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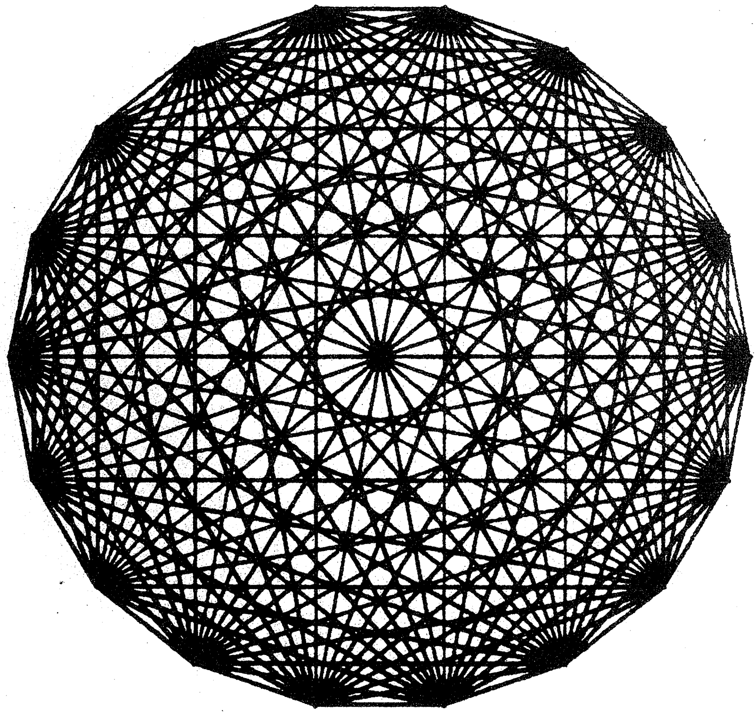

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Many people have judged David Hilbert to be the greatest mathematician who ever lived. Whatever this kind of judgement is worth, few would deny that his influence on twentieth century mathematics has been immense. Many would say that more than half of twentieth century mathematics has directly arisen in response to challenges set by Hilbert. One such set of challenges was given at the International Mathematical Congress in Paris in 1900. He was invited to survey past and future mathematics at this junction of the centuries. He offered twenty-three problems which he thought provided the key to important advances. Our leading article in this issue gives Max Dehn's solution of his third problem. The first part of the article is scissors and paper mathematics - what is discussed describes what can be done by cutting a sheet of paper in various ways. The rest of the article extends the discussion to three dimensions.

Good problems are the heart of mathematics. And Problem 2.6 of Volume 1 has proved this. We have had at least a hundred comments on or attempts at solutions of this problem. We print the elegant solution of Christopher Stuart in this issue. If you have not met the problem before, try to do it before you read the solution.

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HILBERT'S THIRD PROBLEM

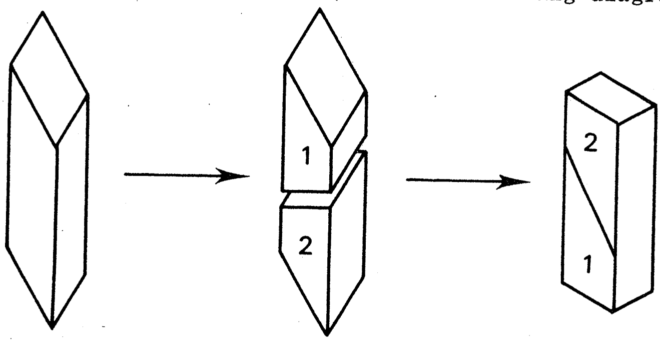
John Stillwell, Monash University

INTRODUCTION

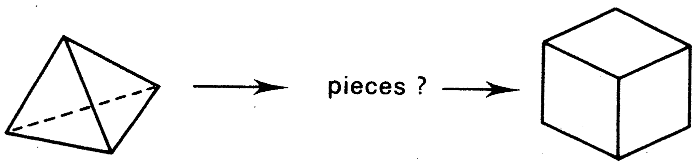
In 1900 the great German mathematician David Hilbert challenged mathematicians of the 20th century with a list of 23 problems. Hilbert's problems have been a great stimulus to mathematics, and some of them are still unsolved. The only one which turned out to be at all easy was the third, and it was solved by Max Dehn a few months after Hilbert proposed it. The problem is the following:

Given two polyhedra P and Q of equal volume, is it always possible to cut P into polyhedral pieces which re-assemble to form Q?

Polyhedra P, Q for which this is possible are called *equidecomposable*. For example, an oblique prism is equi-decomposable with a right prism, as the following diagram shows:



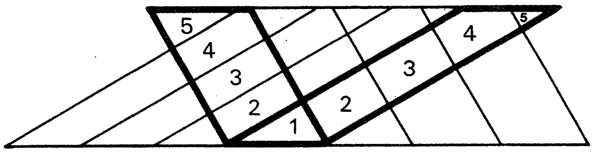
However it was not known (before Dehn) whether even a regular tetrahedron was equidecomposable with a cube.



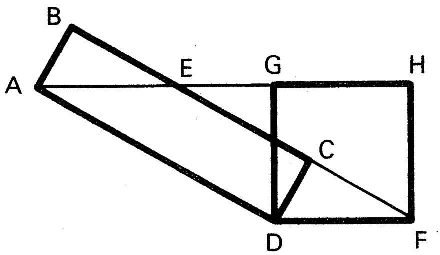
On the other hand, it was known that the corresponding result in the plane was true. Namely, if P and Q are polygons of equal area, then P and Q are equidecomposable. Two special cases of this are

1. *Parallelograms with the same base and height are equi-decomposable.*

As the diagram shows, filling a horizontal strip with copies of the two parallelograms automatically cuts the original parallelograms into corresponding pieces:



2. *Rectangles of the same area are equidecomposable*

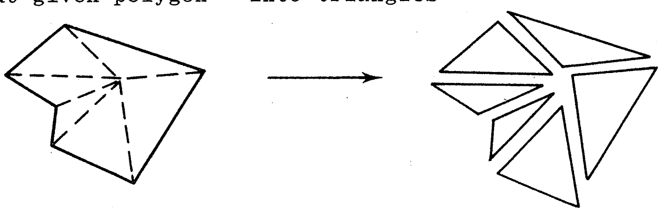


Rectangle *ABCD* is equidecomposable with parallelogram *AEFD*, which has the same base and height as the other rectangle, *GHFD*. Thus both rectangles are equidecomposable with the parallelogram, and hence with each other, as can be seen by superimposing two sets of cuts on *AEFD* - one set dividing it into pieces of *ABCD*, the other set dividing it into pieces of *GHFD*.

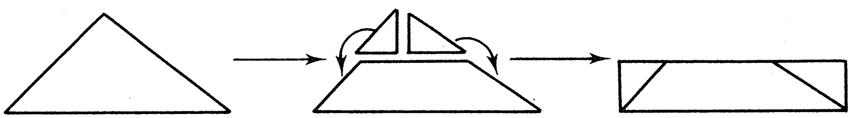
The general case follows from the two special cases in four easy steps.

a) Cut given polygon *P* into triangles

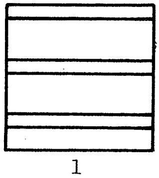
e.g.



b) Convert each triangle into a rectangle



c) Convert each rectangle into a rectangle of width 1 (using special cases 1 and 2) and stack them up. Thus

P $\longrightarrow$  = a rectangle R of height h, say.

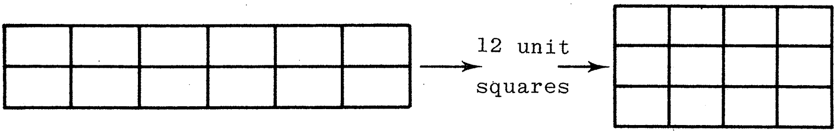
d) Now if Q is a polygon with the same area as P it will also be equidecomposable with R, and hence with P itself (by superimposing two sets of cuts on R, as in special case 2).

A RECTANGLE DECOMPOSITION PROBLEM

In proving that rectangles of the same area are equidecomposable, we first rotated one rectangle, then cut it obliquely. Is it possible that the decomposition can be made

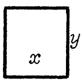
(i) without rotating pieces (i.e. just sliding them)
(ii) with all cuts horizontal or vertical?

Certainly this is possible if the rectangles have sides which are multiples of the same unit, e.g.



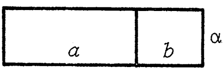
However, it is *not* possible otherwise. For example, it is not possible to cut the unit square into rectangles, then slide them to form the rectangle with sides $\sqrt{2}$ and $1/\sqrt{2}$.

To prove this, we shall convert the problem into algebraic form. Rectangles will all be placed with their sides vertical and horizontal and

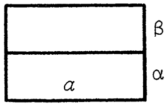


will be denoted by $x \otimes y$.

Since we are not allowing rectangles to be rotated, $x \otimes y$ is not necessarily equal to $y \otimes x$. We can form sum and difference of these symbols (called *tensors*) and two expressions will be considered equal just in case this follows from the rules (1), (2) below, which reflect the results of vertical and horizontal cuts:



$$(a + b) \otimes \alpha = a \otimes \alpha + b \otimes \alpha \quad \cdots \ (1)$$



$$a \otimes (\alpha + \beta) = a \otimes \alpha + a \otimes \beta \quad \cdots \; (2)$$

Similarly, or by deduction from (1) and (2), we can show that

$$(a - b) \otimes \alpha = a \otimes \alpha - b \otimes \alpha$$

$$a \otimes (\alpha - \beta) = a \otimes \alpha - a \otimes \beta.$$

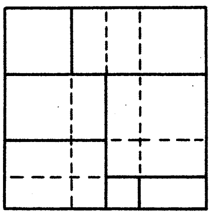
It follows that $a \otimes 0 = 0 \otimes \alpha = 0$, because

$$a \otimes 0 = a \otimes (y - y) = a \otimes y - a \otimes y = 0$$

and $\quad 0 \otimes \alpha = (x - x) \otimes \alpha = x \otimes \alpha - x \otimes \alpha = 0.$

Now observe that the problem of deciding whether the unit square decomposes into the same rectangular pieces as the rectangle with sides x and y is the same as the problem:

Does $\quad 1 \otimes 1 = x \otimes y$?



For the decomposition of the unit square can always be made by a series of vertical and horizontal cuts (dotted lines) corresponding to decompositions of $1 \otimes 1$ by rules (1) and (2), then recombining the resulting pieces into the given rectangles, again by rules (1) and (2).

It is possible to solve this algebraic form of the problem by introducing the ideas of *rational dependence* and *rational basis*.

Non-zero numbers $\alpha_1, \ldots, \alpha_n$ are called *rationally dependent* if there are rational numbers $r_1, \ldots, r_n$, not all zero, such that

$$r_1\alpha_1 + \cdots + r_n\alpha_n = 0.$$

For example, $\frac{1}{2}$ and $\frac{2}{3}$ are rationally dependent, because

$$2 \cdot \frac{1}{2} + (-\frac{3}{2}) \cdot \frac{2}{3} = 0$$

whereas $1, \sqrt{2}$ are rationally independent since any equation

$$r_1 \cdot 1 + r_2 \cdot \sqrt{2} = 0$$

would imply that $\sqrt{2}$ is rational, which is not the case. A *rational basis* for numbers $\alpha_1, \ldots, \alpha_n$ is a set of numbers $\beta_1, \ldots, \beta_m$ such that

(i) $\beta_1, \ldots, \beta_m$ are rationally independent

(ii) each α_i is a rational combination of the β's

i.e. $\alpha_i = p_1\beta_1 + \cdots + p_m\beta_m$ for some rationals $p_1, \ldots, p_m$. This expression for α_i is *unique*, for if two different rational combinations of the β's were equal their difference would be an expression $r_1\beta_1 + \cdots + r_m\beta_m = 0$ with not all $r_i = 0$.

A rational basis for $\alpha_1, \ldots, \alpha_n$ can be chosen from among the α_i's themselves as follows:

Step 1: Put α_1 is the basis.

Step 2: See if α_1, α_2 are rationally dependent. If not, put α_2 in the basis. Otherwise go to Step 3.

⋮

Step i: See if α_i is rationally independent of those previously chosen, and only then put it in the basis. Go to Step i + 1.

⋮

For example, 1, $\sqrt{2}$ is a rational basis for 1, $\sqrt{2}$, $1/\sqrt{2}$, $1/(\sqrt{2} - 1)$ because $1/\sqrt{2} = \sqrt{2}/2 = 0 \cdot 1 + \frac{1}{2} \cdot \sqrt{2}$ and $1/(\sqrt{2} - 1) = (\sqrt{2} + 1)/((\sqrt{2} - 1)(\sqrt{2} + 1)) = (\sqrt{2} + 1) = 1 \cdot 1 + 1 \cdot \sqrt{2}$.

Fundamental Tensor Theorem: If $\alpha_1, \ldots, \alpha_n$ are rationally independent then

$$x_1 \otimes \alpha_1 + \cdots + x_n \otimes \alpha_n = 0$$

holds only if all $x_i = 0$.

Proof. If $x_1 \otimes \alpha_1 + \cdots + x_n \otimes \alpha_n = 0$, the demonstration of this fact will be a series of expressions

$$\left.\begin{aligned} x_1 \otimes \alpha_1 + \cdots + x_n \otimes \alpha_n &= E_1 \\ &= E_2 \\ &\ \vdots \\ &= E_k = 0 \end{aligned}\right\}$$

each following from its predecessor by one of the rules (1), (2).

Now if F is any function with the *additive* property:

$$F(\alpha + \beta) = F(\alpha) + F(\beta)$$

it will satisfy the rules

$$(a + b)F(\alpha) = aF(\alpha) + bF(\alpha) \qquad \ldots (1)'$$

$$aF(\alpha + \beta) = aF(\alpha) + aF(\beta) \qquad \ldots (2)'$$

which have the same form as (1), (2), so we will be able to demonstrate that

$$x_1F(\alpha_1) + \cdots + x_nF(\alpha_n) = 0 \qquad \ldots (3)$$

via a corresponding series of expressions

$$\left.\begin{aligned} x_1F(\alpha_1) + \cdots + x_nF(\alpha_n) &= E_1' \\ &= E_2' \\ &\vdots \\ &= E_k' = 0 \end{aligned}\right\}$$ each following from its predecessor by one of the rules (1)', (2)'.

There is a lot of leeway in the choice of F, and we now proceed to show that it is possible to arrange

$$F(\alpha_i) = 1 \text{ and } F(\alpha_j) = 0 \text{ for } j \neq i$$

for any i we please. What makes this easy is that F *only needs to work on the numbers* α *mentioned in the expressions* $E_1', \ldots, E_k'$. There are only finitely many of these α's, so starting with $\alpha_1, \ldots, \alpha_n$ we can find a rational basis for them, say

$$\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_m.$$

Then if

$$\alpha = r_1\alpha_1 + \cdots + r_m\alpha_m$$

is the unique expression for α we can define

$$F(\alpha) = r_i \text{ (the coefficient of } \alpha_i\text{)}$$

and F is obviously additive, with $F(\alpha_i) = 1$, $F(\alpha_j) = 0$ for $j \neq i$.

Equation (3) then becomes

$$x_1 \cdot 0 + \cdots + x_i \cdot 1 + \cdots + x_n \cdot 0 = x_i = 0$$

and since the choice of i was arbitrary, all $x_i = 0$. QED

This theorem shows that the unit square and the rectangle $(1/\sqrt{2}) \otimes \sqrt{2}$ do not decompose into the same rectangular pieces because $1, \sqrt{2}$ are rationally independent and hence

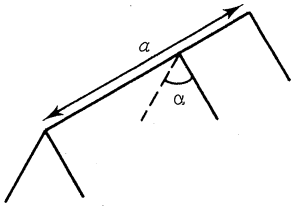
$$1 \otimes 1 - (1/\sqrt{2}) \otimes \sqrt{2} \neq 0$$

i.e. $$1 \otimes 1 \neq (1/\sqrt{2}) \otimes \sqrt{2}.$$

DEHN'S THEOREM

Dehn succeeded in dealing with the polyhedron problem by reducing it to a 2-dimensional problem similar to the rectangle problem above. The use of tensors greatly simplifies Dehn's argument, and was introduced by the Danish mathematician Borge Jessen in the 1960's.

Given a polyhedron P, we associate a tensor $a \otimes \alpha$ with each edge, where a = length of the edge, and
α = angle between the faces which meet along the edge.



The sum

$$a_1 \otimes \alpha_1 + \cdots + a_n \otimes \alpha_n$$

over all the edges of P is called the *Jessen functional* on P, $J(P)$. With these tensors we introduce the additional rule

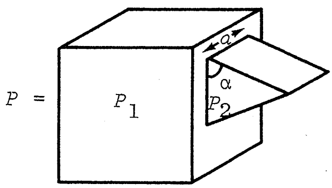
$$a \otimes \pi = 0 \qquad \ldots (4)$$

because it will then follow that if P consists of polyhedral pieces P_1, P_2 then

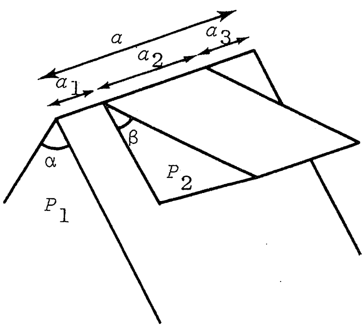
$$J(P) = J(P_1) + J(P_2) \qquad \ldots (5)$$

In other words, the value of the Jessen functional on the whole is the sum of its values on the parts, so if polyhedra P, Q decompose into the *same* parts, $J(P) = J(Q)$.

It is easy to see that (5) is true when none of the edges of P_1, P_2 overlap, e.g. in the example shown the edge a in P_2 with angle α becomes an edge in P with angle $\alpha + \pi$. However $a \otimes (\alpha + \pi) = a \otimes \alpha + a \otimes \pi = a \otimes \alpha$ by (4), so the contributions of edge a to $J(P_2)$ and $J(P)$ are the same. And when the edges do overlap, as in the example at left, the ordinary tensor rules on $J(P_1) + J(P_2)$ give



$$a \otimes \alpha \text{ (contribution from } J(P_1))$$
$$+ a_2 \otimes \beta \text{ (contribution from } J(P_2))$$
$$= (a_1 + a_2 + a_3) \otimes \alpha + a_2 \otimes \beta$$
$$= a_1 \otimes \alpha + a_2 \otimes (\alpha + \beta) + a_3 \otimes \alpha$$



which is the correct contribution of the edge a to $J(P)$ *except* perhaps when $\alpha + \beta = \pi$ and the edge a_2 "disappears". But in this case we are saved by the rule (4) which gives $a_2 \otimes (\alpha + \beta) = 0$, leaving the correct contribution to $J(P)$.

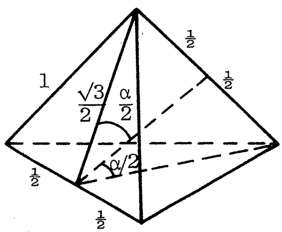
We can now calculate the Jessen functional on the tetrahedron, cube etc., and use information about rational dependence to see if the values are equal. The only difference in the algebra with the additional rule $a \otimes \pi = 0$ is that the fundamental theorem now reads:

If $\alpha_1, \ldots, \alpha_n$ *and* π *are rationally independent then* $x_1 \otimes \alpha_1 + \cdots + x_n \otimes \alpha_n = 0$ *only when all* $x_i = 0$.

The unit cube has 12 edges of length 1 and angle $\pi/2$, hence

$$\begin{aligned} J(\text{cube}) &= 12 \cdot 1 \otimes \pi/2 \\ &= 6 \otimes \pi \\ &= 0. \end{aligned}$$

We shall not worry about the edge length of the regular tetrahedron, however the edge angle α satisfies $\cos\alpha = 1/3$ since the diagram shows



$$\sin \alpha/2 = \tfrac{1}{2}/(\sqrt{3}/2) = 1/\sqrt{3}$$

hence

$$\begin{aligned} \cos\alpha &= 1 - 2\sin^2\alpha/2 \\ &= 1 - 2/3 \\ &= 1/3. \end{aligned}$$

To show that the tetrahedron and cube have different J values (and hence that they are not equidecomposable) it therefore suffices to show that α is not a rational multiple of π. Well if it were,

$$m\alpha = n\pi \text{ for some integers } m,\ n$$

in which case $\cos m\alpha = \pm 1$. Let us compute the first few values of $\cos m\alpha$:

$$\begin{aligned} \cos\alpha &= 1/3 \\ \cos 2\alpha &= 2\cos^2\alpha - 1 = 2/9 - 1 = -7/9 \end{aligned}$$

and in general we can use the identity

$$\cos(m+1)\alpha + \cos(m-1)\alpha = 2\cos m\alpha \cos\alpha$$

to get

$$\cos(m+1)\alpha = 2/3 \cos m\alpha - \cos(m-1)\alpha$$

and hence derive successive values from previous values: $\cos 3\alpha = -23/27$, $\cos 4\alpha = 17/81$, etc. We notice that the results are fractions with denominators $3, 3^2, 3^3, 3^4, \ldots$

and numerators p_1, p_2, p_3, p_4, ... which have no factor 3. This fact is true for all m, for if

$$\cos m\alpha = p_m/3^m, \ \cos(m-1)\alpha = p_{m-1}/3^{m-1}$$

where p_m, p_{m-1} have no factor 3, then

$$\cos(m+1)\alpha = \frac{2}{3}\, p_m/3^m - p_{m-1}/3^{m-1}$$

$$= \frac{2p_m - 9p_{m-1}}{3^{m+1}}$$

and we see that the numerator again has no factor 3, because $2p_m$ has no factor 3, hence leaves a remainder on division by 3, whereas $9p_{m-1}$ does not.

Thus the numerator of $\cos(m+1)\alpha$ will never divide the denominator 3^{m+1}, and so $\cos(m+1)\alpha \neq \pm 1$. The edge angle of the tetrahedron is therefore rationally independent of π, and hence the tetrahedron and cube are not equidecomposable.

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NEWTON'S APPLE

G C Smith, Monash University

Quite often a minor incident in the life of a great man achieves such a notoriety that it is referred to in nearly all later accounts of his life. I am thinking of the episodes of Alfred and the burnt cakes, Galileo and the swinging light, Washington and the cherry tree, and Newton and the apple. Those incidents almost always illustrate a point of character or an aspect of the thought of the person of whom it is told.

What interests me about such stories is whether they are true or not or, put more precisely, whether or not there is sufficient evidence to allow us to conclude that the story can be regarded as history rather than myth.

The story of Newton's observation of the fall of an apple, and the resulting speculation that the force of gravity which caused the apple's fall might be the force that held the moon in its orbit, is a good example of this type of anecdote. It has a curious history, for the first major biographer of Newton, D. Brewster, expressed considerable doubt on the truth of the story; writing is 1855 he said:

> 'the apple is said to have fallen from the tree at Woolsthorpe, and suggested to

> Newton the idea of gravity ... neither Pemberton nor Whiston, who received from Newton himself the History of his first Ideas of Gravity, records the story of the falling apple'.

Brewster's doubts have, I think, influenced quite a number of later writers: often one finds modern books that relate the story (or merely refer to it) in such a way as to suggest doubt about its authenticity. Yet the evidence in its favour is in fact very strong. As the evidence is not at all well known I thought readers of *Function* might be interested to hear about it.

The best evidence would be a written account of the story by Newton himself. However, no such direct account is known to exist. But we do have several accounts by others who knew either Newton or his close relatives, and these accounts were published very soon after Newton's death.

The best such account is due to William Stukeley who collected information on Newton's life with the object of writing a biography of Newton. He met and talked with Newton in 1726 when Newton was in his 84th year. Stukeley's manuscript biography of Newton was not printed until 1936. The relevant part of Stukeley's biography is as follows:

> 'On 15 April 1726 I paid a visit to Sir Isaac at his lodgings in Orbels buildings in Kensington, dined with him and spent the whole day with him, alone ... . After dinner, the weather being warm, we went into the garden and drank thea, under the shade of some appletrees, only he and myself. Amidst other discourse, he told me, he was just in the same situation, as when formerly, the notion of gravitation came into his mind. It was occasion'd by the fall of an apple, as he sat is a contemplative mood. Why should that apple always descend perpendicularly to the ground, thought he to him self. Why should it not go sideways or upwards, but constantly to the earth's center? Assuredly, the reason is, that the earth draws it. There must be a drawing power in matter: and the sum of the drawing power in the matter of the earth must be in the earth's center, not in any side of the earth. Therefore dos this apple fall perpendicularly, or towards the center. If matter thus draws matter, it must be in proportion of its quantity. Therefore the apple draws the earth, as well as the earth draws the apple. That there is a power, like that we here call gravity, which extends its self thro' the universe ...'.

This, then, comes from Newton via Stukeley, and is conclusive provided we believe in Stukeley's honesty; in fact no one has ever doubted that Stukeley reported Newton accurately.

Another early account of the story of the apple comes from Voltaire. Voltaire was an anglophile - he held England and its institutions in high regard. He also wrote a work supporting the Newtonian way of thinking about science. He spent over two years in England, between 1726 and 1729, but did not meet Newton, who died in March 1727. However, it appears that he did meet Newton's niece Catherine Conduitt and her husband John. Catherine Conduitt acted as house-keeper to Newton in the last few years of his life, and so would probably know about his early life through his conversation. Voltaire gave three versions of the story - two in different editions of his *Lectures concerning the English Nation*; but the earliest is contained in a work titled *An Essay upon the Civil Laws of France ... And also upon the Epick Poetry of the European Nation from Homer down to Milton*, which was published in 1727. Voltaire wrote:

> 'Sir Isaac Newton walking in his Gardens had the first Thought of his System of Gravitation, upon seeing an Apple falling from a Tree'.

Stukeley and Voltaire give the best accounts of the apple story. Other briefer accounts provide supporting evidence. A man called Green mentioned the story in a long Latin work called *The Principles of the Philosophy of the Expansive and Contractive Forces* published in 1727. Also Henry Pemberton, who knew Newton, and was the editor of the third edition of the *Principia*, in 1728 gave an account of Newton's early discoveries which starts:

> 'The first thoughts which gave rise to his *Principia*, he had when he retired from Cambridge in 1666 on account of the Plague. As he sat alone in the garden, he fell into speculation of the power of gravity ...'.

The point of this is the mention of the garden; the rest of the piece deals wholly with the discoveries themselves and not with the circumstances surrounding them. One concludes that this sole bit of descriptive detail was of some significance.

Thus the evidence for the truth of the apple story comes from people who were in a position to hear it from Newton directly, or at one remove from his relatives, and these accounts were printed close to the lifetime of Newton. These circumstances allow us to conclude that the story is history rather than myth.

To conclude with a minor query that is rather puzzling: why did Brewer doubt the story? For he knew - and used - Stukely's biographical notes.

MEAN, MODE AND MEDIAN AS DESCRIPTIONS OF FUNCTIONS BY CONSTANTS

P D Finch, Monash University

1. INTRODUCTION

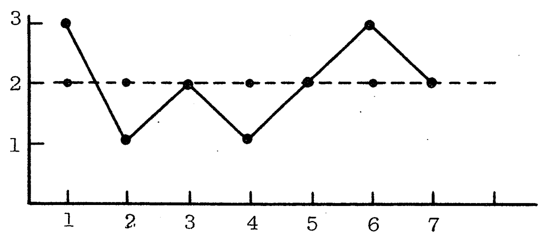
The mean, mode and median are usually introduced into statistics courses as quantities which describe the rough *location* of a frequency distribution as distinct from its *spread* about that location. Unfortunately the concept of "location" is often left somewhat vague and the thoughtful student is rightly puzzled by the fact that a given frequency distribution might have as many as three separate locations. The purpose of this article is to show how these matters can be discussed in a precise though elementary way by means of the function concept. It is to be understood that all the sets to be considered below are finite.

2. FUNCTIONS AND CONSTANTS

We recall that if X and Y are non-empty sets then a function f: $X \to Y$ is a rule which associates with each element x of X exactly one element $y = f(x)$ of Y. The function f is constant when $f(x)$ is the same element of Y for each element x of X. The figure below depicts the graph of two functions mapping $X = \{1, 2, 3, 4, 5, 6, 7\}$ into the set $Y = \{1, 2, 3\}$, viz. the constant function $c(x) \equiv 2$ indicated by the points marked on the broken line and the one given by the rule

x	1	2	3	4	5	6	7
$f(x)$	3	1	2	1	2	3	2

which is indicated by the points marked on the unbroken line.



It is evident from this figure that the values of the variable or non-constant function fluctuate about the value of the constant one and that the straight broken line of the latter provides a rough approximation to the fluctuating unbroken line of the former. This fact suggests the general question of to what extent a given non-constant function can be approximated to or described by a constant function. To simplify matters, fix the sets X and Y, suppose that $X = \{x_1, x_2, \ldots, x_N\}$ has N elements, that $Y = \{y_1, y_2, \ldots, y_n\}$ consists of n real numbers and that $y_1 < y_2 < \cdots < y_n$. There are now only n constant functions to be considered, viz. the functions c_k, $k = 1, 2, \ldots, n$, where, by definition, $c_k(x) = y_k$ for each x in X.

3. PROXIMITY BETWEEN FUNCTIONS

The problem to be discussed is the question of the extent to which a given function $f\colon X \to Y$ can be adequately described by one of the constant functions c_k, $k = 1, 2, \ldots, n$, it being supposed that we have in mind some numerical measure of the discrepancy between two designated functions, f and g say. We shall consider three such measures of discrepancy, viz. Δ_1, Δ_2 and Δ_3 as given by the expressions

$$\Delta_1(f, g) = [\{f(x_1) - g(x_1)\}^2 + \{f(x_2) - g(x_2)\}^2 + \cdots \\ \cdots + \{f(x_N) - g(x_N)\}^2]^{\frac{1}{2}}$$

$$\Delta_2(f, g) = [\{1 - \delta(f(x_1), g(x_1))\} + \{1 - \delta(f(x_2), g(x_2))\} + \cdots \\ \cdots + \{1 - \delta(f(x_N), g(x_N))\}]$$

$$\Delta_3(f, g) = [|f(x_1) - g(x_1)| + |f(x_2) - g(x_2)| + \cdots \\ \cdots + |f(x_N) - g(x_N)|]$$

where, in the expression for Δ_2, δ is the so-called Kronecker delta, that is to say

$$\delta(f(x), g(x)) = \begin{matrix} 1, \text{ if } f(x) = g(x), \\ 0, \text{ otherwise} \end{matrix}$$

and, in the expressions for Δ_3,

$$|f(x) - g(x)| = \begin{matrix} f(x) - g(x), \text{ if } f(x) \geqslant g(x), \\ g(x) - f(x), \text{ if } g(x) \geqslant f(x). \end{matrix}$$

It should be noted that each of these measures satisfies the metric axioms listed by Cameron on page 26 of *Function*, Volume 1, Part 2. In particular each of them has the property $\Delta(f, g) = 0$ if and only if $f = g$ and the reader should take time off to become convinced that there is an intuitive sense in which each $\Delta(f, g)$ becomes larger the greater the discrepancy between f and g. If f is a given function it seems reasonable to describe f by the constant function c for which $\Delta(f, c)$ is a minimum, in that way we choose as a description of f the constant function which is "closest" to f in the sense of the discrepancy measure Δ. Of course we shall get possibly different answers according as Δ is Δ_1, Δ_2 or Δ_3 and, indeed, it is in that way that we are led to the mean, mode and median; to see this we need to define those quantities independently of the measures in question. We do so in the next section.

4. MEANS, MODES AND MEDIANS

Let $f\colon X \to Y$ be a given function and for each y in Y write $s(y)$ for the number of x in X with $f(x) = y$ so that, in particular, $s(y) = 0$ if and only if there is no x in X with $f(x) = y$ and

$$s(y_1) + s(y_2) + \cdots + s(y_n) = N,$$

the number of elements in X. We say that the integer-valued function s is the *spectrum* of the function f and when we wish to draw attention to its dependence on f we write it as s_f. For a constant function c with $c(x) \equiv y_k$ we have $s_c(y_k) = N$ and $s_c(y) = 0$ for $y \neq y_k$.

We write

$$\overline{f} = N^{-1}[f(x_1) + f(x_2) + \cdots + f(x_N)]$$
$$= N^{-1}[y_1 s(y_1) + y_2 s(y_2) + \cdots + y_n s(y_n)]$$

and say that y_k in Y is an *f-mean*, or a *mean* of f, when there is no other element of Y which is closer to $\overline{f}$. An f-mean always exists and there are at most two of them. If there is only one f-mean it is that one of $y_1, y_2, \ldots, y_n$ which is closest to $\overline{f}$. There are two f-means when it happens that, for some k, $1 \leqslant k < n$, $\overline{f} = (y_k + y_{k+1})/2$ in which case, of course, y_k and y_{k+1} are both means of f. The more usual way of defining *the* mean is to take it to be $\overline{f}$ but this is unsuitable for our present purposes because we want *our* "mean" to be an element of Y and $\overline{f}$ may not belong to Y.

An element y of Y is said to be a *mode* of f when

$$s(y) \geqslant s(y_k), \text{ for each } k = 1, 2, \ldots, n.$$

There is always at least one mode but there may be more than one, however if y' and y'' are both modes of f then $s(y') = s(y'')$.

To introduce the concept of a *median* recall that $y_1 < y_2 < \ldots < y_n$, define l to be the smallest integer,

$1 \leqslant l \leqslant n$, such that

$$s(y_1) + s(y_2) + \cdots + s(y_l) \geqslant N/2$$

and let u be the smallest integer such that

$$s(y_1) + s(y_2) + \cdots + s(y_u) > N/2.$$

Clearly $l \leqslant u$. The medians of f are, by definition, the elements $y_l, y_{l+1}, \ldots, y_u$. If $l = u$ there is only one median but if $l < u$ there will be several of them and for each median $y_k < y_n$ one will have

$$s(y_1) + s(y_2) + \cdots + s(y_k) = N/2.$$

In particular, if $u > l + 1$ then $s(y_k) = 0$ for each $k = l + 1, \ldots, u - 1$.

5. DESCRIPTION OF A FUNCTION BY A MEAN

Let f be a given function and consider the problem of finding the constant function c_k for which $\Delta_1(f, c_k)$ is as small as possible. Recalling that $c_k(x) \equiv y_k$ a straightforward calculation yields

$$\begin{aligned}\Delta_1^2(f, c_k) &= \{f(x_1) - c_k(x_1)\}^2 + \{f(x_2) - c_k(x_2)\}^2 + \cdots \\ &\qquad \cdots + \{f(x_N) - c_k(x_N)\}^2 \\ &= (y_1 - y_k)^2 s(y_1) + (y_2 - y_k)^2 s(y_2) + \cdots \\ &\qquad \cdots + (y_n - y_k)^2 s(y_n).\end{aligned}$$

Since

$$\begin{aligned}(y_j - y_k)^2 &= \{(y_j - \bar{f}) + (\bar{f} - y_k)\}^2 \\ &= (y_j - \bar{f})^2 + 2(y_j - \bar{f})(\bar{f} - y_k) + (\bar{f} - y_k)^2\end{aligned}$$

and

$$\begin{aligned}&(y_1 - \bar{f})s(y_1) + (y_2 - \bar{f})s(y_2) + \cdots + (y_n - \bar{f})s(y_n) \\ &= y_1 s(y_1) + y_2 s(y_2) + \cdots + y_n s(y_n) - \bar{f}\{s(y_1) + s(y_2) + \cdots \\ &\qquad + \cdots s(y_n)\} \\ &= N\bar{f} - N\bar{f} = 0\end{aligned}$$

we obtain

$$\begin{aligned}\Delta_1^2(f, c_k) &= N(\bar{f} - y_k)^2 + (y_1 - \bar{f})^2 s(y_1) + (y_2 - \bar{f})^2 s(y_2) + \cdots \\ &\qquad \cdots + (y_n - \bar{f})^2 s(y_n).\end{aligned}$$

Only the first term on the right-hand side of this expression depends on the constant function c_k and we see, therefore, that to minimise $\Delta_1(f_1, c_k)$ we have to make $(\bar{f} - y_k)^2$ as small as possible, in other words we have to choose k so that y_k is an f-mean.

6. DESCRIPTION OF A FUNCTION BY A MODE

Consider now the problem of finding the constant function which minimises $\Delta_2(f, c_k)$. Since

$$\Delta_2(f, c_k) = N - s(y_k)$$

we see that we must make $s(y_k)$ as big as possible, in other words we have to choose k so that y_k is an f-mode.

7. DESCRIPTION OF A FUNCTION BY A MEDIAN

Finally, we determine the constant function c_k which minimises $\Delta_3(f, c_k)$. To do so we start with a straightforward though tedious calculation yielding

$$\Delta_3(f, c_{k+1}) - \Delta_3(f, c_k) = 2(y_{k+1} - y_k)[s(y_1) + s(y_2) + \cdots \cdots + s(y_k) - N/2].$$

Recalling that $y_1 < y_2 < \cdots < y_n$ and, for notational convenience, writing $\Delta(c_k)$ instead of $\Delta_3(f, c_k)$ we find that if l and u are as defined in section 4, so that $y_l, y_{l+1}, \ldots, y_u$ are the medians of f, then we have

$$\Delta(c_1) > \Delta(c_2) > \cdots > \Delta(c_{l-1}) > \Delta(c_l),$$
$$\Delta(c_l) = \Delta(c_{l+1}) = \cdots = \Delta(c_u),$$
$$\Delta(c_u) < \Delta(c_{u+1}) < \cdots < \Delta(c_n).$$

It is now obvious that the desired constant function c_k, viz. one which will minimise $\Delta(c_k) = \Delta_3(f, c_k)$, corresponds to choosing k so that y_k is an f-median.

8. CONCLUDING REMARKS

The upshot of the preceding discussion is that in seeking the "best" constant approximation to a function we are led to means, modes and medians according to the way we assess what is deemed to be "best", that is to say according as we use Δ_1, Δ_2 or Δ_3 to measure discrepancy between functions.

It often happens, of course, that the mean, mode and median all coincide. This is true, for example, of the function f tabulated in section 2 where $X = \{1, 2, 3, 4, 5, 6, 7\}$ and

$Y = \{1, 2, 3\}$, the spectrum of the function in question being

$$s(1) = 2, \quad s(2) = 3 \text{ and } s(3) = 2.$$

Thus 2 is the mode, the median and also the mean. The corresponding values of $\Delta(f, c)$ are

	c_1	c_2	c_3
$\Delta_1(f, c)$	3·32	2	3·74
$\Delta_2(f, c)$	5	4	5
$\Delta_3(f, c)$	11	4	2

exhibiting the fact that $c = c_2$ minimises each of $\Delta_1(f, c)$, $\Delta_2(f, c)$ and $\Delta_3(f, c)$.

A more complicated situation is presented by the function taking values in the set $Y = \{1, 2, 3, 4, 5\}$ with the spectrum

$$s(1) = 40, \; s(2) = 11, \; s(3) = 3, \; s(4) = 1, \; s(5) = 45$$

so that N, the number of elements in X is 100. In this case

$$s(1) + 2s(2) + 3s(3) + 4s(4) + 5s(5) = 300$$

and so the mean is 3. But the mode is clearly 5 and since

$$s(1) + s(2) = 51 > 50$$

the median is at 2. The corresponding values of $\Delta(f, c)$ are

	c_1	c_2	c_3	c_4	c_5
$\Delta_1(f, c)$	752	452	352	452	752
$\Delta_2(f, c)$	60	89	97	99	55
$\Delta_3(f, c)$	200	180	182	222	200

This table illustrates that $\Delta_1(f, c)$ is minimised by $c = c_3$ corresponding to the mean at 3, that $\Delta_2(f, c)$ is minimised by $c = c_5$ corresponding to the mode 5 whereas $\Delta_3(f, c)$ is minimised by $c = c_2$ corresponding to the median at 2.

In statistics, of course, the function f plays a subsidiary role inasmuch as attention is focused on its spectrum s_f and its spectral density $\overline{s}_f$, viz. the normalised version given by

$$\overline{s}_f(y) = N^{-1} s_f(y)$$

which specifies the relative frequency distribution under study. In the statistical context it is customary to refer to the mean, mode and median of s_f rather than to those of f itself as we have done above.

When we have found the constant c which minimises the expression $\Delta(f, c)$ the minimum discrepancy so obtained can be regarded as the "spread" of the function f about that constant. In the case of Δ_1 we are led in that way to the standard deviation as a measure of spread. The analogous "spreads" corresponding to Δ_2 and Δ_3 are less well-known but the student will find it instructive to work them out in a few simple cases.

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SOLUTION TO PROBLEM 1.2.6 (i.e. Problem 2.6 of part 2 of volume 1)

This problem has generated great interest. Many readers have written asking questions about it or offering comments or partial solutions. The solution is {4, 13} and although several correspondents found this, only one, Christopher Stuart, gave all the arguments necessary to show that, as required, this is the unique solution. David Dowe (Form 6, Geelong Grammar School) Geoffrey J. Chappell (Grade 12, Kepnock High School, Bundaberg) and Graham Farr (Form 6 (1977), Melbourne High School) each went a long way towards a solution. The solution we print is except for minor changes that of Christopher Stuart (15 Lois Street, Ringwood East).

The problem is:

> A person A is told the product xy and a person B is told the sum $x + y$ of two integers x, y, where $2 \leqslant x, y \leqslant 200$ [i.e. x and y each lie between 2 and 200 inclusive]. A knows that B knows the sum, and B knows that A knows the product. The following dialogue develops:
>
> 1. A: I do not know $\{x, y\}$.
> 2. B: I could have told you so!
> 3. A: Now I know $\{x, y\}$.
> 4. B: So do I.
>
> What is $\{x, y\}$?

In 1742 a mathematician called Goldbach wrote to Leonhard Euler asking whether every even number (greater than 2) is the sum of two primes. For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$, $10 = 7 + 3$, $12 = 7 + 5$, ..., $80 = 7 + 73$, ... This question has not yet been answered. The result is nowadays known as *Goldbach's Conjecture*. [In the last decade Chinese mathematicians have made significant advances in the attempt to prove it.] For even numbers not greater than 400 which we are dealing with the truth of the conjecture can be easily checked.

From statement 1 by A, the product P ($= xy$) does not have a unique factorization. Hence we conclude

(i) *x and y are not each prime and there is no prime p such that* $\{x, y\} = \{p, p^2\}$.

Thus B can only make statement 2 if the sum S which he is given satisfies

(ii) *every pair of numbers x, y (between 2 and 200, of course) such that $x + y = S$ satisfies condition (i).*

In particular, from Goldbach's result, we can then conclude

(iii) *$S = 2 + b$, say, where b is odd and is not prime.*

Let us apply these results so far to the possibilities 4, 5, 6, ..., 50 for S. All even numbers are excluded by Goldbach's result and, applying result (iii), we are left only with the possibilities

$$S \in \{11,17,23,27,29,35,37,41,43,47\}. \qquad (\alpha)$$

Now we give Stuart's arguments to show that we can assume that $S \leq 50$. He offers two theorems and a lemma.

THEOREM 1 (Stuart). $S < 103$.

Proof. Suppose $S \geq 103$. If $103 \leq S \leq 301$, then S may be written as the sum $x + y$ with $x = 101$ and $y = S - 101$ and with x, y between 2 and 200. But 101 is prime. Since any multiple of 101 is bigger than 200, the factorization $101 \times (S - 101)$ is the only permissible factorization open to A. In view of statement 1 therefore we cannot have $103 \leq S \leq 301$.

A similar argument, using the fact that 199 is prime, disposes of the possibility that $201 \leq S \leq 400$.

Lemma 1 (Stuart). *Suppose that $P = 4pu$ where p is a prime and $p > 33$. Then A can make statements 1 and 3 (assuming that B can make statement 2).*

Proof. Let $u = rs$. Then $4pr$ (or $4ps$) cannot be one of the factors x, y, since $4pr \geq 4p > 132$ and then $S \geq 103$. $2pr$ cannot be a factor, for then $S = 2pr + 2s$ is even. So the only possibilities are $x = pr$, $y = 4s$, where r is odd. If $r \geq 3$ then $S \geq 3p + 4s > 3 \times 33 + 4 = 103$, which is impossible. Hence $r = 1$, and A knows that $x = p$ and $y = 4u$ and can make statement 3.

THEOREM 2 (Stuart). $S < 51$.

Proof. (This proof will be the first place that statement 4 is used.) Suppose that $S \geqq 51$. Since S is odd we may write

$$\begin{aligned} S &= 37 + (12 + 2t) \\ &= 43 + (6 + 2t) \\ &= 41 + (8 + 2t) \\ &= 47 + (2 + 2t) , \end{aligned}$$

where $t > 0$. Note that 37, 43, 41 and 47 are each prime numbers $\geqq 33$. If t is odd then $6 + 2t$ and $2 + 2t$ are divisible by 4 whereas if t is even then $12 + 2t$ and $8 + 2t$ are divisible by 4. In either event, it is possible to divide S into a sum of a prime p and a number $4u$, corresponding to a product $P = 4pu$, in two different ways. For either way A can make his statement 3. But B does not know which way, so B cannot make statement 4.

So we are now left solely with the possibilities listed earlier in (α).

We can use the same procedures as were used to prove the Theorem 2 to eliminate all remaining possibilities for S except 17.

Stuart gives another lemma of great help here.

LEMMA 2 (Stuart). *Suppose that $P = 2^k p$ where $k \geqq 2$ (because of statement 2) and p is a prime. Then, after B has made statement 2, A can make statement 3.*

Proof. Since S is odd the only possibilities for x and y are 2^k and p.

COROLLARY. *S cannot be written in two different ways as a sum $2^k + p$ where $k \geqq 2$ and p is a prime.*

Proof. For, if so, then each of these ways of writing S would correspond to a choice of $\{x, y\}$ which would enable A to make statements 1 and 3; and B would thus be unable to make statement 4.

Observe now that

$$11 = 2^2 + 7 = 2^3 + 3 \qquad 35 = 2^2 + 31 = 2^4 + 19$$

$$23 = 2^2 + 19 = 2^4 + 7 \qquad 37 = 2^2 + 29 = 2^5 + 5$$

$$27 = 2^2 + 23 = 2^3 + 19 \qquad 47 = 2^2 + 43 = 2^4 + 31$$

Hence, by the corollary, each of these is ruled out as a value for S. Hence we now have

$$S \in \{17,\ 29,\ 41\} \quad .$$

S = 29 is impossible

$29 = 2^4 + 13$ gives, by lemma 2, one possible pair $\{16, 13\}$ allowing statements 1 to 3 to be made.

$29 = 4 + 25$ gives another possible pair $\{4, 25\}$ allowing statements 1 to 3 to be made. For A then has the product 100 which offers the possibilities $\{4, 25\}$ and $\{20, 5\}$, since S is odd. He can reject $\{20, 5\}$, for then $S = 35 = 2 + 23$ and B would not be able to make statement 2.

Hence B cannot make statement 4.

S = 41 is impossible

$41 = 2^2 + 37$ gives the pair $\{4, 37\}$ consistent with statements 1 to 3.

$41 = 16 + 25$ gives the pair $\{16, 25\}$ consistent with statements 1 to 3. For from the product 16×25 only $\{16, 25\}$ and $\{80, 5\}$ give an odd value to S and $80 + 5 = 2 + 83$ is inconsistent with B's statement 2, since 83 is prime.

Hence again if $S = 41$, B cannot make statement 4.

Hence we are left with the sole possibility $S = 17$. A little calculation now gives that $\{4, 13\}$ is the only solution.

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INFINITE NUMBERS I

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This article is about giving meaning to the *size*, the *number of elements*, of sets. In everyday life, if you want to know how two collections compare in respect to the number of things in each (for example, are there as many eggs in that basket as there are candles on that cake) then you use the counting numbers 0,1,2,3,... to count up how many there are of each, and from this you find your answer. Since in everyday life, everything is finite, you have to count up only finitely far, and this method (in theory) always gives you the answer. However, in mathematics, you want to count the number of elements in various sets and there are many important sets which have infinitely many members, for example $Z^+ = \{1,2,3...\}$ the set of positive integers, $Z = \{...-2,-1,0,1,2,...\}$ the set of all integers, $Q = \{p/q;\ p,q \in Z, q \neq 0\}$ the set of rational (fraction) numbers, and R the set of all real numbers. I shall consider the question of finding a reasonable way to talk about the size of infinite sets, so that "big" sets have "more" elements than "smaller" ones. In particular, are the sets Z^+, Z, Q, R already mentioned all the same size, or are some of them bigger than the others?

First, let us consider a way which it turns out isn't suitable. There is an idea you might think of immediately: to use the subset relation $\subseteq$ (where $A \subseteq B$ means that every member of A is also a member of B), so set A is smaller than set B if $A \subseteq B$ and $A \neq B$. This seems reasonable: if $A \subseteq B$ and $A \neq B$ there are extra elements in B, over and above those in A, so in a sense B *is* bigger than A. So, for example, this would make Z^+ smaller than Z, since $Z^+ \subseteq Z$ and $Z^+ \neq Z$. However, using the subset relation is no sooner thought of than we realize that it isn't what we want. Which of the sets $\{1,3,5,7...\}$ and $\{2,4,6,8,...\}$ of odd and of even positive integers would be the larger of the two? Answer: neither. Since neither is a subset of the other, we just can't compare them using $\subseteq$. However, definitely one of the properties that any reasonable concept of size for sets should have is that given any two sets, you always *can* compare them - the one is larger than the other (or they have the same size). Since the subset relation doesn't have this property, it is not the concept we are looking for.

To gain inspiration towards a correct idea, let us go back to basics in the comparison of size between two finite collections. Suppose I have a bag of apples and a bag of oranges in front of me, and I want to know if there are more oranges than apples. One way of proceeding, rather than directly counting up the number of oranges and the number of apples, is the following. I put a hand in each bag, and draw out one piece of fruit from each, and put these two pieces of fruit down together in front of me. Then I reach back into the bags and again draw out one piece of fruit from each, and put these together in front of me too. And I keep doing this until one of the bags is emptied. If I run out of apples at exactly the same stage as I run out of oranges, then I know there must have been the same number of apples as of oranges. If I run out of apples before I run out of oranges, then there were fewer apples than oranges. What I have been doing is pairing off apples with oranges: at any stage, when I put the apple and the orange down together in front of me, that apple is paired with that orange. If the apples and the oranges can be exactly paired off in this way, then there is the same number of apples as of oranges; if whenever I try I run out of apples first, then there are fewer apples than oranges.

This idea of pairing off is what turns out to be just what we want when we talk about the size of mathematical sets. We define two sets A and B to be of the *same size* if we can pair off all the elements of A with all the elements of B. We say A is *not larger than* B if we can pair off all the elements of A with elements of B, not necessarily using up all the elements of B. And we say A is *smaller than* B if you can pair off all the elements of A with some of the elements of B, but no matter how you try you can never use up all the elements of B.

Notice that these three definitions are connected in a sensible way; for instance, A is not larger than B if and only if either A is smaller than B or A is of the same size as B. One can show (though it is not easy) that it follows from these definitions that if A is not larger than B and also B is not larger than A then A and B have the same size. Another important fact is that this concept of size does have the comparability property that the subset relation lacks: given any two sets A and B either A and B have the same size or else one is smaller than the other. At this stage, perhaps we should also make this remark: if A is a subset of B then A is not larger than B, for the matching of each a in A with itself, thought of as an element of B, gives a pairing of all the elements of A with some of the elements of B.

Let us look at a number of examples. Consider $Z^+ = \{1,2,3,\ldots\}$ and $A = \{2,3,4\ldots\}$. Since Z^+ has an extra elements over A, you might think Z^+ is bigger. But not with this concept of size, for the diagram

$$\begin{array}{ccccccccc} A & : & \{2, & 3, & 4, & 5, & \ldots, & n+1, & \ldots\} \\ & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ Z^+ & : & \{1, & 2, & 3, & 4, & \ldots, & n\ , & \ldots\} \end{array}$$

shows a pairing off between the elements of A and the elements of Z^+, and so shows that A and Z^+ are of the same size. A similar result is true for all infinite sets: if you have an infinite set and add to it an extra element, then the new set is still the same size as the original. Suppose now we compare Z^+ with $E = \{2,4,6,8,\ldots\}$, the set of all even positive integers. This time infinitely many numbers have been left out (all the odd numbers), so surely E will be smaller? But indeed not; the diagram

$$\begin{array}{ccccccccc} E & : & \{2, & 4, & 6, & 8, & \ldots, & 2n, & \ldots\} \\ & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ Z^+ & : & \{1, & 2, & 3, & 4, & \ldots, & n, & \ldots\} \end{array}$$

shows a pairing between E and Z^+, so in fact E and Z^+ have the same size. In a similar fashion, Z^+ is the same size as Z, as the following pairing shows:

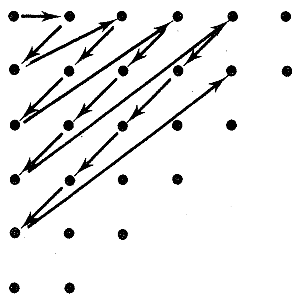
$$\begin{array}{cccccccccc} Z^+ & : & \{1, & 2, & 3, & 4, & 5, & \ldots, & 2n, & 2n+1, & \ldots\} \\ & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \\ Z & : & \{0, & 1, & -1, & 2, & -2, & \ldots, & n, & -n, & \ldots\} \end{array}$$

A similar argument shows that Q^+ (the set of positive fractions) and Q are the same size.

More complicated is the situation with Z^+ and Q^+. Is Q^+ larger? In fact it is not! To see this, start by writing out a square array of all the fractions as follows. In the top line, put all those with denominator 1 (i.e. 1/1, 2/1, 3/1, ...), in the second line all those with denominator 2, in the third line all those with denominator 3, and so on, as shown.

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\[1ex]
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\[1ex]
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\[1ex]
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \cdots \\[1ex]
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

This lists all the members of Q^+ (indeed, each one infinitely often, since e.g. 2/1 = 4/2 = 6/3 = ...). Now think of walking along the following zig-zag path through the array:



As you go, count the different fractions you meet. So you count: one for 1/1, two for 2/1, three for 1/2, four for 4/1, (don't count 2/2 since 2/2 = 1/1 and you have already met 1/1) five for 1/3, ... (the next one you don't count is 4/2 = 2/1). This sets up the pairing

$$\begin{array}{lcccccccccccccl}
Z^+ : & \{1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & \ldots\} \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
Q^+ : & \{\frac{1}{1}, & \frac{2}{1}, & \frac{1}{2}, & \frac{3}{1}, & \frac{1}{3}, & \frac{4}{1}, & \frac{3}{2}, & \frac{2}{3}, & \frac{1}{4}, & \frac{5}{1}, & \frac{1}{5}, & \ldots\}
\end{array}$$

which shows that Z^+ and Q^+ are the same size.

It is certainly surprising, at first sight, that there are the "same number" of positive integers as there are positive rational numbers. Are all infinite sets going to turn out, with the definition of size, or number, that we have chosen, to be of the same size? Fortunately the answer turns out to be no. In the next issue we shall consider, as already promised, the size of the set of real numbers and show that there are "more" real numbers than there are rational numbers.

Interested readers may consult the classical

Bertrand Russell, *An Introduction to Mathematical Philosophy*, 1919, or for a modern version, the small paperback.
S. Swierczkowski, *Sets and Numbers*, Routledge and Kegan Paul, 1972.

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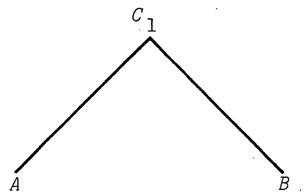
HOW LONG IS A STRAIGHT LINE?

J C Barton, University of Melbourne

Let K_1 denote the "curve" AC_1B consisting of the two straight-line segments AC_1 and C_1B at right angles to each other and of equal lengths. If AB is of unit length, then AC_1 has length $\frac{1}{2}\sqrt{2}$; so K_1 has length $l_1 = \sqrt{2}$. The curve consists of two sides of the triangle AC_1B with base AB and with height $\frac{1}{2}$. If k is any positive number, then $(4k)^0 = 1 = k^0 = 2^0$. Hence we can say that K_1 is a curve consisting of the sides, excluding the base, of 2^0 triangles

of height $\frac{1}{2}k^0$
with length $l_1 = \sqrt{1 + (4k)^0}$.



K_1

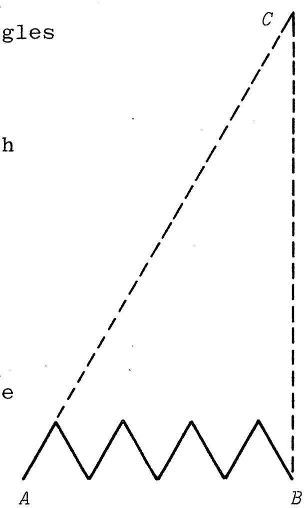
Form K_2, in a similar fashion, connecting A and B, but now consisting of the sides, excluding the bases, of 2^2 equal isosceles triangles

of height $\frac{1}{2}k$
with length $l_2 = \sqrt{1 + (4k)^2}$.

It may be checked that the length l_2 is the length of the straight line segment AC obtained by extending the side of a triangle through A until it meets the perpendicular to AB at C.

Continuing thus, (the reader can supply the appropriate diagram), let K_3 be the curve on the same base, with 2^4 triangles, each of height $\frac{1}{2}k^2$, whose length is $l_3 = \sqrt{(1 + (4k)^4)}$.

And so, on.



K_2

K_n is the curve (imagination now, rather than a diagram), on the same base, with 2^{2n-2} triangles, each of height $\frac{1}{2}k^{n-1}$, and total length

$$l_n = \sqrt{(1 + (4k)^{2n-2})} .$$

Now let the constant k be chosen so that $0 < k < 1$ and $4k > 1$, more succinctly $\frac{1}{4} < k < 1$. For example, $k = \frac{1}{2}$ will do nicely. Let n tend to infinity, and denote the limit of K_n by K_∞, whose graph is shown. Its length, naturally, is

$$\lim_{n\to\infty} l_n = \infty, \text{ or is it?}$$

K_∞

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HARMONIC PROGRESSIONS

Glen Merlo, Taylor's College

We are very familiar with the arithmetic and geometric progressions (or sequences). There is another, closely allied, type of progression that is called *harmonic*.

The sequence of non-zero numbers $p, q, r, s, \ldots,$ is called a *harmonic progression* if and only if the sequence $1/p, 1/q, 1/r, 1/s, \ldots,$ is an arithmetic progression.

Since every arithmetic progression takes the form $a, a + d, a + 2d, a + 3d, \ldots,$ it is clear that every harmonic progression will take the form $1/a, 1/(a + d), 1/(a + 2d), 1/(a + 3d), \ldots,$ where in the general term $H_n = 1/(a + (n-1)d)$.

Three (or more) numbers $a, b, c, (\ldots)$, say, which are successive terms in some harmonic progression are said to be *in harmonic progression*; similarly the phrases *in arithmetic progression* and in *geometric progression* are defined in relation to arithmetic and geometric progressions, respectively.

When a, b, c are in harmonic progression then b is called the *harmonic mean* of a and c. In fact a and c then determine b. To see this, proceed as follows. By definition, since a, b, c are in harmonic progression, $1/a, 1/b, 1/c$ are in arithmetic progression. Thus

$$\frac{1}{b} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) .$$

Hence $2ac/(a + c)$ is the harmonic mean between a and c.

In recent issues of *Function* many situations involving arithmetic and geanetric progressions have arisen and associated problems have been posed. Here are some on harmonic progressions:

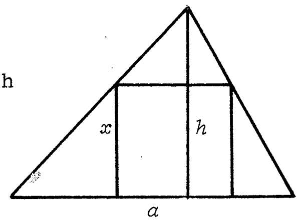
1. If a, b, c, d are in arithmetic progression, show that

(i) $(b + c + d)$, $(c + d + a)$, $(d + a + b)$, $(a + b + c)$ are also in arithmetic progression;

(ii) bcd, cda, dab, abc, are in harmonic progression.

2. If the sum of four numbers in arithmetic progression is 32 and the harmonic mean of the second and third is $7\frac{1}{2}$, find the numbers.

3. In a triangle with base of length a and height h inscribe a square on the base. Let x be the length of the side of the square. Show that $2x$ is the harmonic mean of a and h.



4. Suppose a man travels from A to B at speed u and returns from B to A at speed v. Show that his average speed for the total journey is the harmonic mean of u and v.

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SOLUTION TO PROBLEM 1.5.3 (Solution provided by Geoffrey J. Chappell, Kepnoch High School, Bundaberg)

Theorem 1. If β is a factor of $10^x - 1$ then $10^x/\beta$ must leave a remainder of 1. This is equivalent to saying that $1/\beta$ has a repeating group of x digits.

Example: $10^2 - 1 = 3^2 \cdot 11$. $\frac{1}{3}$ has the repeating group 33 and $\frac{1}{11}$, the group 09, each of two digits.

Theorem 2. If we define k_n as the n-th digit from the left of a number K and if $k_n + k_{n+\alpha} = 9$ then K has $10^\alpha - 1$ as a factor. If α is odd, the other factor is formed by taking the first α digits and adding 1. If α is even, we take the first $(\alpha - 1)$ digits of K and add 1. (Problem: prove this!)

Example: 142 857. $1 + 8 = 9$, $4 + 5 = 9$, $2 + 7 = 9$ so $\alpha = 3$. $142\,857 = (10^3 - 1)y$. α is odd and the first three digits are 142. So $y = 143$.

Using these theorems we can factor the number in

Problem 5.3. Theorem 2 is used first.

$$\begin{aligned} & 5\ 679\ 431\ 432\ 056\ 743\ 205\ 685\ 679\ 432 \\ = {} & 56\ 794\ 314\ 320\ 568 \cdot (10^{14} - 1) \quad \ldots (1) \\ = {} & 5\ 679\ 432 \cdot (10^7 - 1) \cdot (10^{14} - 1) \\ = {} & 568\ (10^4 - 1)(10^7 - 1)(10^{14} - 1) \\ = {} & 568\ (10^2 - 1)(10^2 + 1)(10^7 - 1)^2(10^7 + 1) \end{aligned}$$

since $10^{2k} - 1 = (10^k - 1)(10^k + 1)$.

Using Theorem 1, $10^2 - 1 = 3^2 \cdot 11$, $10^7 - 1 = 3^2 \cdot 239 \cdot 4649$ since the primes 3239 and 4649 all have repeating groups of seven digits, (3 333 333, 0 041 841, and 0 002 151 respectively). $568 = 2^3 \cdot 71$. $10^2 + 1 = 101$ is prime. Any number of the form $10^{2b+1} + 1$ has 11 as a factor and if $b = 3$, the other prime factor is 909 091.

Numbers of the form $10^x \pm 1$ are quite easy to factor, as shown and so the given number, using these factors, is equal to

$$\underline{2^3 \cdot 3^6 \cdot 11^2 \cdot 71 \cdot 101 \cdot 239^2 \cdot 4649^2 \cdot 909\ 091}.$$

Isn't it interesting that the chance of selecting a 28-digit number such that Theorem 2 may be applied as in (1) would give rise to odds of almost 10^{14} to 1 against such a number being chosen. The chances that Theorem 2 can be applied three times must be very small indeed - coincidence?

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LETTER FROM ALASDAIR MACANDREW, FORM 6 (1977), MELBOURNE HIGH SCHOOL

A problem mentioned in the article "Louis Pósa" in the first issue of *Function* (page 4), given by Erdös to Pósa is:

Prove that $\sum_{i=1}^{\infty} \frac{1}{w(i)}$ is irrational, where $w(i)$ is the lowest common multiple of the integers 1, 2, 3, ..., i.

My solution is this:

Suppose on the contrary that $\sum_{i=1}^{\infty} \frac{1}{w(i)}$ is rational, equal to $\frac{p}{q}$, say, and let

$$\frac{1}{w(1)} + \frac{1}{w(2)} + \frac{1}{w(3)} + \frac{1}{w(4)} + \cdots + \frac{1}{w(n)} + \frac{\varepsilon}{w(n+1)} = \frac{p}{q};$$

we can choose n to be as large as we please, and of course ε will depend on n. Suppose that $n > q$ and then multiply both sides of the above equation by $w(n)$. We get

$$\frac{w(n)}{w(1)} + \frac{w(n)}{w(2)} + \frac{w(n)}{w(3)} + \cdots + \frac{w(n)}{w(n)} + \frac{w(n)\varepsilon}{w(n+1)} = \frac{w(n) \cdot p}{q}.$$

Now, if $x < n$, then $w(n) \equiv 0 \pmod{w(x)}$. And as $n > q$, then $w(n) \equiv 0 \pmod{q}$. So $w(n) \cdot p/q$ is an integer, and

$\frac{w(n)}{w(1)} + \frac{w(n)}{w(2)} + \frac{w(n)}{w(3)} + \cdots + \frac{w(n)}{w(n)}$ is an integer. But $\frac{w(n) \cdot \varepsilon}{w(n+1)}$

is not an integer for, with no loss of generality, I have taken ε: $0 < \varepsilon < 1$. Thus, starting with the assumption that our number was not irrational, we conclude that an integer is equal to a non-integer. Therefore, by *reductio ad absurdum*, we conclude that our number is irrational.

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LETTER FROM GEOFFREY J. CHAPPELL, GRADE 12
KEPNOCK HIGH SCHOOL, BUNDABERG

I have recently come across several interesting long divisions in which one or no figures are given:

```
1.          ××8××              2.          ×7×××
       ×××|××××××××                ×××|××××××××
           ×××                         ××××
            ××××                        ×××
             ×××                        ×××
              ××××                      ××××
              ××××                       ×××
                                          ××××
                                          ××××
```

These each have one given number and are very similar. The first is the easier to solve (and the shortest).

```
1.        80809
     124|10020316
          992
           1003
            992
             1116
             1116
```

Problem 1: In long division, when two digits are brought down instead of one, there must be a zero in the quotient. The quotient is therefore ×080×. When the divisor is multiplied by the quotient's last digit, the product is a 4-digit number. The quotient's last digit is a nine because eight times the divisor is a three-digit number. The divisor must be less than 125 because eight times 125 is a four-digit number. The quotient's first digit is larger than 7, because 7 times a divisor less than 125 would give a product that would leave more than two digits after is was subtracted from the first four digits in the dividend. The first digit cannot be 9 (which gives a 4-digit number when the divisor is multiplied by it) so it must be 8. The quotient is therefore 80809. Since 123×80809 is a seven-digit number and the dividend has eight digits, the divisor is less than 125 but more than 123, i.e. the divisor is 124. The division can now be reconstructed.

Problem 2: A solution is left to the reader.

Problem 3: Find the numbers for

```
         xxxx.xxxx
    xxx |xxxxxx
         xxx
          xxx
          xxx
           xxx
           xxx
            xxx
            xxx
             xxxx
             xxxx
```

I have also found problems in which the nine digits, 1-9 have to be arranged to suit certain conditions:

Problem 4: Find the number which, together with its square, shall contain all the nine digits once only, the zero disallowed.

Problem 5: Using the nine digits once only, can you find prime numbers that will add up to the smallest sum possible. Example: A sum of 450 -

```
 61
283
 47   This sum can be reduced much
 59   further.
---
450
```

Solution: The 4, 6, 8 must come in the tens place and the 2 and 5 can only appear in the units place if alone. The rest is easy:

```
 47   or   43   or some similar arrangement
 61        61
 89        89
  2         2   The minimum sum is 207.
  3         5
  5         7
---       ---
207       207
```

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"And as for Mixed Mathematics, I may only make this prediction, that there cannot fail to be more kinds of them, as nature grows further disclosed."

Francis Bacon

PROBLEM 1.1

We have a pack of cards, an even number c of them. By a "shuffle" we shall mean that we divide the pack into a top half and a bottom half, then put the pack back together again by alternately taking one card from each half starting with the bottom half. For instance, if the cards were initially 1, 2, 3, 4, after one shuffle they would become 3, 1, 4, 2. How many shuffles does it take for the cards to return to their original position?

PROBLEM 1.2

I have the following options of depositing $100 for one year. The bank will give me $4 interest at the end of the year. A Housing Co-operative will give me interest at the rate of 2% per half-year (compound, so it pays interest in the second half also on the first half's interest). A Credit Union will give me interest at the rate 1/3% per month, compound. A friend says he will give me interest equivalent to the 4% per annum rate, but compounding *every instant*! Which should I choose, and how much interest do I get?

PROBLEM 1.3

Is 22/2/2022 a Twosday (Tuesday)? How about 2/2/2202? (See *Function* (1977), Volume 1, Part 1, pp 19-23.)

PROBLEM 1.4

We construct a sequence a_1, a_2, a_3, ... by the following method. Choose $a_1 = n$, where n is any positive integer. If n is odd, choose $a_2 = 3n + 1$. If n is even, choose $a_2 = n/2$. Repeat this construction for a_3: if a_2 is odd, choose $a_3 = 3a_2 + 1$; if a_2 is even, choose $a_3 = a_2/2$. Similarly for a_4, a_5, etc.

[Examples: (i) 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, ...
(ii) 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, ...]

Use a simple computer (or calculator) program to show that if you start with any integer up to 1000 then the sequence eventually ends periodically ... 4, 2, 1, 4, 2, 1, ... Can you prove anything in general along these lines? [Actually, not too much is known about such sequences.]

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As long as a branch of science offers an abundance of problems, so long is it alive ...

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It is difficult and often impossible to judge the value of a problem correctly in advance; for the final award depends upon the gain which science obtains from the problem.

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A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.

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... what is clear and easily comprehended attracts, the complicated repels us.

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The mathematicians of past centuries were accustomed to devote themselves to the solution of difficult particular problems with passionate zeal. They knew the value of difficult problems.

David Hilbert: *Mathematical Problems*, 1900

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Front Cover: A computer drawn eighteen-sided polygon with all diagonals. The number of straight lines, vertex to vertex, required to construct an n-sided polygon, as illustrated, can be determined by summing the first $(n - 1)$ terms of an arithmetical progression.