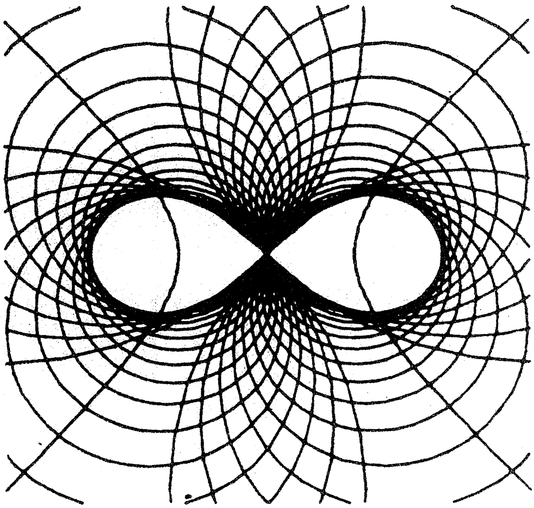
ISSN 0313-6825

FUNCTION

Volume 1 Part 4

August 1977



A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* will contain articles describing some of these uses of mathematics. It will also have articles, for entertainment and instruction, about mathematics and its history. There will be a problem section with solutions invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

EDITORS: G.B. Preston (chairman), N.S. Barnett, N. Cameron M.A.B. Deakin, K.McR. Evans, B.J. Milne, J.O. Murphy, E.A. Sonenberg, G.A. Watterson, N.H. Williams (Queensland)

BUSINESS MANAGER: Dianne Ellis

ART WORK: Jean Hoyle

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
Function,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168

The magazine will be published five times a year in February, April, June, August, October. Price for five issues (including postage): $3.50; single issues 90 cents. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

∞ ∞ ∞

Registered for posting as a periodical - "Category C"

Our leading article this issue is on the mathematics of meteorology and oceanography, areas of science that are highly mathematical. The article, by Dr Fandry, is the text of one of the talks given at Monash this year to an audience of fifth and sixth formers. The article illustrates how a scientist proceeds when he wishes to set up a theory to interpret his observations. He starts with the simplest plausible model and compares its predictions with what actually happens. If the predictions differ too much from the observed data he then tries a refinement of the model; and so on.

No reader has yet attempted Problem 2.6 - or rather has yet sent us his solution. This is a first-class problem. May we invite further efforts?

CONTENTS

THE FRONT COVER

by J. O. Murphy, Monash University

THE LEMNISCATE OF BERNOULLI

This curve has been produced by a set of enveloping circles, of varying radius, all of which pass through the origin of the coordinate system and have their centres on the two branches of the rectangular hyperbola $x^2 - y^2 = a^2$. Historically it is associated with the mathematician James Bernoulli who gave it the name lemniscus in an article (1694) on a curve "shaped like a figure eight".

Alternatively, a lemniscate can be constructed using its polar equation

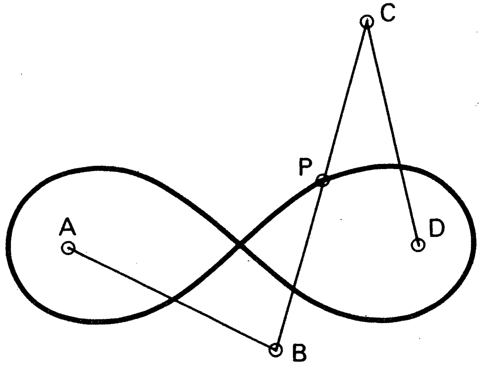
$$r^2 = a^2 \cos 2\theta,$$

or from its Cartesian equivalent

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

Clearly some curves can be constructed by both an envelope method and from plotting a set of points referenced to a set of coordinate axes.

A mechanical† linkage with three sections, as illustrated below, can also be used to draw a lemniscate. If the arms AB and CD are both of length a, the arm BC is $a\sqrt{2}$ (approximately) units long, and the ends A and D are fastened a distance $a\sqrt{2}$ apart then the locus of P, the midpoint of BC, is a lemniscate.



The expression for calculating area in polar form (*Function*, Volume 1, Part 2, p. 2) can be used to establish that the area enclosed by the curve is a^2 square units.

†Suitable materials can easily be obtained to construct this linkage - metal strapping from packing cases for the arms, split paper clips about 3/4" long to secure the movable joints and a stiff piece of cardboard for mounting the system.

MATHEMATICS OF WINDS AND CURRENTS†

by Chris Fandry, Monash University

Without mathematics, the physical sciences of meteorology and oceanography would be sadly underdeveloped subjects with many phenomena lacking precise explanations for their existence. This is not to say that all phenomena associated with physical meteorology or oceanography have already been accounted for by mathematicians, but the enormous progress that has taken place in these fields can be directly attributed to the mathematical modellers (i.e. people who construct mathematical models of physical systems in which the basic relations that hold between the physical quantities in the system hold between the corresponding mathematical variables or constants in the model).

There are two basic branches of each of these disciplines. The first called 'Descriptive Meteorology or Oceanography' deals with the scientific description of the state of the atmosphere or ocean based on observations and measurements. This, of course, is a very important aspect. In meteorology, for example, weather forecasts are made essentially by appeal to data collected daily by weather stations around the globe. Second, we have 'Theoretical Meteorology or Oceanography' which is the art of constructing mathematical *models* to simulate atmospheric or oceanic phenomena which have been observed, or, in a few cases, predict phenomena which have yet to be observed.

This article will attempt to show how an applied mathematician goes about developing a mathematical model for the winds in the atmosphere and the currents in the ocean, and how he refines the model to make it fit what is actually observed. Since both the atmosphere and ocean are composed of fluid (air and water respectively) it may not be too unreasonable to expect that very similar mathematical models apply to both the atmosphere and ocean, and it is with this basic assumption that we construct the simplest model.

The primary cause of fluid motion is pressure difference within the fluid. In the absence of all other forces the consequent *pressure gradient force* will move fluid from an area of high pressure to one of low pressure.

When the atmosphere or ocean is in equilibrium, the weight at any level is balanced by the pressure at that level, and this balance, called the *hydrostatic balance*, may be expressed mathematically by the differential equation

$$\frac{dp}{dz} = \rho g, \qquad (1)$$

† This is the text of a talk to fifth and sixth formers given at Monash University on June 10, 1977.

where p and ρ are the pressure and density respectively, z is a co-ordinate measuring distance vertically downward and g is the gravitational acceleration constant. Departures from this balance in the atmosphere or ocean are very small and this accounts for the relatively small vertical currents in the ocean or vertical winds in the atmosphere. Thus we make the first assumption in this simplest model, that the hydrostatic balance exists, with the resulting consequence of zero vertical velocity.

Having taken care of the vertical direction, we must now look at the balance of forces in the horizontal direction. To do this we first calculate the horizontal force on a block of fluid associated with a horizontal pressure gradient. If we remember that pressure is force per unit area, then the force associated with a pressure p acting over a plane area A is pA in a direction perpendicular (inwards towards the fluid) to the area. Thus considering the block of fluid in Figure 1 with a pressure $p(x)$ exerted on the face of area A at x and a pressure $p(x + h)$ on the face of area A at $x + h$, the total force due to the pressure gradient in the x-direction is

$$p(x)A - p(x + h)A.$$

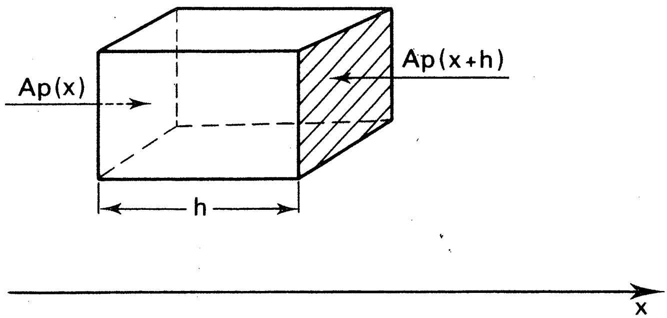


Figure 1: Horizontal forces due to pressure, p, on a block of fluid of lengh h and cross sectional area A.

Thus if ρ is the density of the fluid within the block, the net force per unit mass in the x-direction is given by

$$\frac{p(x)A - p(x + h)A}{\rho A h} = -\frac{1}{\rho}\left(\frac{p(x + h) - p(x)}{h}\right)$$

If p varies continuously in the x-direction, we can write this force per unit mass in terms of the derivative of $p(x)$ (letting $h \to 0$) as

$$F_x = -\frac{1}{\rho}\frac{dp}{dx}. \qquad (2)$$

Note that the negative sign indicates that the force is always in a direction from high pressure to low pressure.

We are now in a position to calculate the horizontal velocity associated with horizontal pressure gradients by applying Newton's second law of motion (see the article 'Cyclones and Bathtubs, Which Way Do Things Swirl?', by K.G. Smith, pp. 15-20, Volume 1, Part 2, of *Function*). Consider the pressure distribution over Australia as depicted in Figure 2. We shall use the above simple model to calculate the average speed and direction of the wind at Melbourne. Let us denote by H (see diagram) the centre of the high in the Southern Ocean. Assuming this system to remain in action

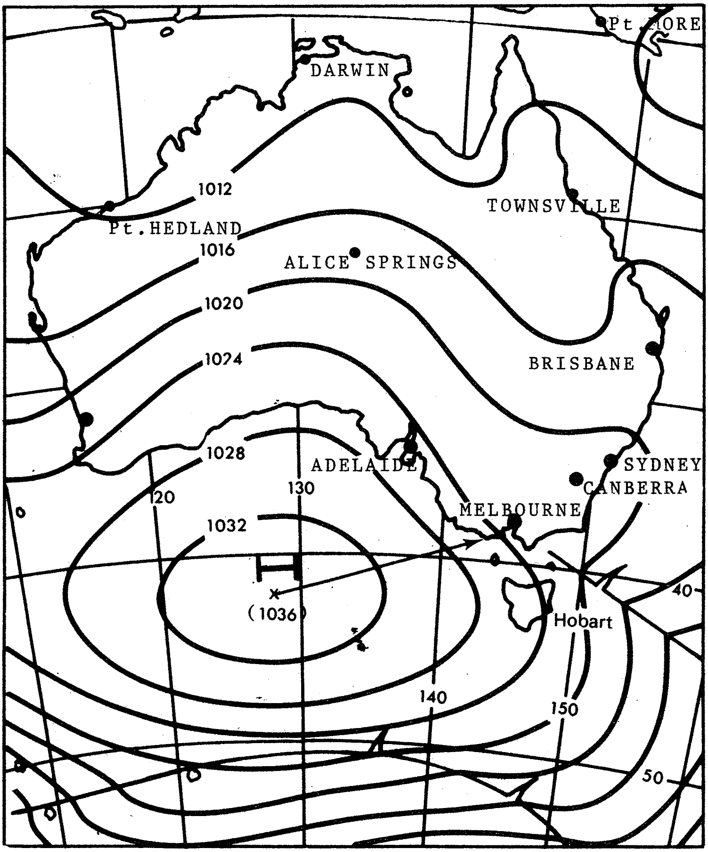


Figure 2

for a period of 24 hours, the average rate of change of velocity, v, in the direction from H to Melbourne is given by

$$\frac{v}{t} = F_x . \qquad (3)$$

We can approximate the pressure gradient $\frac{dp}{dx}$ by $\frac{(1024 - 1036)}{1200}$ mb/km, taking 1200km as the distance from H to Melbourne. Using the fact that 1mb = 1000dynes† $/\text{cm}^2$ and $\rho = 1{\cdot}2 \times 10^{-3}\text{gm/cm}^3$ we find from (2) that

$$F_x = 10{\cdot}8\text{km/hr}^2$$

and hence using (3) with t = 24 hours, we get v = 259km/hr. Our own experience tells us that this velocity is much too large for such weather systems. Moreover, only a little experience in meteorological observations would be necessary to realize that the direction of the wind is wrong. Clearly, therefore, our very simple model for the winds in the atmosphere has failed and a refinement of the basic model is required.

What we have shown is that there cannot be a simple balance between horizontal acceleration and horizontal pressure gradient in the atmosphere, and this is also true in the ocean. Some other force must be important and this, of course, is the Coriolis force; that is, the force associated with the rotation of the earth. The article by K.G. Smith 'Cyclones and Bathtubs, Which Way Do Things Swirl?' in *Function* 1(2), already referred to, contains an explanation of the effects of this force on air particles and an analogous explanation is valid for the ocean. He shows that particles (air or water) will be deflected to the left (right) of their path in the Southern (Northern) Hemisphere due to the Coriolis force.

Consider the situation depicted in Figure 3, with parallel isobars in the Southern Hemisphere. A particle beginning at A experiences a pressure gradient force F_p perpendicular to the isobars from High to Low, and starts to move in that direction. The Coriolis force F_c then acts to move the particle to its left (right as seen on the page), and it therefore moves in a curved path to B where the pressure gradient force is still unchanged, but the Coriolis force F_c has now rotated to a new direction. The particle continues to move in a curved path until the Coriolis force and pressure gradient force balance (at C) which occurs when the particle is moving parallel to the isobars.

---

†One Newton equals 10^5 dynes.

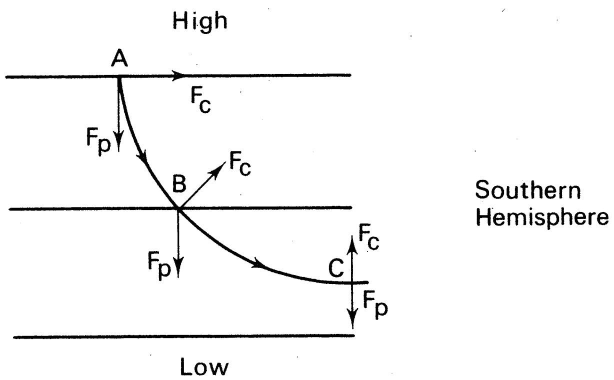


Figure 3: Particle moving under the influence of a pressure gradient force F_p and Coriolis force F_c.

Our revised mathematical model must now take account of the earth's rotation and should express a balance between the pressure gradient force and the Coriolis force, which is termed the *geostrophic balance*.

It can be shown that for a particle moving with speed v at a latitude θ, the Coriolis force has magnitude

$$F_c = 2\Omega v \sin \theta \tag{4}$$

where Ω is the earth's rotation rate (2π radians per 24 hours).

Thus if we recalculate the wind speed associated with the weather system in Figure 2, based on the geostrophic balance, (i.e. putting $F_x = F_c$ and using equations (3) and (4)) it is seen that

$$v = \frac{F_x}{2\Omega \sin \theta}$$

Taking $\theta = 40^\circ$, gives v = 32km/hr in an anticlockwise direction, which is a much more realistic value.

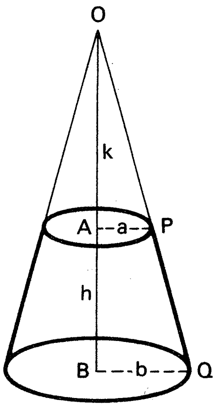
Geostrophic ocean currents are calculated similarly. If the pressure distribution in the ocean is known at a certain level, then the geostrophic ocean current can be calculated at any latitude except near the Equator, where the geostrophic balance breaks down.

Geostrophic winds and currents are the most common form of large scale motion in the atmosphere and ocean and as is the case with many naturally occurring physical phenomena are described by fairly straightforward mathematical equations. There are other forces which come into play in the analysis of atmospheric and oceanic dynamics. The most important ones not mentioned so far are friction and buoyancy forces (caused by vertical density variations), and the addition of these naturally complicates our simple mathematical model.

Further reading on atmospheric dynamics may be found in the Pelican book "Understanding Weather" by O.G. Sutton and in a small book by D.H. McIntosh and A.S. Thom called "Essentials of Meteorology". An extremely well illustrated account of oceanography may be found in the book "Oceanography", a selection of articles from the Scientific American, compiled by J. Robert Moore.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

SOLUTION TO PROBLEM 1.3



Suppose that the frustum has base of radius b and top of radius a. Let h be the height of the frustum and $h + k$ the height of the cone (see picture) of which the frustum is part. The volume of this cone is $\frac{1}{3}\pi b^2(h + k)$, while that of the top cone with base the top of the frustum is $\frac{1}{3}\pi a^2 k$. Hence the volume of the frustum is

$$\frac{1}{3}\pi b^2(h + k) - \frac{1}{3}\pi a^2 k$$
$$= \frac{1}{3}\pi b^2 h + \frac{1}{3}\pi(b^2 - a^2)k. \qquad (1)$$

Since triangles OAP and OBQ, in the picture, are similar,

$$\frac{OA}{OB} = \frac{AP}{BQ},$$

i.e. $$\frac{k}{k + h} = \frac{a}{b},$$

whence $k(b - a) = ha$.

Substitute for $k(b - a)$ in expression (1) and we get that the volume of the frustum is

$$\frac{1}{3}\pi h(b^2 + ba + a^2). \qquad (2)$$

Method (1) of Nesbit gives the volume as

$$\frac{1}{3}h(\pi a^2 + \pi b^2 + \sqrt{\pi a^2 \cdot \pi b^2}),$$

which is easily checked to be the same as (2).

Method (2) gives the volume as

$$(2b \times 2a + (2a)^2 + (2b)^2) \times h \times 0{\cdot}2618,$$

i.e. $$h(b^2 + ba + a^2) \times 4 \times 0{\cdot}2618,$$

which is correct if $4 \times 0{\cdot}2618 = \frac{1}{3}\pi$. Indeed this is an approximation:

$$4 \times 0{\cdot}2618 = 1{\cdot}0472,$$

which is the value of $\frac{1}{3}\pi$ correct to 4 decimal places.

SOLUTION TO PROBLEM 1.7 (Solution from Christopher Stuart)

Denote the terms t_{n-4}, t_{n-3}, t_{n-2}, t_{n-1}, t_n by a, b, c, d, e, respectively. In this notation, the problem is to show that

$$e^2 - c^2 = d(2d + a).$$

Any fibonacci sequence, once two consecutive terms are known, is determined by the recurrence relation $t_m = t_{m-1} + t_{m-2}$. Here we have $c = a + b$, $d = b + c$, $e = c + d$; whence

$$\begin{aligned} e^2 - c^2 - d(2d + a) &= (c + d)^2 - c^2 - d(2d + a) \\ &= d(2c - d - a) \\ &= d(2c - (b + c) - a) \\ &= d(c - b - a) \\ &= 0, \end{aligned}$$

as required.

SOLUTION TO PROBLEM 1.9 (Leonardo's rabbits)

There is one pair of rabbits initially. At the end of one month one pair of offspring is produced. At the end of one further month each of these pairs produces one pair of offspring. The initial pair of rabbits then produces no more offspring. In sequence, we have initially 1 pair; in the first month 1 pair is produced; in the second month, (1 + 1) pairs are produced; in the third month those born in the first and second months will produce, so 1 + 2 pairs are produced; and so on. The sequence of numbers produced in successive months is

$$1, 1, 2, 3, 5, 8, \ldots,$$

which is the Fibonacci Sequence.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

"... the different branches of Arithmetic - Ambition, Distraction, Uglification, and Derision."

The Mock Turtle in Lewis Caroll's
Alice's Adventures in Wonderland

SURE TO WIN
by Chris Ash, Monash University

"I suggest", said George to his father "that tonight we play a game even more in your favour than poker has been the last few times we've played. We throw one die and, if the numbers 1 or 2 come up, I win, otherwise, if 3, 4, 5 or 6 come up, you win an equal amount."

"That's obviously in my favour, so I agree", replied his father.

"In exchange for giving you better odds", added George "I think that I should be allowed to decide on the stake before each throw and when to stop."

"I don't see how that will help you, so I'll agree", said George's father "provided we don't go on all night."

"Let's say that I stop as soon as I have won 200 times", answered George. "Since I should win once in every three throws, that should only take about 600 throws."

"That still sounds a lot."

"Well then", suggested George "let's work out the probable outcome and then settle up, without going to all that trouble."

"By all means. Let's assume you bet at least 10¢ each time." George nodded agreement. "Well, each time we play, I have twice your chance of winning, so I should win about 400 throws to your 200. That way I win 200 throws more than you, which is 200 times 10¢ or $20. The more you bet, the more you lose, so we'll just say you owe me $20", he finished happily.

"Oh no!" George produced paper and pencil. "Since I may choose the stakes, I choose to play as follows. The first throw we play for 10¢. If I win, we play for 10¢ again. But if I lose, we play for 20¢, and if I lose again we play for 40¢, and so on, doubling the stakes each time until I eventually win, and then we go back to 10¢.

"Playing this way, if I win the first throw, I collect 10¢; if I lose the first throw then you get 10¢, but then if I win the second throw, I get 20¢, making an overall profit of 10¢ to me. If you win the first two and I win the third, you get 10¢ plus 20¢, but then I get 40¢, so I still net 10¢. In fact, however long it takes for me to win, when I do win I get back everything you have taken from me since my previous win plus a further 10¢." George demonstrated the point to his father at greater length on the back of his Monash lecture notes. "So you see, each time I win I'm a further 10¢ up, and 200 wins means

\$20 profit. The more I bet the more I win, but I'll settle for \$20. And please may I use the car?"

George's father refused the request, rather irritably, and sat deep in thought for a few moments. Then he brightened. "I see a flaw in your argument" he said.

DO YOU?

"How much cash have you got, George?" he asked. "\$26.00", was the reply. "Well then, if I were to win 8 throws in succession you would have lost, as you just showed me, \$25.50, leaving you with 50¢ plus something less than \$20 which you had won so far. So you wouldn't have enough for your next stake, which should be \$25.60. Of course there's not much chance of this happening before you win once but, if you insist on pushing your luck 200 times, I should say it might happen with substantial probability - at a rough guess something like $1 - \left(1 - (\frac{2}{3})^8\right)^{200}$. We can go into more detail if you like, but you might rather call it quits."

"I'm sure I could rely on your generous nature to give me credit", tried George.

"I must be firm for the sake of your character", answered the patriarch, with a dignity which well became him. "However, you may borrow the car."

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

SOLUTION TO PROBLEM 1.8

The number

$$\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log_e n\right)$$

is usually denoted by γ and is called Euler's constant. It is important in calculus and is involved in the properties of the so-called gamma function $\Gamma(n + 1)$ which is a generalization to non-integral values of n of the factorial function $n!$. The value of γ correct to 7 decimal places is 0·577 2157.

In Norah Smith's article, 'Infinite Series', in this issue, it is proved that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ increases without limit as n increases. Perhaps the most important aspect of the above result is that it shows that $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is approximately equal to $\log_e n$ for large n.

Denote $1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log_e n$ by γ_n. The convergence of γ_n to γ is very slow. You have to work hard with a hand calculator to get close to 0·577 2157. Here is a table of γ_n.

n	γ_n	β_n
1	1	·5
2	·806 8528	·556 8528
3	·734 7210	·568 0544
4	·697 0390	·572 0390
5	·673 8954	·573 8954
10	·626 3832	·576 3832
15	·610 1788	·576 8455
20	·602 0074	·577 0074
40	·589 6636	·577 1636
100	·582 2073	·577 2073
200	·579 7136	·577 2136
400	·578 4652	·577 2152
700	·577 9298	·577 2155
1000	·577 7156	·577 2156

The above values were worked out on an HP25 programmable calculator. The final values were calculated just for multiples of 100 and the program used was:

100, STO 3, 1, STO 1, RCL 1, g 1/x, STO +2, RCL 3, RCL 1, -, g x=0, GTO 18, 1, STO +1, GTO 07, RCL 1, f ln, STO 4, RCL 2, RCL 4, -, R/S, 1, STO +1, 100, STO +3, GTO 07.

We have not yet explained the column headed β_n. As you see β_n approaches γ more rapidly than does γ_n. We define

$$\beta_n = \gamma_n - \frac{1}{2n},$$

so that

$$\beta_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \log_e n.$$

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

FROM THE COVER OF A PACKET OF MAHATMA LONG GRAIN RICE

"Drain in a colander and serve. Add a pat of butter if desired. To keep warm, return to saucepan and cover. Makes about 3 cups of fluffy white rice. (Double all ingredients for more rice. Halve for less rice.)"

The little more, how much it is,
The little less, what worlds away.

∞ ∞ ∞

"Logic, like whiskey, loses its beneficial effect when taken in too large quantities."

Lord Dunsany

SOLUTION TO PROBLEM 1.10

This problem asked you to show that if $(1 + \sqrt{2})^n = a + b\sqrt{2}$, where a, b and n are positive integers, then a is the integer closest to $b\sqrt{2}$.

This is a beautiful and most interesting result and can be used as the basis of a very efficient program to compute $\sqrt{2}$: for example it may be shown that, if $(1 + \sqrt{2})^{20} = a + b\sqrt{2}$, then $\frac{a}{b}$ gives $\sqrt{2}$ correct to 14 decimal places. (See the print-out opposite which gives b_{20} (= 15 994 428), and use equation (7)).

The problem also asks you to use a computer to print a, $b\sqrt{2}$ and the difference between a and $b\sqrt{2}$ as n increases. Doing this highlights the importance of paying attention to round-off errors when using a computer. We used the program on the opposite page on the Monash ECS and have reproduced below it the print-out from $N = 2$ to $N = 30$.

Notice how the errors have accumulated. As we show, in our theoretical discussion below, the difference $a - b\sqrt{2}$, for increasing n, is alternately positive and negative. This sign difference has disappeared as early as $N = 11$, where the difference is shown as $0{\cdot}488\ 281\ 25 \times 10^{-3}$, whereas this difference should be negative. The correct differences are in fact all between -1 and +1. For $N = 30$, as you see, the computer print-out gives the difference as 16 384.

Now for the theoretical discussion. It will be convenient to change the notation and now write

$$(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}.$$

Since, if the left hand side of this equation is multiplied out, powers of $\sqrt{2}$ contribute to the value of a_n only if they are even powers and to the value of b_n only if they are odd powers, we also have

$$(1 - \sqrt{2})^n = a_n - b_n\sqrt{2}.$$

Hence, multiplying,

$$((1 + \sqrt{2})(1 - \sqrt{2}))^n = (a_n + b_n\sqrt{2})(a_n - b_n\sqrt{2}),$$

i.e.
$$(-1)^n = a_n^2 - 2b_n^2. \tag{1}$$

From this, we have immediately that,

$$\begin{aligned} a_n - b_n\sqrt{2} &= (-1)^n/(a_n + b_n\sqrt{2}) \\ &= (-1)^n/(1 + \sqrt{2})^n. \end{aligned} \tag{2}$$

Since $(1 + \sqrt{2})^n > 2$ for all positive integers n, a_n is always, as we had to show, the integer nearest to $b_n\sqrt{2}$.

```
MONASH EDUCATIONAL COMPUTER SYSTEM
RUN AT MONASH UNIVERSITY

*JOB,1234,FTN,JOM1

      C     FUNCTION PROBLEM 1.10
      C     EXPANSION OF (1 + SQRT(2))**N
            EN=1.
            FN=1.
            SR2=SQRT(2.)
            MAXN=100
            DO 1 N=1,MAXN
            EN1=EN+2.*FN
            FN1=FN+EN
            DIFF=EN1-FN1*SR2
            PRINT,EN1,FN1,DIFF
            EN=EN1
            FN=FN1
          1 CONTINUE
            STOP
            END

TOTAL ERRORS  =  0000

TIME =      8
```

3.0000000	2.0000000	0.17157292
7.0000000	5.0000000	-0.07106781
17.000000	12.000000	0.02943802
41.000000	29.000000	-0.01219177
99.000000	70.000000	0.50506592E-02
239.00000	169.00000	-0.20904541E-02
577.00000	408.00000	0.85449219E-03
1393.0000	985.00000	-0.36621094E-03
3363.0000	2378.0000	0.24414062E-03
8119.0000	5741.0000	0.48828125E-03
19601.000	13860.000	0.00000000
47321.000	33461.000	0.39062500E-02
114243.00	80782.000	0.78125000E-02
275807.00	195025.00	0.03125000
665857.00	470832.00	0.06250000
1607521.0	1136689.0	0.12500000
3880899.0	2744210.0	0.25000000
9369319.0	6625109.0	0.00000000
22619538.	15994428.	2.0000000
54608396.	38613968.	0.00000000
0.13183634E 09	93222368.	8.0000000
0.31828109E 09	0.22505870E 09	32.000000
0.76839853E 09	0.54333978E 09	64.000000
0.18550781E 10	0.13117384E 10	128.00000
0.44785551E 10	0.31668165E 10	512.00000
0.10812189E 11	0.76453719E 10	1024.0000
0.26102934E 11	0.18457561E 11	0.00000000
0.63018058E 11	0.44560495E 11	4096.0000
0.15213906E 12	0.10757856E 12	16384.000

Moreover, as we asserted earlier, the differences $a_n - b_n\sqrt{2}$ are with increasing n, alternately positive and negative. Again, since $1/(1 + \sqrt{2})^n \to 0$ as $n \to \infty$, the difference $a_n - b_n\sqrt{2} \to 0$ as $n \to \infty$. Equation (2) measures how fast this difference tends to 0. For example, if $n = 30$, the difference is approximately $10^{-12} \times 3{\cdot}29$. For $n = 11$, the difference is approximately, $10^{-5} \times 6{\cdot}16$.

Another comment on the problem was made by Bruce Rasmussen (Grade 11, Kingaroy High School, Queensland). He computed the value of the determinant $\begin{vmatrix} a_n & b_n \\ a_{n+1} & b_{n+1} \end{vmatrix}$, i.e. $a_nb_{n+1} - a_{n+1}b_n$, and showed it to be equal also to $(-1)^n$. [Mr Rasmussen in fact dealt with the more general situation in which $(c + \sqrt{d})^n = e_n + f_n\sqrt{d}$, say, and showed that $\begin{vmatrix} e_n & f_n \\ e_{n+1} & f_{n+1} \end{vmatrix} = (c^2 - d)^n$.] For our case Mr Rasmussen's result may be shown as follows. From

$$(a_n + b_n\sqrt{2})(1 + \sqrt{2}) = a_{n+1} + b_{n+1}\sqrt{2}$$

we get

$$a_n + 2b_n + (a_n + b_n)\sqrt{2} = a_{n+1} + b_{n+1}\sqrt{2},$$

whence (these equations explain the computer programs)

$$a_{n+1} = a_n + 2b_n, \tag{3}$$

$$b_{n+1} = a_n + b_n. \tag{4}$$

Multiplying (4) by a_n and (3) by b_n and subtracting, we get

$$\begin{aligned} a_nb_{n+1} - a_{n+1}b_n &= a_n^2 - 2b_n^2 \\ &= (-1)^n, \end{aligned} \tag{5}$$

by equation (1).

From (5) we have, dividing by b_nb_{n+1},

$$\frac{a_n}{b_n} - \frac{a_{n+1}}{b_{n+1}} = \frac{(-1)^n}{b_nb_{n+1}}. \tag{6}$$

And from (2), dividing by b_n, we get

$$\frac{a_n}{b_n} - \sqrt{2} = \frac{(-1)^n}{(1 + \sqrt{2})^n b_n}. \tag{7}$$

Equation (7) shows that $\frac{a_n}{b_n}$ tends to $\sqrt{2}$ as $n \to \infty$, the right hand side of (7) showing how fast $\frac{a_n}{b_n}$ approaches $\sqrt{2}$. It is indeed very fast and, as commented earlier, can be used as an iterative process to compute $\sqrt{2}$. Equation (6) shows the difference between successive approximations when $\sqrt{2}$ is computed in this way.

To compute successive values of a_n, b_n, and a_n/b_n you may use either the Basic or the Fortran program below.

Fortran

```
      REAL M,N,L,M1
      M=1.
      N=1.
      DO 1J=1,50
      L=M/N
      PRINT,M,N,L
      M1=M+2.*N
      N=M+N
      M=M1
    1 CONTINUE
      STOP
      END
```

Basic

```
 20 LET A=1
 30 LET B=1
 40 LET D=1
 45 PRINT "N","A","B","A/B"
 50 FOR N=1 TO 50
 60 PRINT N,A,B,D
 70 A1=A
 80 A=A+2*B
 90 B=A1+B
100 D=A/B
110 NEXT N
999 END
```

As a final comment, let us look again at equations (3) and (4). Dividing, we get

$$\frac{a_{n+1}}{b_{n+1}} = \frac{a_n + 2b_n}{a_n + b_n}$$

$$= \frac{a_n/b_n + 2}{a_n/b_n + 1} .$$

Put $a_n/b_n = x_n$. Thus

$$x_{n+1} = \frac{x_n + 2}{x_n + 1} . \tag{8}$$

We have already shown that x_n tends to a limit as $n \to \infty$. Denote this limit by x. Since x_n and x_{n+1} both tend to x, equation (8) tells us that x must satisfy the equation

$$x = \frac{x + 2}{x + 1},$$

i.e. $$x^2 + x = x + 2,$$

i.e. $$x^2 = 2.$$

Thus we have another argument to show that the limit of a_n/b_n is $\sqrt{2}$.

EXPECTATION AND THE PETERSBURG PROBLEM

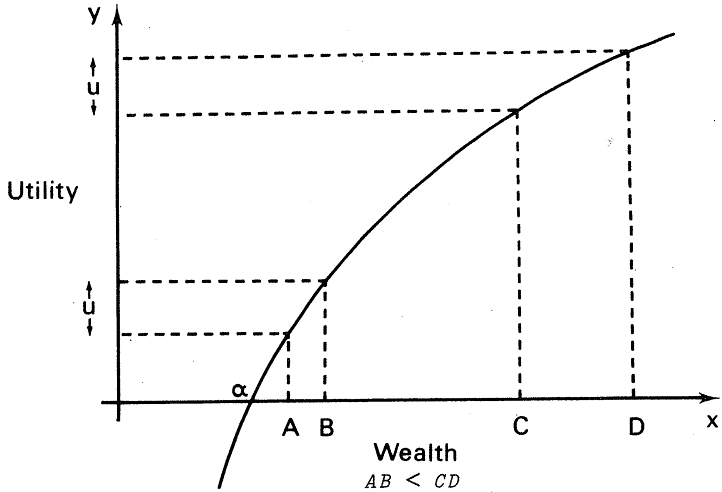
by N. S. Barnett, Monash University

Before embarking on a game of chance a wise man will seek to estimate, in some way, his "likelihood" of eventually winning. One commonly accepted method uses the idea of mathematical expectation. If the gambler is able to calculate his probability of winning at any one play of the game as p then the quantity $py - (1 - p)x$ is called his expected return: here x represents the amount which he must pay to enter the game and y the amount he will win if successful. If this expected return is positive then it represents an amount won, if negative then an amount lost. If the gambler's expected return is zero then the game he is playing is said to be fair. Another way of looking at a fair game is as follows. Suppose that with probability p the gambler wins y at any one play; then the amount he should pay on losing ought to make his expected return zero. To say that his probability of winning at any one play is p is to say that if the game is continually repeated then the proportion of wins will approach the number p. A fair game can thus be interpreted as a game which, if repeated many times, will give each player zero average return. Although the expected gain may be zero, in the short term anything can happen. Extensive past experience in circumstances similar to the one in hand will be needed in order to estimate accurately the value of p (on which the statement of fairness depends).

In 1738 the distinguished Swiss mathematician Daniel Bernoulli proposed some alternative criteria for deciding the fairness of a game of chance. Bernoulli observed that the foregoing definition of expectation does not allow for the difference in value of the same amount of money to two or more people: \$10 000 would mean much more to a pauper than to a millionaire. Bernoulli proposed the calculation of an expectation which takes into account this difference in value. What he said, in essence, was that the expectation of two people embarking on an identical game is different, not only because of probabilistic differences but also because of differences in their initial means (wealth). He maintained that the value of a sum of money is not based on its numerical value but on its utility, meaning its significance to the people concerned. Bernoulli cited numerous examples in which the utility of a certain amount would be higher for a rich man than for a poor one; these he passed off as exceptional and not the norm. He put his proposal in numerical form by suggesting that the utility resulting from any small increase in wealth be regarded as inversely proportional to the quantity of goods previously possessed and directly proportional to the actual small increase in wealth. We can write this mathematically as

$$\delta y = \frac{K\delta x}{x}\ , \tag{1}$$

where K is the constant of proportionality, δy is the increase in utility resulting from the small increase δx in wealth, and x is the wealth previously possessed. Using equation (1) we can see that the graph of utility against wealth has the property that at a high wealth level a larger increase in wealth is needed to produce the same increase in utility than is needed at a lower wealth level. We would expect the graph to be similar to the following



$AB < CD$

An interpretation of equation (1) in terms of the graph is that the gradient of the tangent to the curve at the wealth level $x = a$ is $\frac{K}{a}$. In fact equation (1) is sufficient to enable us to write down the equation of the curve as

$$y = K \log_e x - K \log_e \alpha$$

where α is the value of x at which the curve cuts the x axis. Since this notion of utility is based on an increase in wealth then it is logical to take the utility to be zero for the wealth level corresponding to the amount with which the game starts. We can thus interpret α as the individual's original wealth.

By way of an example consider a person who embarks on a game for which there are only three possible outcomes, namely that he can win amounts x_1, x_2 or x_3 with respective probabilities p_1, p_2, p_3. Wishing the game to be fair we calculate the amount he should pay in order to participate.

Using the definition of fairness based on mathematical expectation we obtain $p_1x_1 + p_2x_2 + p_3x_3$ as the amount he should pay to enter the game. Alternatively, we can use Bernoulli's definition of moral expectation or mean utility, the essence of which we have already discussed.

It is the utility of each possible profit expectation multiplied by its probability of occurrence and summed for all possible profits.

For the current example, the moral expectation is

$$\begin{aligned} & p_1K\log_e(x_1 + \alpha) - p_1K\log_e(\alpha) \\ & + p_2K\log_e(x_2 + \alpha) - p_2K\log_e(\alpha) \\ & + p_3K\log_e(x_3 + \alpha) - p_3K\log_e(\alpha) \end{aligned}$$

$$= K(p_1\log_e(x_1 + \alpha) + p_2\log_e(x_2 + \alpha) + p_3\log_e(x_3 + \alpha)) - K\log_e\alpha,$$

since $$p_1 + p_2 + p_3 = 1.$$

Now the actual wealth corresponding to this moral expectation is found by solving the following equation for x.

$$(p_1\log_e(x_1 + \alpha) + p_2\log_e(x_2 + \alpha) + p_3\log_e(x_3 + \alpha)) - \log_e\alpha$$
$$= \log_e x - \log_e\alpha,$$

i.e. $$\log_e((x_1 + \alpha)^{p_1}(x_2 + \alpha)^{p_2}(x_3 + \alpha)^{p_3}) = \log_e x,$$

giving $$(x_1 + \alpha)^{p_1}(x_2 + \alpha)^{p_2}(x_3 + \alpha)^{p_3} = x.$$

Therefore, the actual increase in wealth needed to produce this moral expectation is $x - \alpha$ i.e.

$$(x_1 + \alpha)^{p_1}(x_2 + \alpha)^{p_2}(x_3 + \alpha)^{p_3} - \alpha.$$

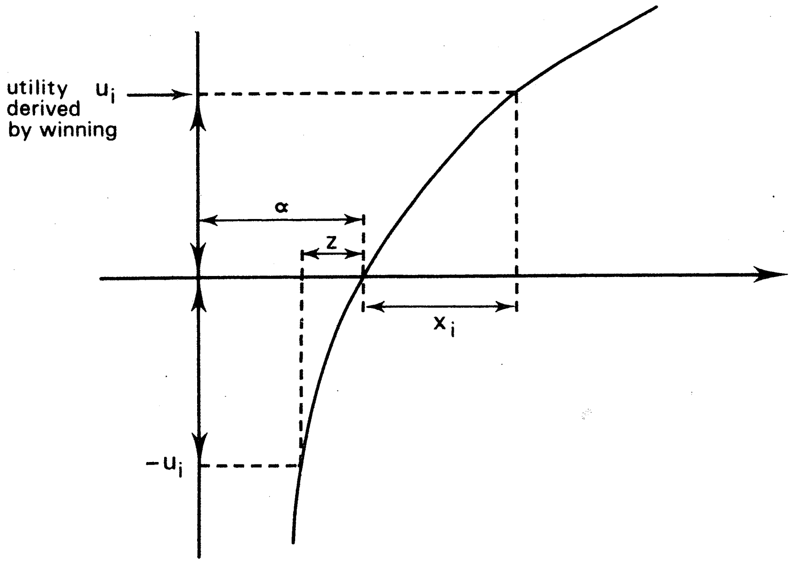
Bernoulli suggested that for a game to be fair the disability to be suffered by losing must equal the utility derived by winning.

From the graph†, adopting Bernoulli's definition of fairness, z should be paid to enter the game. Thus for this example the gambler ought to pay an amount z to enter the game where

$$(x_1 - z + \alpha)^{p_1}(x_2 - z + \alpha)^{p_2}(x_3 - z + \alpha)^{p_3} = \alpha.$$

---

† See next page



It should be noted that under Bernoulli's definition of moral expectation, the idea of a 'fair game' depends on the particular gambler playing. The other gambler may not be playing what to him is a fair game although his opponent has calculated his own game to be fair. On the other hand, using mathematical expectation being zero as the criterion of fairness, ensuring 'fairness' for one player automatically ensures it for his opponent. The utility/wealth curve for the mathematical expectation is merely the straight line $y = x$.

Daniel Bernoulli's cousin, another distinguished mathematician, Nicolas Bernoulli wrote to him asking his opinion on a problem of expectation which had arisen in what has since been called the 'Petersburg Problem'. The problem is as follows.

Two people are involved in a coin tossing game in which one of them receives, from the other, an amount $\$2^{n-1}$ if he obtains his first head at his nth attempt ($n = 1, 2, 3, \ldots$). Obviously the person who will pay out in the event of a head will want to charge his opponent a sum for entering the game which will render the game fair.

Let us assume that the coin in question is fair, i.e. we can assume that the probability of obtaining a head at any one toss is $\frac{1}{2}$. The person tossing the coin thus has a probability of $\frac{1}{2}$ of winning \$1 at the first toss. For his probability of winning at the first or second toss we consider the three events

(heads at first toss),
(tails at first toss and heads at second toss),
(tails at first toss and tails at second toss).

The first of these corresponds to the player winning at his first throw, the second to him winning at his second throw and the third to the event that he continues on beyond the second throw. We can see that the respective probabilities for these events are $\frac{1}{2}$, $(\frac{1}{2})^2$, $(\frac{1}{2})^2$. Developing this process we are able to obtain the probability of the player getting his first head at the nth toss (and of thus winning $\$2^{n-1}$) as $(\frac{1}{2})^n$. Therefore, his expected return (mathematical expectation) is $\sum_{n=1}^{\infty} (\frac{1}{2})^n 2^{n-1}$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

which is infinite in magnitude! Hence following our previous line of reasoning, in order to make the game fair the player should stake $\$\infty$! This result conflicts with our common sense and it was this paradox that prompted the communication between Nicolas and Daniel Bernoulli. Daniel Bernoulli's reply involved using his concept of moral expectation. We have that the amounts by which the player in question can increase his wealth are

$$x_1 = 1,\ x_2 = 2,\ x_3 = 2^2,\ x_4 = 2^3,\ x_5 = 2^4 \cdots$$

with respective probabilities $p_1 = \frac{1}{2}$, $p_2 = (\frac{1}{2})^2$, $p_3 = (\frac{1}{2})^3 \cdots$.

The increase in wealth needed to produce his expected winnings (moral expectation) equals

$$(1 + \alpha)^{\frac{1}{2}}(2 + \alpha)^{\frac{1}{4}}(4 + \alpha)^{\frac{1}{8}}(8 + \alpha)^{\frac{1}{16}} \cdots - \alpha.$$

This can be shown to be

$$\begin{aligned} & 2 \text{ if } \alpha = 0, \\ \doteq\ & 3 \text{ if } \alpha = 10, \\ \doteq\ & 4\tfrac{1}{3} \text{ if } \alpha = 100, \\ \doteq\ & 6 \text{ if } \alpha = 1000. \end{aligned}$$

We can see then, that the amount that should be paid by the player tossing to enter the game (in order to make the moral expectation zero) is z, where

$$(1 + \alpha - z)^{\frac{1}{2}}(2 + \alpha - z)^{\frac{1}{4}}(4 + \alpha - z)^{\frac{1}{8}} \cdots = \alpha.$$

For α very much bigger than z we have that

$$z \doteq (\alpha + 1)^{\frac{1}{2}}(\alpha + 2)^{\frac{1}{4}}(\alpha + 4)^{\frac{1}{8}} \ldots - \alpha. \text{ (Can you see why?)}$$

Yet another famous Swiss mathematician, Gabriel Cramer, independently considered the Petersburg paradox and reached conclusions similar to those of Bernoulli.

The amount won in the Petersburg game is an example of a random variable with an infinite mathematical expectation and such variables gave a lot of difficulty in the early formulation of probability theory. It is possible to determine an amount which a player should pay to enter the Petersburg game so as to make it fair in the sense of the mathematical expectation being zero, provided that the amount paid to enter the game depends on the number of tosses the player makes.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

SOLUTION TO PROBLEM 2.1

If $y = 3x^4 - 4x^3 - 6ax^2 + 12ax$, then

$$y' = 12x^3 - 12x^2 - 12ax + 12a$$
$$= 12(x - 1)(x^2 - a).$$

Stationary points are at $x = -\sqrt{a}$, $x = \sqrt{a}$ and $x = 1$ (assuming $a \geqslant 0$).

(i) If $0 < a < 1$, then $x = -\sqrt{a}$ and $x = 1$ are minima, and $x = \sqrt{a}$ is a (local) maximum of y.

(ii) If $a = 1$, then $x = 1$ is a point of inflexion and $x = -1$ is a minimum.

(iii) If $a < 0$, then $x = 1$ is the unique minimum.

(iv) If $a = 0$, then $x = 0$ is a point of inflexion, $x = 1$ is a minimum.

(v) If $a > 1$, then $x = -\sqrt{a}$ and $x = \sqrt{a}$ are minima, $x = 1$ is a maximum.

Note: $y'' = 12[3x^2 - 2x - a] = 12(1 - a)$ for $x = 1$, $= 12[2a \mp 2\sqrt{a}]$ if $x = \pm\sqrt{a}$. Observe how the nature of the stationary point changes as a varies.

SOLUTION TO PROBLEM 2.3

If $d(P, Q) = 0$ when $P = Q$, and $d(P, Q) = 1$, when $P \neq Q$,

$B((0, 0), 2) = \{P: d(P, 0) < 2\}$ = whole plane,

$B((0, 0), \frac{1}{2}) = \{P: d(P, 0) < \frac{1}{2}\}$ = origin only.

INFINITE SERIES

by Norah Smith, Monash University

In early childhood we become familiar with the addition of finitely many numbers. This process can be conceived as the joining, end to end, of pieces of string whose lengths are measured with respect to some unit. Now let us take things one step further and sum an infinite array of positive numbers $a_1, a_2, \ldots, a_n, \ldots$. What value, if in fact any, can we assign to $a_1 + a_2 + \cdots + a_n + \cdots$? It certainly cannot be the value attained by adding on the last term in the sequence, as in the finite case, since no last term exists!

Consider the infinite series,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots.$$

In order to assign an appropriate value to this sum we shall proceed to obtain some insight into its behaviour. In Figure 1, a unit square *ABCD* is bisected by the diagonal *AC*. One half of the square is shaded and then half the remaining area also. Continuing to do this as in Figure 1, we observe that at the n^{th} stage in this procedure an area equal to $(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n})$ of the unit square has been shaded and as n increases, the entire square is almost attained.

Thus we conclude that the sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

is 1.

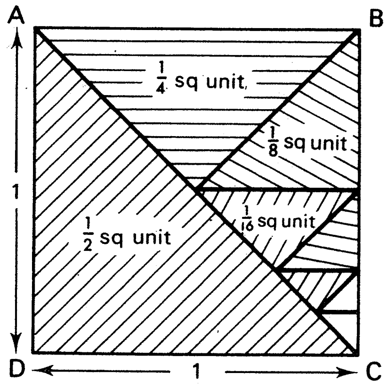


Figure 1

Hence we say that the sum of an infinite series is the value towards which the partial sums (the partial sum to n terms is the sum of the first n terms: $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$) tend but never quite reach unless only a finite number of terms are non-zero.

Another familiar infinite series is

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots + \frac{1}{4^n} + \cdots .$$

To visualize what interpretation may be given to this sum we construct an equilateral triangle $A_1B_1C_1$ with sides of length 1 unit (see Figure 2). The sides of the triangle in Figure 2 are bisected and the centroid is labelled by X.

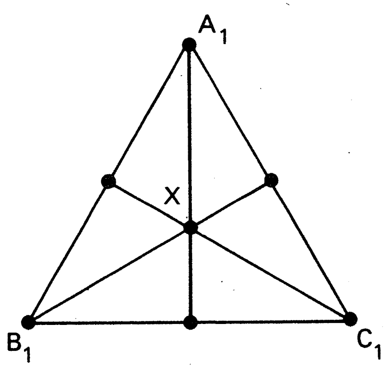


Figure 2

The following procedure is then adopted:

Step 1 Labelling the midpoints of the sides A_1B_1, A_1C_1 and B_1C_1 by C_2, B_2 and A_2, respectively, another equilateral triangle $A_2B_2C_2$ is constructed which divides triangle $A_1B_1C_1$ into four congruent triangles with side of $\frac{1}{2}$ a unit in length. The region indicated on Figure 3 is then shaded. This is equal in area to $\Delta A_2B_2C_2$, that is $\frac{1}{4}\Delta A_1B_1C_1$.

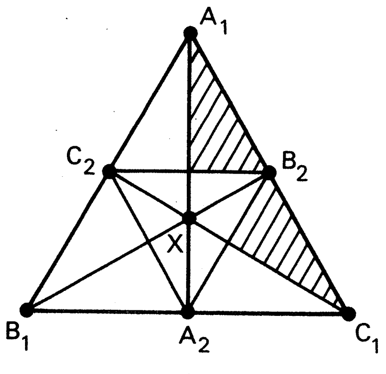


Figure 3

Step 2 Constructing $\Delta A_3B_3C_3$, where A_3, B_3 and C_3 are the midpoints of the sides of $\Delta A_2B_2C_2$, divides $\Delta A_2B_2C_2$ into four equilaterial triangles of side length $\frac{1}{4}$ of a unit. The region equal to $\frac{1}{4}$ of the area of $\Delta A_2B_2C_2$, and therefore $\frac{1}{16}A_1B_1C_1$, is then shaded as indicated in Figure 4.

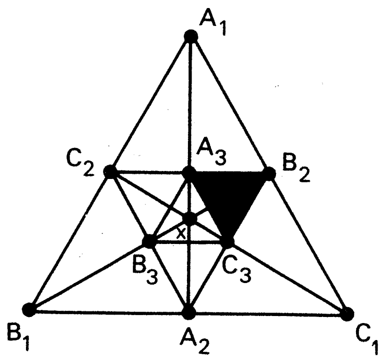


Figure 4

Now applying the process to $\Delta A_3B_3C_3$ and continuing, we obtain, on the K^{th} repetition, triangles $A_{2K-1}B_{2K-1}C_{2K-1}$ and $A_{2K}B_{2K}C_{2K}$ with shaded regions as indicated on Figures 3 and 4. Clearly for any integer n, $\Delta A_nB_nC_n$ has edges of length $\frac{1}{2^{n-1}}$ of a unit and shaded region equal in area to $\frac{1}{4^{n-1}}A_1B_1C_1$.

Accumulating the shaded regions on $\Delta A_1B_1C_1$ we see that with each application of the above process, more and more of ΔA_1C_1X is being claimed, although this entire triangle is never quite shaded in.

So we may conclude that the sum of the areas of the infinite array of triangles $A_2B_2C_2$, $A_3B_3C_3$, ..., $A_nB_nC_n$ ... is the area of triangle A_1C_1X, which is precisely one third of $\Delta A_1B_1C_1$, that is,

$$\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^n} + \cdots \text{ is } \frac{1}{3}.$$

Perhaps you would like to try for yourselves, interpreting geometrically the sums of other infinite geometric series. Not every geometric series behaves in the same way. For example,

$$1 + 2 + 4 + 8 + 16 + \cdots + 2^n + \cdots$$

grows larger and larger as each term is added on and evidently the values of the partial sums hover around no fixed number. In fact, writing down a general geometric sequence a, ar, ar^2, ..., ar^n, ..., (with r positive) and putting $S_n = a + ar + ar^2 + \cdots + ar^n$, we see that since

$$rS_n = ar + ar^2 + \cdots + ar^{n+1},$$

$$S_n = \frac{a(1 - r^{n+1})}{1 - r} \quad \text{if } r \neq 1. \tag{1}$$

By inspection of formula (1) we notice that if $r < 1$ the value of S_n grows closer and closer to $\frac{a}{1 - r}$ as n becomes larger, as r^{n+1} gets closer to 0. (Check this if you have any doubts!) In conclusion we may say that the sum of the infinite series is $\frac{a}{1 - r}$ if $r < 1$, and that there exists no number which the partial sums approach whenever $r \geqslant 1$.

Recognition that the sum of an infinite series is a value which is almost reached but never attained dates as far back in history as the fifth century B.C., as reflected in the paradoxes of the Greek philosopher Zeno of Elea (c. 450 B.C.). Zeno's observations were disturbing to the philosophers of the times as recorded by Aristotle (384-322 B.C.), since they conflicted with their intuitive concepts of the infinitely large and the infinitely small.

One paradox may be presented as follows:

'A man wishes to journey from A to B. In order to do so he must first cover half the distance, then half the remaining distance and so on. In this manner there always exists a fraction of the path for him to cover.'

The distance he travels, in this way of dividing it up, is just the sum of the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$ which we know from (1) has sum equal to 1. Imagine how disconcerting it was for Zeno to find that by his reasoning any destination would be impossible to reach even though within grasping distance! The dilemma should not be as quickly brushed aside as it was by Diogenes the Cynic, who refuted it by his jubilant walk from A to B in record time - less than a lifetime! Can you supply a more constructive explanation of his error in thinking? The paradox and another equally famous one of Zeno's which follows, have daunted mathematicians over the centuries. In fact not until the early nineteenth century, when Cauchy developed mathematical analysis in connexion with the theories of light and mechanics, was there a beginning to a definite solution.

In Zeno's second paradox† he chose earth's fastest creature Achilles and the slowest animal the tortoise to demonstrate that however fleet o'foot Achilles might be he could never overtake another animal in motion. He reasoned as follows:

Suppose Achilles moves 100 times faster than the tortoise. If the tortoise is a unit distance ahead of him, then by the time Achilles has reached this spot, the tortoise will have moved on a distance of 1/100th of a unit further. Achilles then has to cover this distance only to find the tortoise still in front by 10^{-4} of the unit distance; and so on. In this way Achilles can never overtake the tortoise.

The fallacy which emerges is that any infinite sequence of time intervals will always sum to eternity. It is therefore not surprising that such an explanation by Zeno's contemporaries was not forthcoming since it was the thinking of the time that, irrespective of whether the terms grow infinitely small or not, an infinite sum would always be infinitely large.

It is worth commenting, furthermore, that Zeno's contemplation of infinite series is thought to have influenced the development of Greek geometry. The paradoxes laid open the intuitive concepts of "point" and "line", since he displayed the breaking up of a line into infinitely many small segments each of finite length, which aroused uncertainty in describing lines as composed of points.

Having indulged in this historical excursion, let us now return to the mathematics of the infinite series. We have

---

†There are in fact four paradoxes attributed to Zeno. They are entitled the *Dichotomy*, the *Achilles*, the *Arrow* and the *Stadium*. See: Cajori, F., 'The History of Zeno's Arguments on Motion'. *Amer. Math. Monthly*, Vol. 22, 1915.

looked at the infinite series

$$\frac{1}{4} + \frac{1}{16} + \cdots$$

$$\text{and} \quad 1 + 2 + 4 + \cdots .$$

It was seen that the first of these *converges* (grows close to some number) whereas the latter *diverges*. A natural question arises: When does a series converge? It is tempting to conclude that all series whose terms approach zero converge. This is not so, however, as we see by considering the so-called harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots . \tag{2}$$

The terms certainly decrease to zero, yet careful inspection yields that

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} > \frac{1}{16} + \cdots + \frac{1}{16} = \frac{1}{2};$$

and so on.

Hence the sum in (2), if it exists, exceeds the sum of as many $\frac{1}{2}$'s as we please. So the partial sums just get bigger and bigger. The infinite sum does not exist. We say the series *diverges*.

It is an interesting consequence of the harmonic series being divergent that, assuming a supply of bricks, as large as is necessary, we may stack them in such a way that the top brick is any distance we desire from the brick at the bottom. This is achieved as follows.

Take two bricks. To ensure stability, yet have the top brick protruding as far as possible, we have an offset of $\frac{1}{2}$ a brick's length. Placing these two bricks on a third brick results in a maximum offset of the top brick of $\frac{3}{4}$ of a brick length (see Figure 5). Proceeding to stack the bricks by placing the centre of mass of the n bricks above the end of the $(n + 1)$th brick we have an offset of size

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} = \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right),$$

which we have seen, from (2), may grow as large as desired, with increasing n,

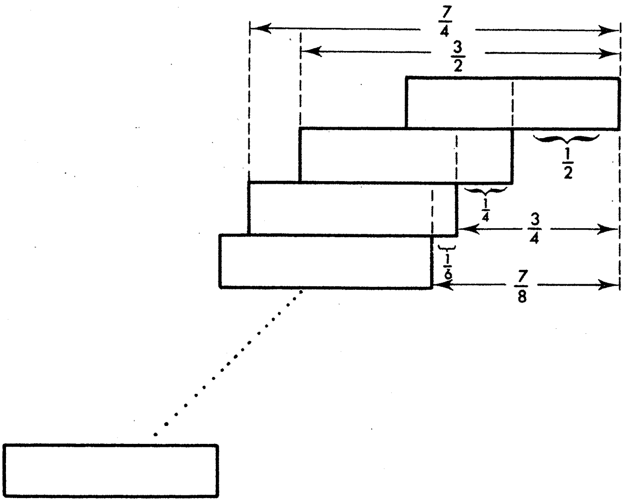


Figure 5

As an exercise, calculate the number of bricks needed to ensure an offset of 1, 5 and then 10 brick lengths. You should observe the steep increase in number of bricks required.

If any person is wondering what would happen if divergent series were assigned "numbers" as their sums according to some definition, they should be warned to take care. For instance, suppose we put a "number" x, say, equal to

$$1 + 1 + 1 + \cdots + 1 + \cdots.$$

Then, since $1 + 1 + \cdots + 1 + \cdots = 1 + (1 + 1 + \cdots)$, we would have $x = 1 + x$. Consequently we cannot cancel x, for this would give $0 = 1$. Thus x cannot be one of our usual real numbers. It is often desirable in calculations to say that the sum of such a series is ∞. But be careful how you manipulate the symbol ∞.

Perhaps in rounding off our survey, something should be said concerning those infinite series with some negative terms. Our above discussion has neglected to account for such series. Their behaviour differs from those series having only positive terms in one striking aspect; namely, the same sum is attained in the latter case irrespective of the arrangement and grouping of the terms whereas in the former, it makes a considerable difference. For instance, consider

$$1 + (-1) + 1 + (-1) + \cdots + (-1)^n + \cdots \qquad (3)$$

Grouping the terms thus, $(1 + (-1)) + (1 + (-1)) + + (1 + (-1)) + \cdots$ we would conclude that the sum is 0, whereas on the other hand, writing it $1 + ((-1) + 1) + + ((-1) + 1) + \cdots$ the resulting sum would be 1.

This series, whose successive partial sums oscillate between 0 and 1, caused the Italian mathematician and priest Guido Grandi (1671 - 1742) of Pisa to claim this as the symbol of creation for in his eyes it illustrated God's ability to start with nothing and by simple arrangement create everything! Grandi desired a value for the sum of (3) and declared it to be $\frac{1}{2}$, supporting his view by the following parable:

A man dies leaving to his two sons a priceless gem, and stipulates in his will that it should be shared between them by each having it in his possession on alternate years. In conclusion he says that ownership of the gem by the sons is precisely halved.

This parable is not a particularly convincing argument for the adoption of $\frac{1}{2}$ as the sum, since there are infinitely many terms in (3) whereas any lifetime spans but a finite number of years. However there are in fact mathematical theories of divergent series in which it is useful to give this series the value of $\frac{1}{2}$.

Changing the signs of some of the terms in infinite series can change a divergent series to a convergent series. For instance transforming the harmonic series (2) by multiplying the nth term by $(-1)^{n+1}$ results in the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots - (-1)^{n+1}\frac{1}{n} + \cdots,$$

which is convergent. Its sum is $\log_e 2$.

On this note it is left to the reader to extend and experiment with a concept which has occupied mathematicians throughout the centuries and led them to conclusions both erroneous and profound.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

THE PRECISION OF NUMERICAL STATEMENTS

A recent television commercial states:

"Two out of three dentists, responding to a survey, recommend ..."
or is it
"Two out of three dentists responding to a survey recommend..."?
Did two dentists out of every three surveyed respond and recommend, or did it happen that out of those who responded two thirds recommended? Or possibly they surveyed three dentists, of whom two responded and recommended, or what?

A BOOK REVIEW

"I am unable to commend this book on any grounds, or to understand why it should have any prospect of competing with much better books already on the market. It is not attractively written nor, so far as I can see, particularly "practical," and the author's knowledge of the theory may be estimated in ten minutes by any competent critic. Such a critic I would refer in particular to the first four pages, to a discussion of the differentiation of x^n (pp. 9-10), to the treatment of differentials (p. 68), or that of areas (pp. 88-92).

Mr Baker confines himself for the most part to the reproduction of other people's mistakes, but occasionally indulges in the expression of his own opinions, as when he defines an "independent variable" as "a quantity to which we may assign any value" (p. 1), or says that the differential coefficient "always exists in functions of every kind" (p. 6)."

A review, in the *Mathematical Gazette*, 1913, by G.H. Hardy of *Calculus for Beginners*, by W.H. Baker, 1912

Readers are invited to submit reviews of their own textbooks.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

SOLUTION TO PROBLEM 3.5 (from Alisdair McAndrew, Sixth Form, Melbourne High School)

We must find, first, the probability that a five was not thrown, given that, on the nth throw, a six was thrown. This is equal to the probability that no fives were thrown, and a 6 was thrown on the nth throw, divided by the probability that no fives were thrown.

Thus our probability is $\dfrac{(4/6)^{n-1}(1/6)}{(5/6)^{n-1}}$. However this is not the answer to the problem. Our answer is equal to:

$$1/6 \sum_{i=1}^{\infty} (4/5)^{i-1}, \text{ as } \frac{(4/6)^{n-1}}{(5/6)^{n-1}} = (4/5)^{n-1}.$$

This comes out to be 5/6.

SOLUTION TO PROBLEM 3.1 (from Christian Cameron, Year 8, Glen Waverley High School)

By inspection 23 is the smallest number which works, the next is 50. Trying $23 + 27x$, this works. It must be because dividing by 3 three times means dividing by 27, so we have all the numbers.

SOLUTION TO PROBLEM 3.4 (from Magnus Cameron, Grade 5, Glen Waverley Heights School)

9 pages use 9 digits, 10 to 99 use 180 digits, 100 to 999 use 2700 digits. A book with 999 pages uses 2889 digits.

We only have 1890 digits so we need 999 digits less so 333 pages less. Our book must have 666 pages.

(Also solved by Christian Cameron and Alisdair McAndrew.)

PROBLEM 4.1 (supplied by Alisdair McAndrew)

Imagine a circle rolling, without slipping, on a flat surface. At the same time, a plank rolls (without slipping) along the top of the circle. What is the ratio of the speed of the plank to the speed of the centre of the circle?

PROBLEM 4.2

The CSIRO division of Mathematics and Statistics Newsletter for August 1977 asks whether or not $n^4 + n^3 + n^2 + n + 1$ is a perfect square for any integer n greater than 3. Show that, in fact, the only integral values of n making $n^4 + n^3 + n^2 + n + 1$ a perfect square are $n = 3$, $n = 0$, $n = -1$.

PROBLEM 4.3

A student believed that $\frac{d}{dx}\{u(x)v(x)\} = u'(x)v'(x)$. Using his formula, he correctly differentiated $(x + 2)^2x^{-2}$. What relation must hold between a pair of functions $u(x)$, $v(x)$, for him to get a correct answer? Give some other examples.

PROBLEM 4.4

Prove that no number in the sequence

11, 111, 1111, 11111, ...

is the square of an integer.

PROBLEM 4.5

A repair shop has three boxes, one containing left-foot bicycle pedals, another containing right-foot bicycle pedals, and a third containing both left and right-foot pedals. Labels describe the contents of the boxes. A naughty customer changed all the labels around. You are allowed to inspect *one* pedal from *one* box. Which box should you choose it from in order to identify which box is which?

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

APPROXIMATIONS TO e AND γ (Euler's constant)

$e \cong \left(3^2 + 9^2 + (14^2 + 4^2)/(112^2 - 13^2)\right)^{2/9}$ (accurate to 9 decimal places)

$\gamma \cong 1/(15 + 202/333)^{1/5}$ (accurate to 6 decimal places).

Both were invented by Alisdair McAndrew.

"He was 40 years old before he looked on Geometry; which happened accidentally. Being in a Gentleman's Library, Euclid's Elements lay open, and 'twas the 47 *El. libri* I [Pythagoras' Theorem]. He read the Proposition. *By G——*, sayd he (he would now and then sweare an emphaticall Oath by way of emphasis) *this is impossible*! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. *Et sic deinceps* [and so on] that at last he was demonstratively convinced of that trueth. This made him in love with Geometry.

I have heard Mr Hobbes say that he was wont to draw lines on his thigh and on the sheetes, abed, and also multiply and divide."

Thomas Hobbes John Aubrey's *Brief Lives*, 1669-1696
Edited by Oliver Lawson Dick, 1958

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

"One of the chief ends served by mathematics, when rightly taught, is to awaken the learner's belief in reason, his confidence in the truth of what has been demonstrated, and in the value of demonstration."

Bertrand Russell: *The Study of Mathematics*, 1902

∞ ∞ ∞ ∞ ∞ ∞ ∞

The Learning of this People is very defective; consisting only in Morality, History, Poetry and Mathematicks; wherein they must be allowed to excel. But, the last of these is wholly applied to what may be useful in Life; to the Improvement of Agriculture and all mechanical Arts; so that among us it would be little esteemed. And as to Ideas, Entities, Abstractions and Transcendentals, I could never drive the least Conception into their Heads."

Jonathan Swift: *Gulliver's Travels: A Voyage to Brobdingnag*, 1726

∞ ∞ ∞