CATASTROPHE THEORY

by M. A. B. Deakin, Monash University

According to its inventor, the French topologist René Thom, Catastrophe Theory is more an attitude of mind than a mathematical theory. Even if he is right in this, however, there is a lot of valid and interesting mathematics in the area. Not only is the mathematics new and exciting in its own right, but it has also attracted a lot of attention because of its apparent potential for applications.

Underlying this potential is a postulate that constitutes Thom's "attitude of mind". We may express this epigrammatically as *Nature is almost always well-behaved.* In a very general way, we can represent a natural process as in Figure 1. The output, or result, generally depends in a smooth and unremarkable way on the input, or conditions applied.

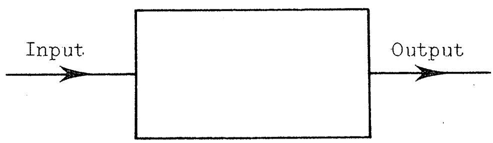

Figure 1

For instance, the density of a liquid depends in a smooth and relatively simple way on its temperature. The precise formula expressing the dependence may be very complicated, not expressible in familiar form, or even unknown, but the curve is an ordinary graph, such as Figure 2 which shows the behaviour of water.

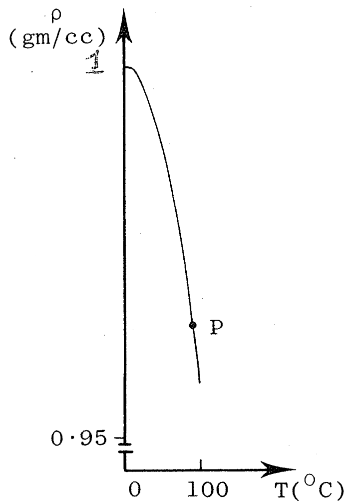

Figure 2

Smooth functions such as this are studied by the use of calculus. For our present purposes, we may say that the essential insight provided by calculus is the result: *sufficiently close to any point P on its graph, a function may be approximated to arbitrarily good accuracy by a straight line, namely the tangent at P.*

In our example, ρ is the density, T the temperature, and ρ_0, T_0 are the values of these variables at P. The line then has the formula

$$\rho - \rho_0 = -\beta\rho_0(T - T_0),$$

where β is a constant known as the coefficient of bulk expansion. Formulae such as this are everyday fare and depend, as this one does, on the tangent (or local linear) approximation.

In the neighbourhood of certain special points *(stationary points)* the local linear approximation is constant - i.e. the tangent is flat. (The curve of Figure 2 has a stationary point at $T = 4$.) However, the local linear approximation tells us little about the shape of the curve. For further information we must look at a local quadratic approximation. If the curve can be approximated locally by a parabola whose shape is like that of $y = x^2$, we have a minimum point; if the shape resembles that of $y = -x^2$, we have a maximum point. (We can also use a similar analysis at points other than singular points to achieve a better approximation than the local linear one.)

As a matter of fact, all but a few rather unusual maxima and minima may be locally approximated by parabolae. (The n-dimensional generalisation of this statement is known as Morse's Lemma; it is the starting point of a branch of mathematics known as Morse Theory which is closely related to Catastrophe Theory. Morse Theory is not, however, related in any way to Morse Code. It is named after its inventor, the American mathematician Marston Morse.)

A related result (which you should be able to prove) states that every quadratic $y = ax^2 + bx + c$ may, by suitable choice of origin and by suitable scaling of the axes, be reduced to the form $y = x^2$. (So all parabolae can be drawn using a single template!) This theorem, incidentally, was in the Victorian school curriculum 25 years ago.

When we come to cubics, something more interesting happens. Any cubic equation $y = Ax^3 + Bx^2 + Cx + D$ can be reduced to the form $y = x^3 + ax$ by suitable placement of the origin and suitable scaling of the axes. (Can you prove this?)

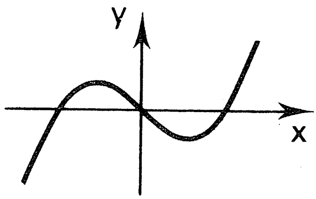

Figure 3

However, the shape of the graph now depends upon the sign of a. If $a < 0$, then the graph is as shown in Figure 3, while if $a > 0$, the appearance is that of Figure 4.

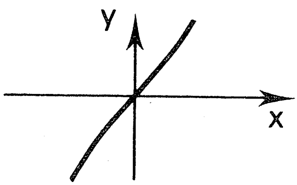

Figure 4

We may now imagine a situation such as that shown in Figure 5. Here a particle is able to rest at the minimum point P as long as that minimum exists; that is, as long as $a < 0$.

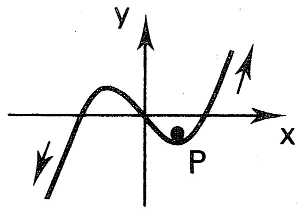

Figure 5

But now suppose that the value of a is progressively increased, e.g. by slowly pulling on the curve in the direction of the arrows. Eventually, we reach a configuration for which $a = 0$ and the minimum ceases to exist. The particle now falls to the left. A quite continuous input has produced a sudden jump in the output. The particle will fall until it encounters another minimum (beyond the range of validity of the local cubic approximation).

Such a sudden change is termed a *catastrophe*. Catastrophes are the exceptions to the orderly behaviour of nature.

The density of water manifests a similar behaviour, for the water boils if heated sufficiently. Figure 6 shows a more complete graph than Figure 2. (Note that the scale has had to be altered. The density of steam is about 6×10^{-5} gm/c.c., and this is so small as to be indistinguishable from zero on either scale. The slight lean of the apparently vertical line is barely visible.)

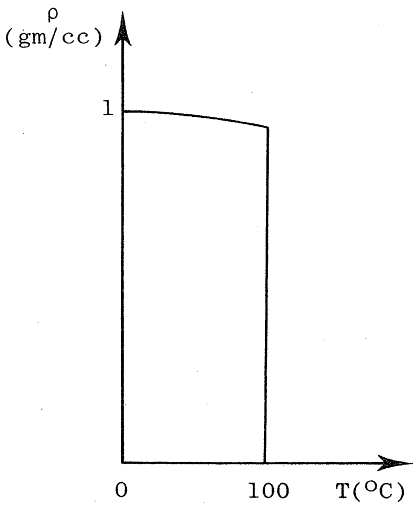

Figure 6

The boiling point is immediately obvious on this diagram because of the abnormal appearance of the graph there. This is an example of a catastrophe also.

The systematic study of such exceptions to the orderly behaviour of nature forms the subject matter of the new science of Catastrophe Theory. It is a very recent area of mathematics, which first burst into prominence with the publication of Thom's book *Structural Stability and Morphogenesis* in 1972.

The basic theorem of Catastrophe Theory gives a classification of the various types of catastrophe that can occur in certain important (but still far from completely general) situations. Its basic assumption is that the state of the system under study can be determined by maximising or minimising some suitable function. In statics, for example, the configuration achieved *minimises potential energy*; in dynamics, a quantity termed *action* *is minimised*; in thermodynamics, *entropy is maximised*; in geometrical optics, the *total travel time of a light ray is minimised*; in many areas of genetics, *mean population fitness is maximised*, and in managerial science, many firms seek to *maximise profit*. (It is the thermodynamic case that covers our example of boiling water.)

A catastrophe occurs in such a context when the system adjusts from one maximum or minimum to another, or when a previously available maximum or minimum suddenly ceases to exist.

The very simplest catastrophe is the one we saw earlier with the stretching cubic. The position of the minimum point P can be found by a straightforward differentiation. Its x-coordinate, x_P, is given by

$$x_P = \sqrt{\frac{-a}{3}} .$$

This gives the graph drawn in Figure 7. (The dotted line corresponds to the maximum in Figure 3.) The shape of this graph has led to catastrophes of this type being termed *fold catastrophes*, also referred to sometimes as *threshold catastrophes*.

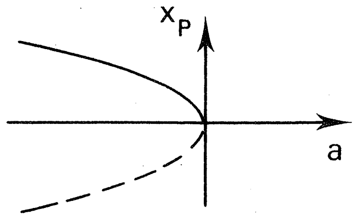

Figure 7

In this example, the value of x_P defines for us the state of the system. It is referred to as a *state variable*. The value of x_P depends upon the value of a. This latter is under our control, and is termed a *control variable*. The control variable (or more generally, variables) correspond to the input of Figure 1, while the state variable (or variables) may be thought of as the output.

All situations that involve a single control variable can, as a consequence of Thom's classification theorem, be approximated locally by a cubic. The fact that the approximation is *local* stops the particle P in Figure 5 from necessarily falling forever; it is allowed that another minimum exist in what my 5-year old son would call "the far distance". With this proviso, it is possible to set up an exact analogy between the boiling water and a variant of the stretching cubic.

However, we shall follow a different road here, for we are still oversimplifying the behaviour of the water. The density of a fluid depends not only on the temperature, but also on the pressure. Hence a full analysis uses two control variables (temperature and pressure) and one state variable (density).

According to Thom's theorem, this new situation can always be approximated locally by a quartic function

$$y = x^4 + ax^2 + bx, \qquad (*)$$

where the minima of y will give values of a state variable x_P and a, b are the control variables. (Can you show that all quartics can be reduced to the above form by suitable choice of scales and origin?)

Putting the derivative $\frac{dy}{dx}$ equal to zero, we find the stationary values of the quartic function $(*)^{\dagger}$. Denote by x_P the corresponding value(s) of x. Then

$$4x_P^3 + 2ax_P + b = 0.$$

This equation is a cubic in x_P. For some values of a, b, it will have three solutions (2 minima and a maximum), for others there will be only a single solution (which is a minimum). The graph of x_P as it depends upon a, b is the complicated-looking pleated surface shown in Figure 8. It is

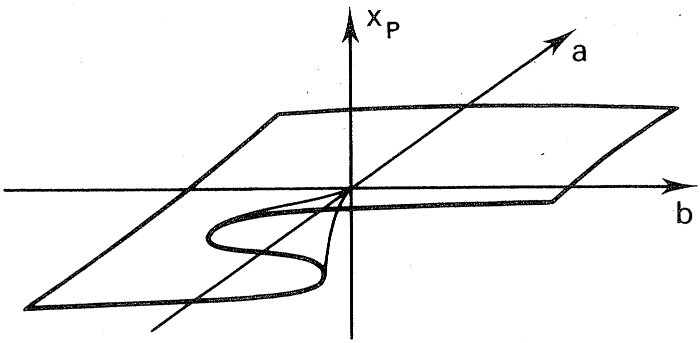

Figure 8

this figure which has become almost the trademark of Catastrophe Theory. (Although whether it is capable of supporting all the alleged applications being hung on it - from behaviour of dogs to stock exchange crashes, from fashions in curricula to optical caustics, from prison riots to falling in love - is another, and moot, point.)

The two different behaviours of the quartic function are separated by the curve

$$8a^3 + 27b^2 = 0,$$

$\dagger$ See Problem 2.1.

shown in Figure 9. When $8a^3 + 27b^2 < 0$, two minima exist (i.e. for values of (a, b) lying under the curve); in the case where the inequality is reversed, there is a single minimum. The point on the curve of Figure 9 is called a cusp and hence this catastrophe is termed the *cusp catastrophe*.

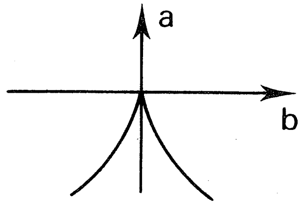

Figure 9

In our case of the boiling water, a quantity called "entropy" is maximised - or equivalently its negative, the negentropy, is minimised. The exact definitions of entropy and negentropy need not concern us here. All we need to know is that the negentropy can be locally approximated by a quartic (∗), where the control variables a, b are functions of temperature and pressure.

Now, for some quartics, such as that of Figure 10, there are two minima, while for others (as in Figure 11), there is only one.

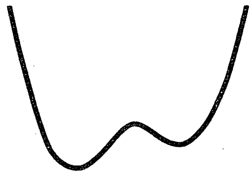

Figure 10

On the intuitive level, we might guess that where two minima are possible, these correspond to two possible states of our system (liquid and gaseous), while the unique minimum allows only one state to exist. In the case where two states are possible, *the one that is achieved is the one for which the minimum value is the less*. This rule is termed the Maxwell convention and it applies to our liquid-gas system. (Other cases, including mechanical systems like Figure 4, require somewhat different rules, which is why a direct analogy between boiling water and stretching cubics is not quite exact.)

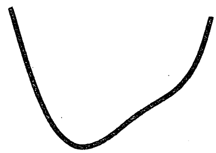
Figure 11

This allows us to make a number of predictions which are in fact borne out by experiment. First, there are critical values for p (the pressure) and for T, above which only one state can be observed and below which two are possible. Secondly, there are two ways to produce liquid-gas transitions: we might go smoothly "round the back" of the critical point P (see Figure 12) as we might come "round the front" and see a sudden change of state.

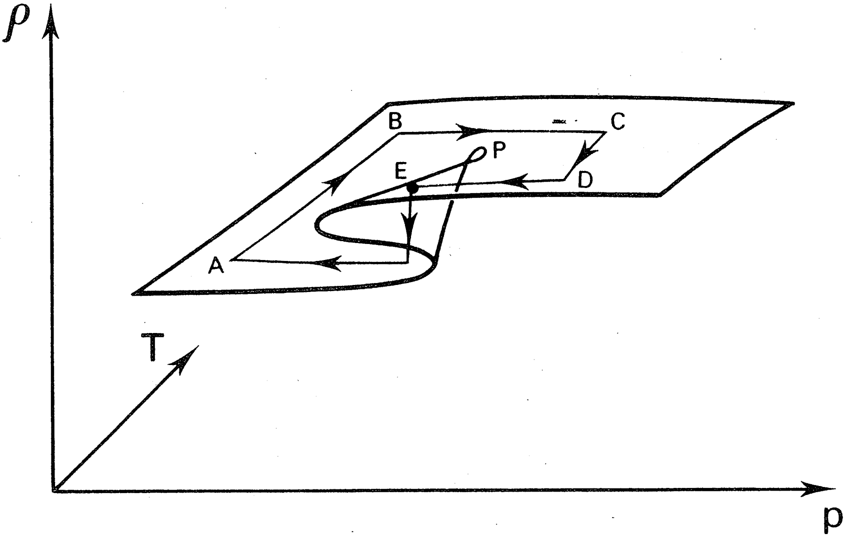

Figure 12

Figure 12 diagrams a situation in which a gas is first heated at constant pressure, then ~~expanded~~ compressed at constant temperature, then cooled back to its original temperature. This corresponds to the path *ABCD*. The point representing the state of the system has been moved from the lower surface to the upper, **i.e.** the system has changed from the gaseous to the liquid state.

The quartic negentropy function has gone through the sequence of shapes shown in Figure 13. The gas has become a liquid without appearing to liquefy in any sudden or dramatic way.

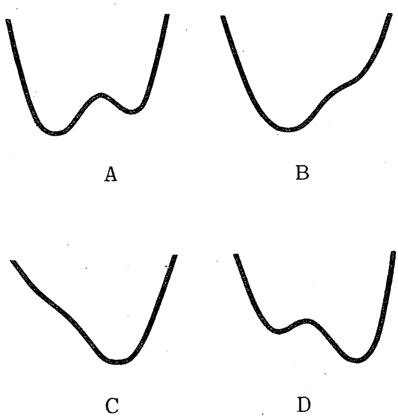

Figure 13

If we now, however, decrease the pressure, we reach a point *E* at which the liquid becomes unstable (the left-hand trough in Figure 13D increases in size until the corresponding minimum is the lower; so that Figure 13A is recovered).

There is then a sudden readjustment (boiling) and the system is once again represented by the original point A.

All this has actually been observed. There is an excellent account, in fact, in Chapter 5 of Glasstone and Lewis' *Elements of Physical Chemistry*, which goes into a lot more detail. The experiment requires fairly high pressures and is difficult (and dangerous) to do without special apparatus. Nevertheless the predictions of Catastrophe Theory correspond exactly to the experimental facts.

The mathematician who pointed out the relationship described above was D.H. Fowler, who also performed another service to the mathematical community when he translated *Structural Stability and Morphogenesis* from the original French. It was in this book that Thom stated his famous classification theorem, known as the *Theorem of the Seven.*

This result lists all the possible situations that arise for one, two, three or four control variables. There are exactly seven such cases. We have seen two already, and two more are easy to predict. These are

$$y = x^5 + ax^3 + bx^2 + cx$$

and

$$y = x^6 + ax^4 + bx^3 + cx^2 + dx$$

These have one state variable and three and four control variables respectively.

The other cases involve two state variables and are more complicated. Remarkably, there are no cases at all involving more than two state variables.

A complete list of the seven catastrophes is given in the table opposite. This lists the state and the control variables for each catastrophe and then gives the function of these that is to be minimised (or maximised). The names that follow are used as reference labels. Finally, some of the applications or hoped-for applications are listed.

The Theorem of the Seven was not fully proved by Thom. In fact the first complete proof was published as recently as 1974. More recently still, extensions have been proved for up to 16 control variables. In the main, however, the further cases that emerge are less important for applications.

From this, it will be seen that Catastrophe Theory is a very new branch of mathematics. It is a very exciting one, with wide-ranging applications and great intrinsic mathematical interest. If pursued in its full detail, it is still a difficult subject. However, a number of good expositions have appeared. If you want to learn more of this subject, you could begin with Christopher Zeeman's article *Catastrophe Theory* (Scientific American, April 1976), Ian Stewart's *The Seven Elementary Catastrophes* (New Scientist, 20 November 1976) or pages 277-285 of Stewart's Pelican book *Concepts of Modern Mathematics*. Stewart is currently writing a Pelican book on Catastrophe

State Variable(s)	Control Variable(s)	Function to be Minimised	Name	Application
x	a	$x^3 + ax$	Fold	All simple threshol phenomena.
x	a, b	$x^4 + ax^2 + bx$	Cusp	Very many; see text
x	a, b, c	$x^5 + ax^3 + bx^2 + cx$	Swallowtail	Embryology; the psychology of drunk driving.
x	a, b, c, k	$x^6 + ax^4 + bx^3 + cx^2 + kx$	Butterfly	Treatment of *Anorex Nervosa*; internatio disputes.
x_1, x_2	a, b, c	$x_1^3 - 3x_1x_2^2 + a(x_1^2 + x_1^2) + bx_1 + cx_2$	Elliptic Umbilic	Speculative applica in hydrodynamics an embryology.
x_1, x_2	a, b, c	$x_1^3 + x_2^3 + ax_1x_2 + bx_1 + cx_2$	Hyperbolic Umbilic	Buckling of stiffen panels.
x_1, x_2	a, b, c, k	$x_1^2x_2 + x_2^4 + ax_1^2 + bx_2^2 + cx_1 + kx_2$	Parabolic Umbilic	Speculative applica in linguistics and embryology.