THE MAGIC HEXAGON

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Many readers will be familiar with the magic squares - arrangements like that shown in Figure 1. The nine (in this case) small squares form a larger square with the property that for each row, each column and both diagonals, the sum of the numbers involved is 15.

4	9	2
3	5	7
8	1	6

Figure 1

Another feature in this arrangement is the use of each of the consecutive numbers one to nine exactly once.

There are other magic squares. A 4×4 square is depicted in Durer's famous engraving *Melencolia I.*† Here the numbers 1 to 16 are arranged in such a way that each row, column or diagonal sums to 34.

The study of such magic squares can hardly be said to be a major theme of mathematics, but it is an interesting and widely known recreational topic. It becomes more and more complicated as the size of the square is increased, and much remains to be discovered, even for relatively small squares. Often amateurs surprise professional mathematicians by finding previously unknown results.

You might like to try your own hand exploring this area. For a start, calculate what the sum of the numbers should be in a 5×5 square, and an $n \times n$ square.

Apparently more complicated than the magic squares are the magic hexagons. Regular hexagons pack neatly as in Figure 2. Here 19 small hexagonal cells are placed together to form a shape which, while not a hexagon, has the same six-sided symmetry as a hexagon. By a slight, but allowable, misuse of language,

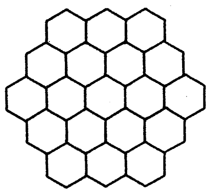


Figure 2

† Made in 1514. On display at the British Museum.

this figure is referred to as a hexagon: in this case a hexagon of order three, as there are three cells on each side.

If we examine the structure of the hexagon in Figure 2, we see that there are 5 horizontal rows of cells, 5 rows slanting from top left to bottom right of the page and 5 slanting from top right to bottom left. There are 19 cells in all arranged in a total of 15 rows.

The problem is to arrange the numbers 1 to 19 in the cells of Figure 2 so that the sum along each row is the same as the sum along every other row. We're not asking you to do this (for reasons which will become obvious), but you could try to see why each row must add up to 38.

The answer to the arrangement problem is usually attributed to Clifford Adams, an amateur mathematician who may be said, without exaggeration, to have devoted half a lifetime to its solution.

Adams, a railway clerk, began his search in 1910. He had a set of hexagonal ceramic tiles specially made, each bearing a number from 1 to 19, and used these in an experimental effort of mammoth proportions. (Disregarding the different points of view achieved by rotations and reflections, how many combinations are there?)

His spare time was devoted to this problem for 47 years. He finally found a solution while convalescing following an operation and jotted it down on a piece of paper. When he returned home, however, he found that he had mislaid the solution.

It attests to his determination that for five years, he continued (he had by then retired) his efforts to reconstruct the solution. He never succeeded. Instead, he had the good luck to locate the missing piece of paper.

He forwarded a copy to Martin Gardner, the Scientific American columnist, in December 1962. (If you don't know Gardner's columns and the Problem Books he compiles from them, you have a treat in store.)

Let Gardner now take up the story:

"When I received this hexagon from Adams, I was only mildly impressed. I assumed that there was probably an extensive literature on magic hexagons and that Adams had simply discovered one of the hundreds of order-3 patterns. To my surprise a search of the literature disclosed not a single magic hexagon. I knew that there were 880 different varieties of magic squares of order 4, and that order-5 magic squares ... [had not then] ... been enumerated because their number runs into millions. It seemed strange that nothing on magic hexagons should have been published."

Gardner contacted Charles W. Trigg, a United States mathematician with a wide reputation in the area of combinatorics (the branch of mathematics involved) and asked for his opinion. Trigg took a month to reply, but the answer was worth waiting for.

Apart from trivial alterations caused by rotation or reflection, *Adams' magic hexagon was the only one that could exist.*

Well, not quite. There is one other. Here it is: 

This is so trivial that we don't count it. It is easy to see also that there is no magic hexagon of order 2. Suppose we have an order-2 hexagon as shown in Figure 3. The numbers one to seven must be arranged in the cells so that nine different sets of numbers all add up to the same figure. Suppose the top entries are a and b, as shown. Then all rows must add up to $a + b$, whatever that may be. But now what are we to put in the far left cell? We have:

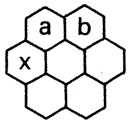


Figure 3

$$a + x = a + b$$

so that

$$x = b,$$

and the number b is used twice, which is against the rules.

(An alternative impossibility proof notices that the row sum must be 28/3. Can you produce this proof?)

Two more things remain to be proved in order to show that Trigg's theorem (if we may so term the uniqueness claim) is true. We need to be assured that:

(1) There is no magic hexagon of order n, if $n > 3$.

(2) Among all the (how many did you get?) possibilities of order 3, only one is magic.

At first sight, we would think that the first statement, which comprises infinitely many cases, would be harder to prove than the second. In point of fact, this is not the difficult part. The proof is a little long to include in this article, and contains some ideas that will be new to, but not above the capabilities of, the readers of *Function*. Interested readers will find it on pages 71-73 of Ross Honsberger's *Mathematical Gems*. (A more cryptic account is given by Martin Gardner, *Scientific American*, August 1963, p. 116.)

It remained to Trigg to show that of all the (?) possibilities of order 3, only one was magic. This he accomplished in a proof that, on Gardner's account, "..... used a ream and a half [750 pages] of sheets on which the cell pattern had been reproduced six times", i.e. the "answer was obtained by combining brute force ..... with clever short cuts".

That short cuts were necessary may be seen easily enough. There are (?) possible combinations, of which Trigg needed to discuss only 6 × 750.

The case is somewhat reminiscent of that discussed by John Stillwell in the first issue of *Function*. In discussing the four colour problem, he referred to the Haken-Appel solution as "a barbarous way to do mathematics", and our editorial indicated that some check was necessary before the result could be unprovisionally accepted.

Trigg's theorem provides a similar case. The result is not important enough for anyone to pay for its publication. It is no slight on Gardner to say that he probably did not check all the details. Are we then to hold Trigg's theorem unproved, or only probably right?

In this case, the answer is "no". The result was proved independently by Frank Allaire (in 1969). Allaire was then a second-year student at the University of Toronto. Using an elegant computer programme, Allaire reduced the problem to 70 cases (each involving many sub-cases) and confirmed Trigg's theorem in 17 seconds of computer time. Enough of his method is now public (see, e.g., pp. 73-76 of *Mathematical Gems*) that any bright young mathematician with a flair for combinatorics and computing can check the result.

Trigg and Allaire thus not only duplicated the result of Adams' search, but extended it. Trigg (without a computer by the way) did more in a month than Adams achieved in 47 years. However, just as Allaire knew from Trigg's work what he had to aim for, so Trigg knew from Adams' more pioneering efforts where he was going. (Trigg's "clever short cuts", the result of a well-practised mathematical mind, had much to do with this also.) Did Adams himself have some guiding star? It appears now that he may have done. Gardner more recently (in *University of Chicago Magazine*, Spring, 1975) shows a picture of a puzzle incorporating the magic hexagon. This was patented in 1896 by William Radcliffe, a schoolteacher on the Isle of Man. Was Adams influenced by a (possibly subconscious) memory of Radcliffe's puzzle?

Two other possible discoverers exist, although they too may owe a debt to Radcliffe. An unpublished manuscript from wartime Germany (1940) contains the result. The author is Martin Kühl of Hanover.

More ironically, a lot of the time Adams was agonising over his lost paper, the result was in print, widely distributed, but unrecognized. It is published, as a diagram, with no words at all, in *Mathematical Gazette* (1958), p. 291. The author of the strangely silent article was Tom Vickers. Perhaps the reason that Vickers' result was overlooked is the one which will strike any reader who cares to look it up. Visually, it's quite different from Figure 2. (For a start, it's a *hexagon* and not just a courtesy hexagon.)

Another reason is more subtle, but perhaps more important. Vickers' paper was overlooked, just as Adams' paper was almost dismissed by Gardner. Both were seen at first as quaint numerical curiosities. These are not, as such, of great mathematical interest. They merit attention, however, when we can say something truly surprising about them - when we "embed" them in deeper, more general, results. This is what Trigg did in this instance. Once he had done that, people took notice.

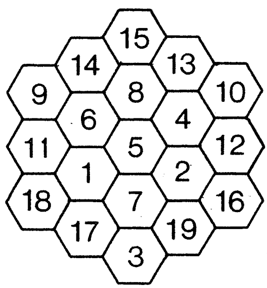
At the beginning of this article, I referred to magic hexagons as being "apparently more complicated than magic squares". They are not really more complicated, however. There is only one of them. Meanwhile, results on magic squares proliferate and we have no comprehensive framework in which to place them. Provision of such frameworks is what mathematics is about.

Here are two problems to consider:

(a) Equilateral triangles can fit together to form larger equilateral triangles; small squares combine into larger squares; regular hexagons interlock to make up "hexagons". No other regular plane figures can do this. Why not?

(b) We have talked of magic hexagons and magic squares. What of magic triangles?

P.S. You didn't really think I was condemning you to 47 years' hard labour, did you? Here is the magic hexagon:



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