions, whatever number n may be."

(Maybe a kind reader would consider writing a future article on the models for the relation of three dimensions to four, or n to $n + 1$!),

In briefly considering three-dimensional curve stitching I regret that our two-dimensional paper is unable to accurately illustrate the excellent models which are possible. However, if we extrapolate the concept of curve-stitching slightly, to include rigid or semi-rigid materials forming the envelope, then we are all able to see some excellent examples in architecture. One current example is the spire to be constructed on the Melbourne Arts Centre.



In a recent interview with "The Age"[3] the Architect, Sir Roy Grounds, described something of its creation. The open-lattice-constructed version of the single-layered space frame was chosen in order that the wind load did not crush the buildings below. In a remarkable insight to the beauty, simplicity and humble birth of the spire, Sir Roy constructed the final concept model at home with string and pieces of wood. Following this "the drawings, hyperbolic paraboloids, had to be done by computer because almost every dimension was different." (How far curve-stitching has come from Mary Boole's kindergarten children!). And in reference to the beauty we feel for these forms: "I kept thinking of the Hudson River Bridge, and spider webs that I have seen time and time again with dew on them in the early morning".

A hyperbolic paraboloid is formed when a parabola is moved so that its vertex travels along another parabola (Figure 8). The surface so formed has the property that its intersection with a horizontal plane is a hyperbola; hence its name. A number of hyperbolic paraboloids can be pieced together to make attractive surfaces for roofs, spires, domes etc. especially when constructed by the relatively new reticulated space frame method. This allows for a thin, light surface requiring relatively little support and is therefore

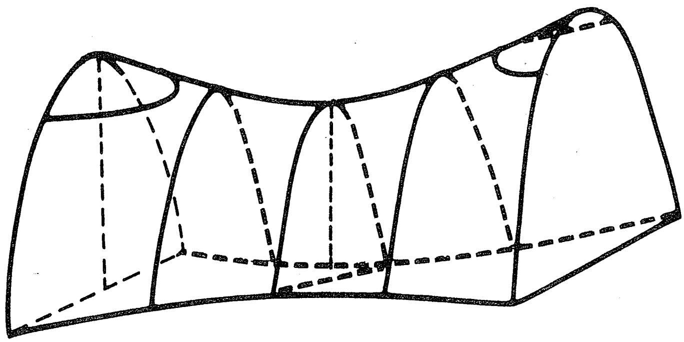


Figure 8.

more aesthetically pleasing than the older concrete construction.

Curve-stitching, therefore, is a topic which ranges from an interesting and educational activity for pre-school children to the most sophisticated forms of draughting and engineering possible today. At any point in this spectrum there is a curve, or a surface, the investigation of which is stimulating, interesting and always aesthetically pleasing. I leave this for the reader to explore.

References and Further Reading.

1. Bell, E.T. *Men of Mathematics.* Pelican.

2. *A Boolean Anthology - Selected Writings of Mary Boole,* published by the Association of Teachers of Mathematics.

3. "The Age", 25th October, 1979.

4. Fielker, D. *Mathematics and Curve Stitching.* Mathematics Teaching, September 1973.

5. Lowry, H.V. and Hayden, H.A. *Advanced Mathematics for Technical Students.*

WHO WERE THE FIRST MATHEMATICIANS IN THE SOUTHERN HEMISPHERE ?

Hans Lausch
Monash University

I still recall vividly a conversation that took place 11 years ago between myself and a young Aboriginal woman in Coober Pedy where I found myself stranded for five days with motor damage. While we were talking, a number of people entered or left the room, and on numerous occasions she pointed out to me that this or that person was her brother or her sister. To take her remarks literally would have meant to accept that she had some 30 brothers and sisters between the ages of 7 and 70. Was she just telling a story? Or is generous adoption a feature of indigenous societies? Or was she a missionary's victim, being told that we were all brothers and sisters? As it happens, there is a much more plausible answer to this question: applied algebra!

Let us first consider some of our own possible ancestors: the Romans, or the Anglosaxons and other Teutonic tribes. The Roman grandfather was "avus", the Roman uncle "avunculus", which incidentally is the origin of our term "uncle". "Uncle" thus was "little grandfather". Nephews as well as grandsons were called "nepos". The Anglosaxons had similar "confusions" with their relatives. In 18th century German, "Ohm" meant grandfather, "Oheim" uncle. A number of North American Indians still address grandfathers and uncles in the same way. Is there an explanation for this phenomenon?

In modern Western society it is commonly assumed that marriage is entirely a matter of free choice between individuals. There are, however, a number of constraints aimed at the avoidance of what is known as "incest": father-daughter marriages, mother-son marriages, and brother-sister marriages are outlawed, marriages between first cousins, uncle and niece, aunt and nephew are at least discouraged by various versions of Christianity. The comparatively large number of eligible individuals in our society facilitates the choice of spouses from outside the group of closest relatives. This situation changes drastically if a given society is small, e.g. consists of a few hundred individuals. Incest prohibition in both its narrower and wider meaning will lead to more rigid regulations and constraints on the choice of spouses. A very common solution is to divide the society into subsections which, for brevity's sake, we will henceforward call clans, and establish certain "contracts" between the clans pertaining to marriage and procreation. To be more specific, let S be the set of all clans, say $S = \{\alpha, \beta, \gamma, \ldots\}$, and let us subject S to the following "laws":

1. There is a permanent rule fixing the <u>single clan</u> among whose women the men of a given clan must find their wives. In other words, there is a rule or "wife function" W which assigns to each clan α a clan $W(\alpha)$ from which α-men have to take their wives.

2. Men from two different clans cannot marry women of the same clan. Note that without this rule there would be at least one clan whose females could not find husbands. Hence if $\alpha \neq \beta$, then $W(\alpha) \neq W(\beta)$.

3. All children of a couple are assigned to a <u>single</u> clan, uniquely determined by the clans of their mother and their father. This means that there is a rule or "child function" C which assigns to each clan α a clan $C(\alpha)$ to which the children of an α-man belong.

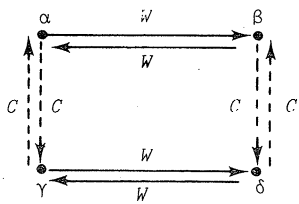
4. Children whose fathers are in different clans must themselves be in different clans. Were this not so, then within one generation there would be at least one clan which could not regenerate itself by way of newcomers, i.e. newborn children. Hence if $\alpha \neq \beta$, then $C(\alpha) \neq C(\beta)$.

5. (i) A man can never marry a woman of his own clan. Otherwise his own sister would be an eligible spouse - incest! Hence $W(\alpha) \neq \alpha$ for all clans α.

   (ii) $W(\alpha) \neq C(\alpha)$ for all clans α: this prohibits father-daughter marriages.

   (iii) $C(\alpha) \neq \alpha$ for all clans α: this effectively prohibits mother-son marriages. For if mother-son marriages were possible, then mother would be eligible to both father and son which by "law" 2 would mean that father and son belong to the same clan, α say; i.e. $C(\alpha) = \alpha$, violating the constraint we started with.

Before introducing our sixth and final law, let us briefly reflect upon the functions W and C . Laws 2 and 4 tell us that they must be so-called *1-1-functions*, and since there is only a finite number of clans, both W and C have the effect of permuting clans. An example which comes from some Aboriginal societies, will illustrate this:
Let $S = \{\alpha, \beta, \gamma, \delta\}$ be a 4-clan society. A solid arrow labelled W pointing from α to β will mean $\beta = W(\alpha)$, a broken arrow labelled C pointing from α to γ will mean $\gamma = C(\alpha)$, etc. The "picture" of this society then is



We may table the two functions W, C as follows:

x	$W(x)$	$C(x)$
α	β	γ
β	α	δ
γ	δ	α
δ	γ	β

We observe that all four clans appear in both the W- and the C-column, but in different arrangements, i.e. permuted. If we wish to find the clan to which a given clan gives wives away, we have to look for the given clan in the W-column and then cross over to the x-column, e.g. clan β gives its females to clan α since $W(\alpha) = \beta$. This allows us to define a "husband function" W^{-1}: $W^{-1}(\beta) = \alpha$ means $W(\alpha) = \beta$. Similarly, using C, we may define a "father" function C^{-1} by: $C^{-1}(\gamma) = \alpha$ if and only if $C(\alpha) = \gamma$. It is obviously true for all clans x that $W^{-1}(W(x)) = W(W^{-1}(x)) = x$, and $C^{-1}(C(x)) = C(C^{-1}(x))$. Brackets are a little bit cumbersome, and also unnecessary, so we may write $W^{-1}Wx$ for $W^{-1}(W(x))$, etc. How can we find the clans of other relatives, e.g. mother's brother's daughter's clan for an α-individual? Mother's clan is father's wife's clan, in short $WC^{-1}\alpha$. This is also mother's brother's clan. Mother's brother's daughter's clan is then $CWC^{-1}\alpha$, which is β in our example. We see that any string consisting of the functions W, W^{-1}, C, C^{-1} designates a relative, and vice-versa, any relative by blood or marriage can be described by such a string. This leads us to our last "law":

6. Whether two people who are related by marriage and descent links are in the same clan depends only on the kind of relationship, not on the clan either one belongs to. E.g. if it happens that my son's son is in my own clan, then every male's son's son will be in the same clan as this male. In other words, if for two relatives' strings R, S it is true that $R\alpha = S\alpha$ for one clan α, then $Rx = Sx$ for all clans x. This means that designating a clan amounts to designating a specified class of relatives. In our example, $x = CCx = WWx$ for all clans x, i.e. sisters, a male's son's daughter (CC), and a wife's brother's wife (WW) will all belong to the same clan, and to a member of this clan it will not be absurd to refer to all these individuals by the same term, e.g. "sister". Remember now the Aboriginal woman's story.

We have associated with each relative a certain string R consisting of functions W, W^{-1}, C, C^{-1}. If S is another string designating another relative, then by simply applying R first, and then S we can evaluate for each clan x a new clan, namely the S-relatives of the R-relatives of x, SRx; e.g. if $R = CW$ (wife's brother's children), $S = W^{-1}C$ (a male's daughter's husband), then $SRx = W^{-1}CCWx$ (wife's brother's son's daughter's husband of members of x), a rather intriguing relationship. For convenience we may invent the "empty" string I which stands for a blank, i.e. $Ix = x$, for all clans x . With this notation it is now easy to show that

(1) $(RS)T = R(ST)$, for any three strings R, S, T.

(2) $RI = IR = R$, for all strings R.

(3) for each string R there is a string R^{-1} such that

$$RR^{-1} = R^{-1}R = I;$$

e.g. if $R = WC^{-1}W^{-1}$ we take $R^{-1} = WCW^{-1}$, then $RR^{-1} = WC^{-1}(W^{-1}W)CW^{-1}$. Since $W^{-1}Wx = x = Ix$, we can simplify ("cancel out") the expression to $RR^{-1} = W(C^{-1}C)W^{-1}$; again $C^{-1}C$ cancels out, thus $RR^{-1} = WW^{-1} = I$, again by cancellation. Similarly we argue for $R^{-1}R = I$.

Statements (1), (2), (3) are nowadays known as the axioms of group theory, a mathematical subject that was initiated by the French mathematician Galois (1810-1832) about whom there is an article in another FUNCTION-issue (Vol 3, Part 2, April 1979). The graphic presentation of groups, i.e. the way we depicted our clan system as a dots-and-arrow pattern dates back to the British mathematician Cayley (1821-1895). A group is called commutative (or abelian) if for any two elements R, S of the group (strings of W, W^{-1}, C, C^{-1} in our model), $RS = SR$ is true. Our 4-clan society is such an example (check!). But in a number of other Australian societies this fails to be so. Why? Suppose $RS = SR$ for any two strings. Then, in particular, $WC = CW$ and if we put C^{-1} on the right of each side of the equation, we obtain $WCC^{-1} = CWC^{-1}$. As CC^{-1} can be deleted, we have $W = CWC^{-1}$. We remember that for any clan x, $CWC^{-1}x$ is the clan of mother's brother's daughter, a certain type of first cousin. Our equation tells us that this clan coincides with Wx, the "wife-givers" of x. In other words, a certain type of first cousin is an eligible wife. To some societies this is not objectionable, to others it borders on outright incest.

The mathematical theory of groups produces a number of important theorems that can be applied to our societies. Here is one which when applied reads as follows: Suppose a society is such that

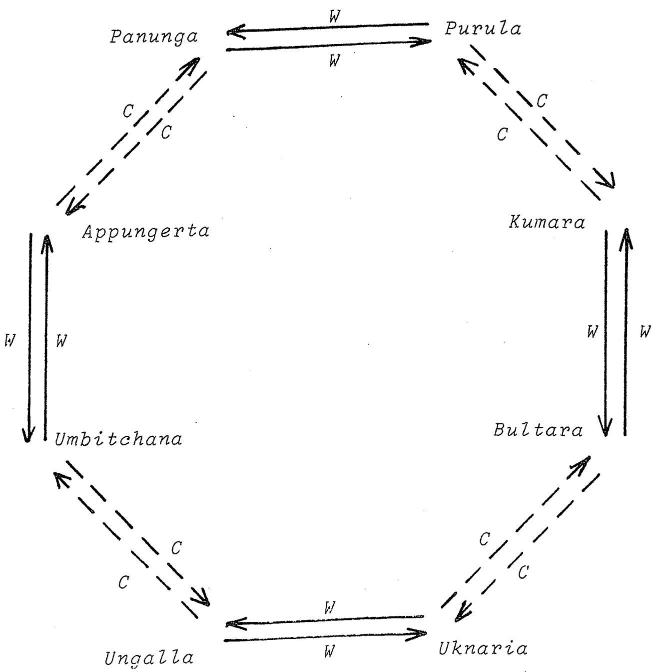
(1) any clan member can be a relative of any other clan member, i.e. for any two clans α, β there is a string R such that $\beta = R\alpha$;

(2) the number of clans in the society is either a prime number, or the square of a prime number.

Then a mother's brother's daughter is an eligible spouse.

In a sense the Aborigines were the first mathematicians of the southern hemisphere: they laid down formal rules which nowadays are most adequately described in mathematical terms. It is quite appropriate to note that modern Australian mathematics has a strong bias towards group theory. Many new and exciting results in group theory originate from Australia.

Finally, here is one society which you yourself may analyse. It is the model of the Arunta society of Central Australia whose most famous member was the painter Albert Namatjira. The model presented here is the one as conceived by the Aruntas themselves. It has eight clans. Anthropologists found, by analysing their kinship terms, that the society is more accurately described by a 16-clan model, each clan splitting into two subclans. Here we stick to the Aboriginal conception, and also present the original clan names instead of the symbols α, β, γ ...



FURTHER READING

Robin Fox, *Kinship and Marriage*. Penguin Books.

This paperback which has been reprinted a number of times informs the reader about the social use of kinship and marriage laws within societies and studies a wide variety of systems from all over the world. Easy to read for the non-expert.

Harrison C. White, *An Anatomy of Kinship*. Prentice-Hall, 1963.

A book for the anthropologist with little mathematical background, develops basic group theoretical ideas for the purpose of analysing societies and compares models with actual observations.

Claude Lévi-Strauss, *Elementary Structures of Kinship*. Presses Universitaires de France, 1949. Not too long ago, the author of this book gave a series of lectures on ABC radio on structural anthropology. He is one of the world's leading anthropologists who through his "structural" approach has taken anthropology one step closer to becoming an exact science: his ideas facilitate mathematical model building. In the appendix of his book, the famous French mathematician André Weil uses group theory the first time for the study of marriage systems.

Kemeny, Snell and Thompson, *Introduction to Finite Mathematics*. Prentice-Hall, 1960. This book offers a range of interesting applications of group theory; kinship systems are amongst them. The six "laws" for clans of this article are a variation of the axioms mentioned in this book.

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PROBLEM SECTION

MORE ON PROBLEM 1.2.6.

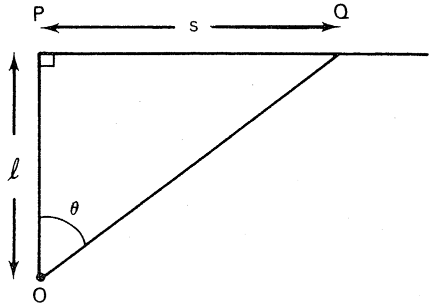
This was perhaps our best problem so far (depending on your taste !). Henry Finucan (University of Queensland) has drawn our attention to its being published in *Scientific American*, December 1979 p.20. There, it was slightly modified and the solution given was consequently wrong ! Martin Gardner promised to write at length on the problem in a later Scientific American, but we haven't seen that article yet. Incidentally, our version of the problem was told to us by Hans Lausch, who has an article in this issue on another matter. He heard it in Vienna.

SOLUTION TO PROBLEM 3.5.2

MANIKATO AND THE TV CAMERA

by M.A.B. Deakin and C.B. Fandry, Monash University.

This problem is based on Problem 25, Section 10(a) of Fitzpatrick and Galbraith's text for Victorian HSC Applied Mathematics.



A TV camera at O focusses on a horse entering the straight at P, with speed u, and accelerating along it at constant acceleration a. OP has length ℓ. If Q is a subsequent position of the horse $\angle$ O PQ is a right angle and $\angle$ POQ is θ (radians). If PQ = s, find the value of s for which $\dot{\theta}$ is maximised. (The dot indicates differentiation with respect to t, the time).

We have

$$s = \ell \tan \theta, \tag{1}$$

so that

$$\dot{s} = \ell\, \dot{\theta} \sec^2 \theta = \ell\, \dot{\theta}(1+\tan^2 \theta) = \ell\, \dot{\theta}\left(1+\frac{s^2}{\ell^2}\right). \tag{2}$$

We also have

$$\dot{s}^2 = u^2 + 2as. \tag{3}$$

From Equations (2), (3), we find:

$$\dot{\theta}^2 = \frac{\ell^2(u^2 + 2as)}{(s^2 + \ell^2)^2} \tag{4}$$

It would now be possible to write $s = ut + \frac{1}{2}at^2$, find $\dot{\theta}$ as a function of t, differentiate and set $\ddot{\theta} = 0$ and so on. This makes the problem very complicated. A much better method is to observe that if $\dot{\theta}$ is maximised, then $\dot{\theta}^2$ is also maximised.

Thus, use Equation (4) directly and set

$$\frac{d\dot{\theta}^2}{ds} = 0. \tag{5}$$

This gives, after some simplification,

$$3as^2 + 2u^2s - a\ell^2 = 0, \tag{6}$$

so that

$$s = \frac{-u^2 + \sqrt{u^4 + 3a^2\ell^2}}{3a}, \tag{7}$$

the required answer.

The problem, as stated by Fitzpatrick and Galbraith, refers to a car rather than a horse, but it gives values for u (13.2 ms^{-1}) and a (1.1 ms^{-2}) more appropriate to a good racehorse (say Manikato). The given value of ℓ is 60 m, so that we may use Equation (7) to deduce the given answer (10.35m). An extension of the problem might ask for the maximum value of $\dot{\theta}$. An explicit formula could be given here, but is clumsy. It is best to calculate s from Equation (7) and then use Equation (4) to find max $\dot{\theta}$ = 0.22 radius per second (2.1 rev./min).

It is best to solve the problem generally, as we have done here, and insert numerical values at the end. For one thing, this allows a number of checks on the working. The easiest of these is to compute the required s in the case $a = 0$. Equation (7) is inapplicable directly, but Equation (6) gives $s = 0$, which is clearly correct on geometrical grounds.

We have shown this problem to a number of mathematicians. Interestingly, many different techniques were used, some of them very elegant. The solution presented above, however, is probably the simplest derivation of Equation (7).

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MORE ON PROBLEM 3.4.1.

The solution given in the last issue assumed that in travelling through traffic lights, it was sufficient to start across the intersection whenever the lights are green. But if you are cautious and wise, you should also travel at sufficient speed to cross during the green and amber phases, and not get caught with the lights turning red. If you can use all the green and amber period to cross, your speed should be at least $\frac{20}{28+2} \times \frac{3600}{1000} = 2.4$ k.p.h. for the situation given in the problem. If you start out at the end of the green phase, your speed should be at least $\frac{20}{2} \times \frac{3600}{1000} = 36$ k.p.h.

SOLUTION TO PROBLEM 3.5.3.

Set $S_k(n) = 1^k + 2^k + 3^k +\ldots+ n^k$. When does n divide $S_1(n)$? $S_2(n)$? $S_3(n)$?

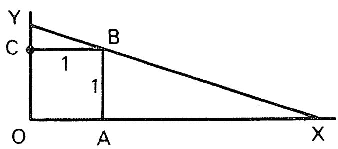
It can be shown, with some difficulty, that $S_1(n) = n(n+1)/2$, $S_2(n) = n(n+1)(2n+1)/6$ and $S_3(n) = n^2(n+1)^2/4$. Thus $S_1(n)$ is divisible by n if $(n+1)/2$ is an integer, i.e. if n is odd. $S_2(n)$ is divisible by n if $(n+1)(2n+1)$ is divisible by 2×3. But this requires that n be odd, otherwise neither $n + 1$ nor $2n +1$ is divisible by 2. Further, n must not be divisible by 3 so that one of $n + 1$ or $2n + 1$ is divisible by 3. $S_3(n)$ will be divisible by n if n is divisible by 4 or if n is odd.

The same condition as for $S_3(n)$ ensures that $S_k(n)$ is divisible by n for k odd, $k > 3$. $S_k(n)$ is divisible by n, for k even, if n is not divisible by $2,3$ or any other prime p for which $p - 1$ divides k. These general results are considerably harder to prove, though.

SOLUTION TO PROBLEM 3.5.4.

If the number of kilometres in a mile is given approximately as $8/5$, this presumably means that the simpler ratios $3/2$ and $5/3$ are less accurate. Thus $\frac{48}{30}$ is closer than either $\frac{45}{30}$ or $\frac{50}{30}$, so that the true ratio is between $\frac{46.5}{30} = 1.55$ and $\frac{49}{30} = 1.6\dot{3}$. (It is about 1.609, in fact.)

SOLUTION TO PROBLEM 4.1.1.



Find conditions under which XY and OY are both rational.

Following M. & J. Hirshorn, Austn. Math. Soc. Gazette 1979 Vol 6 No 2, let $AX = x$ and $CY = y$. We need $OY = y + 1$ and $XY = \sqrt{(x+1)^2 + (y+1)^2}$ to be rational. But YCB and BAX are similar triangles so that $y/1 = 1/x$. Thus we need y and $\sqrt{(\frac{1}{y} + 1)^2 + (y + 1)^2}$ to be rational. Now if $y = \frac{m}{n}$, m and n integers, then we need

$$\sqrt{\left(\frac{n}{m}+1\right)^2+\left(\frac{m}{n}+1\right)^2} = \frac{m+n}{mn}\sqrt{m^2+n^2}$$

to be rational. This will be the case only if m^2+n^2 is a perfect square, i.e. $m^2+n^2=r^2$ for some integer r.

This means the triangle OYX has sides

$$OY = y+1 = \frac{(m+n)m}{mn},\ OX = 1+\frac{1}{y} = \frac{(m+n)n}{mn},\ XY = \frac{(m+n)r}{mn}$$

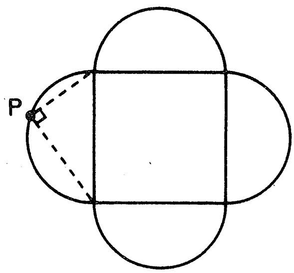
in the ratios $m:n:r$, that is, proportional to a "Pythagorean triad" of integers such that $m^2+n^2=r^2$. This is the required condition.

SOLUTION TO PROBLEM 4.1.2.

Find the positive numbers x, y, z such that $xy = z$, $xz = y$ and $yz = x$. Multiplying both sides of the first two equations yields $xyxz = zy$, and using the third, this reduces to $xxx = x$. The only positive solution is $x = 1$ (and similarly $y = z = 1$). The problem also asked whether it was possible to find six different positive numbers such that each is the product of two of the others. We leave this open for readers to attempt.

SOLUTION TO PROBLEM 4.1.3.

A point P, from which a given square subtends an angle of 90°, has a locus consisting of four semicircles, as in the diagram. An angle in a semicircle *is* a right angle.



SOLUTION TO PROBLEM 4.1.4.

In choosing a lady to be a wife, out of three presented in random order, a man would have probability $1/3$ of getting the *best* wife if he chose the first, or the second, or the third regardless of the other ladies. But if he ignores the first, chooses the second if she is better than the first, but otherwise chooses the third, the man has probability $1/2$ of getting the best wife. This is therefore the best strategy. The reason is that, if we label the wives g (good), b(better), and B (best), there are 6 orders of presentation possible:

(1) *gbB* (2) *gBb* (3) *bgB* (4) *bBg*

(5) *Bgb* (6) *Bbg*

In cases (2), (3), and (4), the strategy yields him the best wife. They have probability $3/6 = 1/2$.

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PROBLEM 4.3.1.

(Submitted by Colin Wright, Monash Science student.)

Given a cube, is it possible to cut a hole in it through which a larger cube can be passed?

PROBLEM 4.3.2.

In connection with the article on curve stitching in this issue, the following questions arise.

(i) Prove equation (1) on p.16, and its applicability to Figure 2.

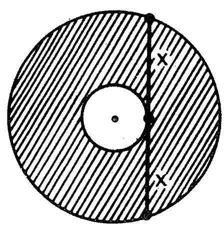
(ii) Show that the lines $y = \pm x$ are tangents to the circles $(x-a)^2 + y^2 = \frac{1}{2} a^2$ as a varies. [See front cover.]

(iii) The mid-point of a ladder sliding down a wall does *not* trace out an astroid. What *is* its curve?

(iv) The ladder in Figure 6 on p.18 has constant length and so the curve-stitching to produce the astroid is difficult. One cannot just sew stitches from equally spaced points on each axis. What curve *would* be obtained by joining equally-spaced points?

PROBLEM 4.3.3.

You wish to paint the shaded region in the target.



(i) Supposing that it is sufficient information to know the length, $2x$, of the chord in the diagram, what is the *area* to be painted? (No complicated calculations!)

(ii) Prove that $2x$ *is* sufficient information to find the shaded area.

MORE ON "SOME POWER-FULL SUMS" (FUNCTION 4.2 P.25)

The result $144^5 = 133^5 + 110^5 + 84^5 + 27^5$ was found by a computer, in a search initiated by Lander and Parkin, 1966, (Bulletin of Amer. Math. Soc., Vol 72, p.1079). The example disproves a conjecture of Euler, who thought that at least n n^{th} powers would be required to sum to an n^{th} power, $n > 2$. The other examples we quoted do not disprove the conjecture.

Garnet J. Greenbury, Brisbane. has sent us some examples of equalities in which the same digits occur on both sides. For instance,

$$153 = 1^3 + 5^3 + 3^3,$$

$$1233 = 12^2 + 33^2$$

$$438\ 579\ 088 = 4^4 + 3^3 + 8^8 + 5^5 + 7^7 + 9^9 + 0^0 + 8^8 + 8^8.$$

(You have to interpret $0^0 = 0$ for this last equation.). See if you can discover some others.

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> I do hate sums. There is no greater mistake than to call arithmetic an exact science. There are permutations and aberrations discernible to minds entirely noble like mine; subtle variations which ordinary accountants fail to discover; hidden laws of number which it requires a mind like mine to perceive. For instance, if you add a sum from the bottom up, and then again from the top down, the result is always different.
>
> — Mrs La Touche (19th century)

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There was a young fellow from Trinity,
Who took $\sqrt{\infty}$.

But the number of digits
Gave him the fidgets;

He dropped Maths and took up Divinity.

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