Kenichi Miura's Water Wheel
or
The Dance of the Shapes of Constant Width

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1 The water wheel

One of the many memorable presentations at the 2010 Gathering for Gardner was that by Kenichi Miura. In it he reported about a fun idea of his that Martin Gardner would have loved to have heard about: a water wheel with buckets in the shape of Reuleaux triangles; Figure 1 is an example. In this water wheel water is being added from the top. This water then drains out of the buckets on the ways down, through holes in the sides facing us (not shown).[1] This water wheel has the remarkable property that, as the wheel turns, any two adjacent buckets always touch in a single point while maintaining their downward orientation.

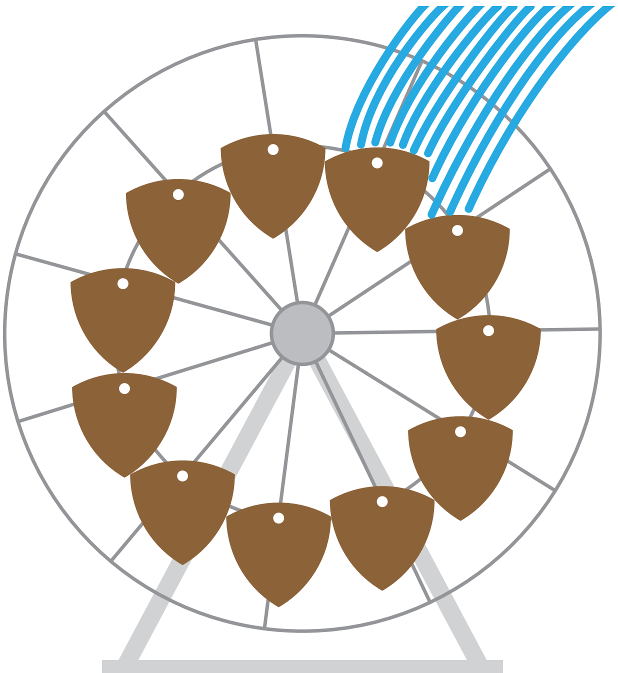

Figure 1: A water wheel with a circle of ever-touching buckets.

One practical consequence of this property mentioned by Kenichi Miura in his talk is that, as water is added from the top, at no time will it fall through any gap in between the buckets. However, much more exciting for me was that this idea represented a striking new way of putting the famous Reuleaux triangle to work.

Reuleaux triangles are the simplest examples of plane shapes of constant width that are not circles, and we'll show that, in fact, water wheels like

[1] Another incarnation of the same idea would be a Ferris wheel, with the cabins playing the role of the buckets.

Kenichi Miura's can be built using any shape of constant width.

A number of real-life gadgets exhibiting the surprising counterintuitive properties of shapes of constant width have been devised that feature prominently in real-life and virtual math exhibitions: non-circular rollers, square drills, carts with non-circular wheels, etc.

In the following we'll quickly summarize the main properties of shapes of constant widths and then proceed to exhibit some new interesting gadgets and animations that were inspired by Kenichi Miura's water wheel.

Before you read on, please check out `www.qedcat.com/waterwheel.html` *for animations of the water wheel and some of the other gadgets described on the following pages.*

2 Shapes of constant width

The width of a compact plane shape in a certain given direction is the width of the thinnest infinite parallel strip perpendicular to that direction and containing the shape. A *shape of constant width (SCW)* is a compact convex plane set whose width is the same in all directions.

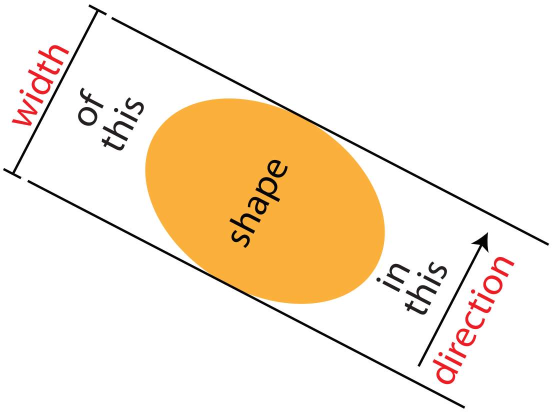

Figure 2: The width of a shape in a certain given direction.

Obviously any circle has constant width. However, contrary to intuition there are also non-circular SCWs. The simplest examples are the *Reuleaux triangles* which are constructed from equilateral triangles, as indicated in Figure 3.

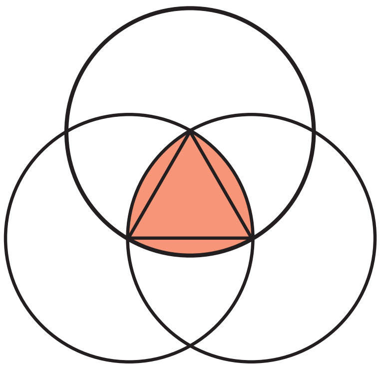

Figure 3: Constructing a Reuleaux triangle from an equilateral triangle.

Just like circles and Reuleaux triangles, all SCWs turn out to be *strictly convex*, which means that their boundaries do not contain line segments. For many beautiful, elementary constructions and real-world applications of SCWs see, for example, [2, Chapter 10]. Unless indicated otherwise, all the basic results about shapes of constant width mentioned in the following can be found in [5, Chapter 7]. Of course, there is also a wealth of interesting and relevant information about shapes of constant width available online.

Turn! Turn! Turn!

To effectively exhibit the defining property of SCWs, you can make rollers with cross sections that are SCWs of the same width to transport an object smoothly along them. Another immediate consequence of the defining property of SCWs is that any SCW of width w can be turned around inside a square of side length w such that each of the sides of the square is always in contact with exactly one point of the SCW; see Figure 4.

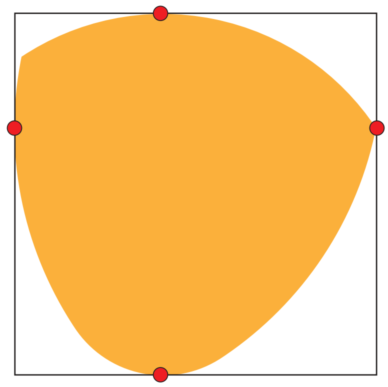

Figure 4: A SCW can be rotated inside a square. Throughout the rotation each side of square is always in contact with the shape.

To be able to explain the curious touching buckets of our water wheel, let's first have a closer look at a SCW inside its surrounding square, as shown in the left diagram in Figure 5.

It is well-known that no matter how the SCW is oriented, its two contact points with the vertical sides of the square are always connected by a horizontal line segment.[2] These connecting intervals also turn out to be the *diameters* of our SCW. This means that the connecting intervals are the longest intervals contained in the shape. The diagram on the right shows a few of these diameters.

Here is a useful (animated) picture to keep in mind: If a SCW has width w, we can rotate an interval of length w inside it in exactly one way, with the interval coinciding with one of the diameters at each moment of its rotation.

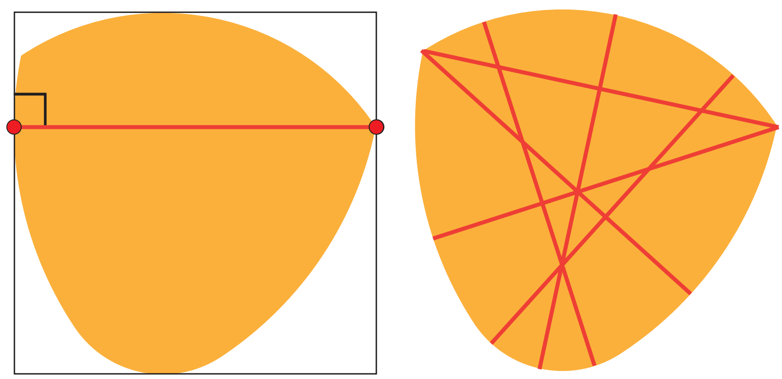

Figure 5: The interval/diameter connecting the contact points with the two vertical sides of the square is always horizontal (left). A few diameters of our SCW (right).

Let's put two copies of the diagram on the left next to each other and let's imagine that the two SCWs are rotating in unison inside their respective square; see Figure 6. Then it is clear from what we just said that the two shapes will always touch in exactly one point (the blue point).

We've also highlighted some green points in corresponding positions inside the shapes. These are supposed to be part of the shapes and are supposed to rotate together with the shapes. It is clear that they are always exactly the width of the shape apart.

Now, let's look at the same scenario, but from within a frame of reference in which the two green points are fixed. Then what you see is the two shapes rotating in unison, each around its associated green point and, of course

[2]And, of course, the corresponding statement is true for the contact points of the shape with the horizontal sides of the squares.

always still touching in exactly one point at all times.

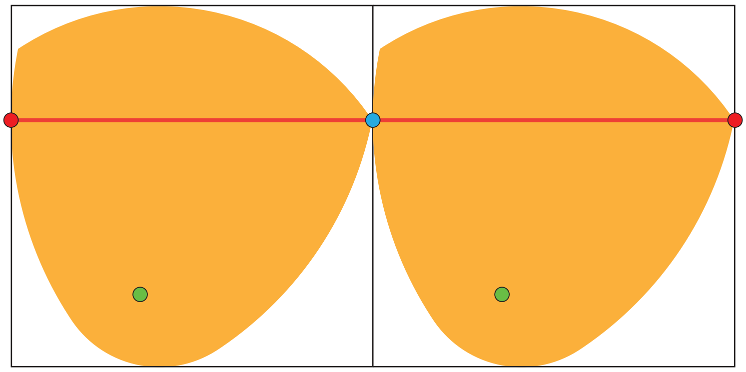

Figure 6: As the two (translated) copies of the SCW rotate in unison in their respective surrounding square, the rotating SCWs always touch at exactly one point.

This then implies the following *touching property* of SCWs: *Take any SCW and fix a* center *(the green dot in the example above). Let's call this setup a centered SCW. Note that the center can be anywhere in the plane and does not even have to be inside the shape. Make a copy of the centered SCW and translate it in some direction by the width of the shape. Start spinning the two centered SCWs in unison around their respective centers. Then the two shapes will touch in exactly one point at all times.*

We can now arrange multiple copies of a centered SCW into "dance" formations in which the dancers will touch at all times while they spin in unison. For example, in Figure 7 we show three instances of such a dance.

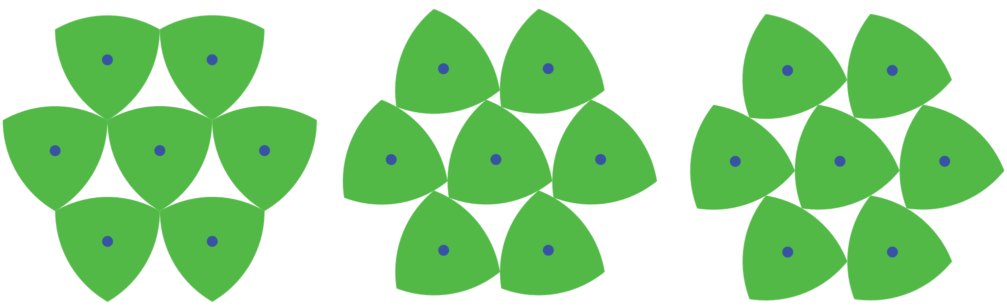

Figure 7: Three instances of a dance performed by seven Reuleaux triangles.

Using the touching property we can also explain the kissing buckets of Kenichi Miura's water wheel. We simply start with a centered Reuleaux triangle and make up a regular n-gon with vertices that are exactly the width of the Reuleaux triangle apart. Then we make n copies of this centered triangle and translate them such that their centers come to coincide with the vertices of the regular n-gon; see Figure 8.

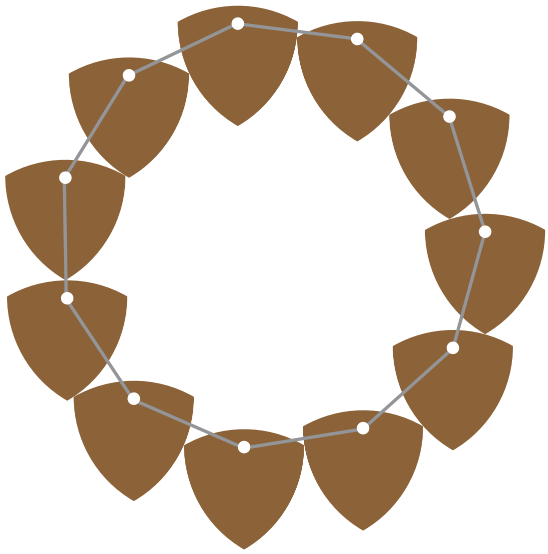

Figure 8: Constructing the water wheel.

Now the touching property of SCWs guarantees that, as we spin the Reuleaux triangles around their white centers, adjacent triangles will touch at all times. Of course, if we rotate the n-gon at the same time to compensate for the rotation of the Reuleaux buckets nothing will change in terms of touching, and we end up seeing in front of us the real movement of the water wheel.

Parallel shapes

Associated with any convex compact set are its (outer) parallel shapes, one for each positive real number r. The parallel shape at distance r is the shape bounded by the outer envelope of all the circles of radius r centered on the boundary of the SCW; see the left diagram in Figure 9.

If the convex compact set is a SCW, an alternative way of producing the same parallel shape is to simply extend all diameters of the SCW by a distance R on both ends. Then the end points of these extended diameters form the boundary of our parallel shape; see the right diagram in Figure 9.

Our parallel shape turns out to be a SCW as well and the extended diameters that we used to produce it are just its diameters.

Now, imagine a SCW and one of its parallel shapes glued together while the SCW is spinning it its square (keep looking at the right diagram in Figure 9). The construction of our parallel shape makes it clear that:

(1) The midpoints of the surrounding squares of the parallel shapes and the original SCW coincide. (2) The original and the parallel shapes can rotate in unison in their respective squares. As they do so, all points of contact with the vertical sides of the squares are contained in a horizontal line.

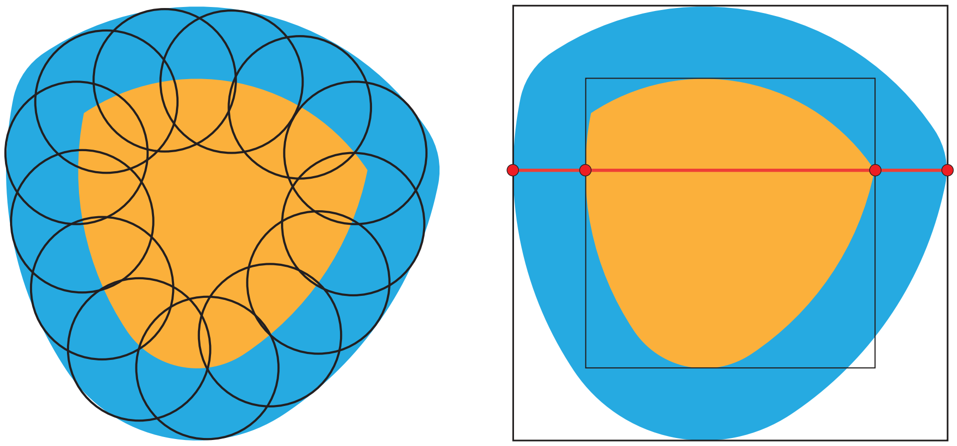

Figure 9: All parallel shapes touch the vertical sides of their surrounding squares at the same height.

Let's choose a center common to the original SCW and its parallel shapes; see the left diagram in Figure 10. Now, we separate things by translating the original SCW to the left until it touches the parallel shape as in the diagram on the right.

It is clear that as we simultaneously rotate the SCW and its parallel shape in their squares, they will always touch in exactly one point. Furthermore, their centers will always stay a constant distance apart.

Therefore we can conclude, as in the previous section, that *if a SCW and one of its parallel shapes touch, then they will do so at all times when rotated in unison around their respective centers.*

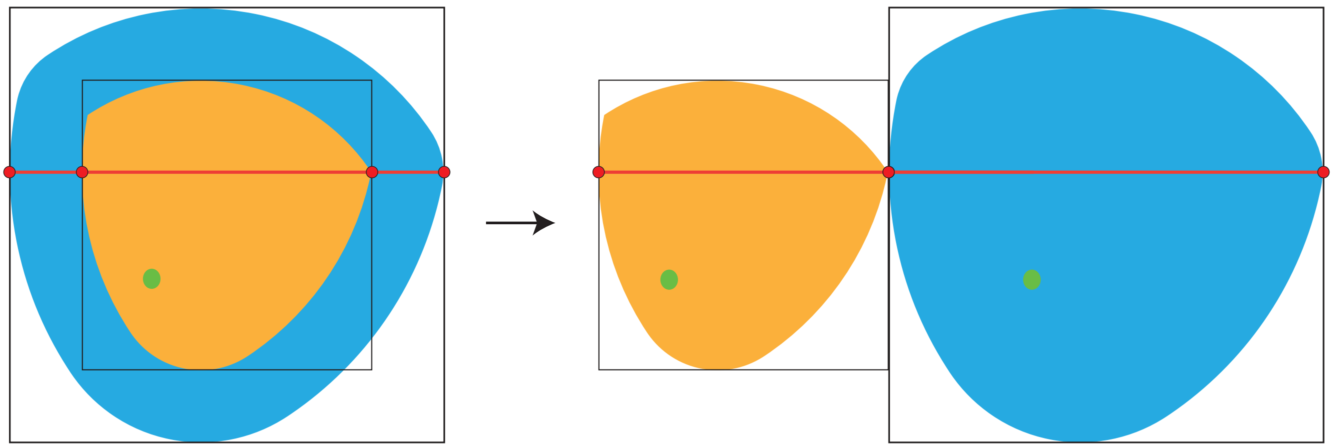

Figure 10: A centered SCW and one of its parallel shapes, ready for a dance.

Again we can arrange parallel shapes of a given SCW in lots of interesting dance formations. For example, in Figure 11 we've surrounded a Reuleaux triangle with 6, 5, 4, and 3 parallel shapes of just the right sizes to guarantee a maximum number of contacts throughout their dances.

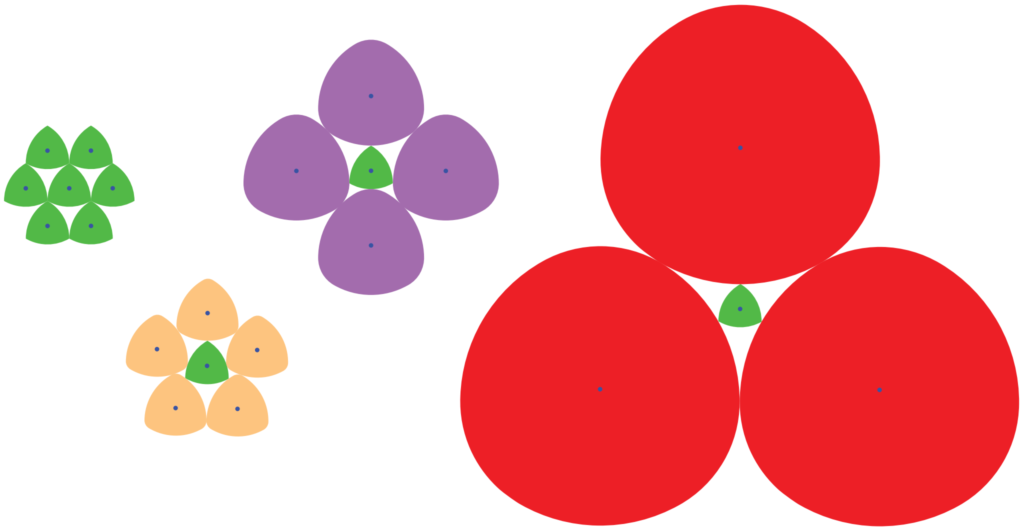

Figure 11: A centered Reuleaux triangle surrounded by different groups of its parallel shapes poised for some mesmerizing dance performances.

Here is another intriguing observation. Returning to our water wheel, we suspend another bucket at the axis of the water wheel; see Figure 12. Then there are two parallel curves of this triangle that all buckets touch at all times. Can you see why this is the case?

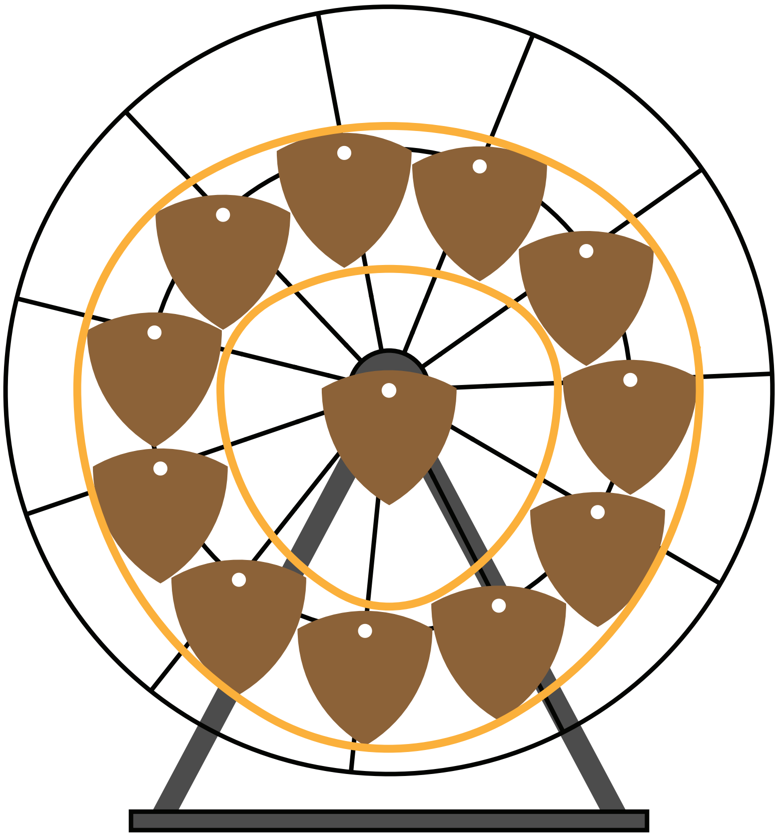

Figure 12: The buckets of the water wheel touch the two parallel curves of the Reuleaux triangle suspended at the axis of the wheel.

2.1 Non-circular wheels

In their 2005 article *Reinventing the Wheel: Non-Circular Wheels* [4] Claudia Masferrer León and Sebastián von Wuthenau Mayer, presented their beautifully simple idea for fitting non-circular wheels to a cart to achieve a smooth ride. Figure 13 shows how this works. One of the wheels consists of a Reuleaux triangle glued together with one of its parallel shapes. Then, when the triangles are restricted to rotating inside their associated squares, the top of the car will smoothly move along a horizontal without up or down movement.

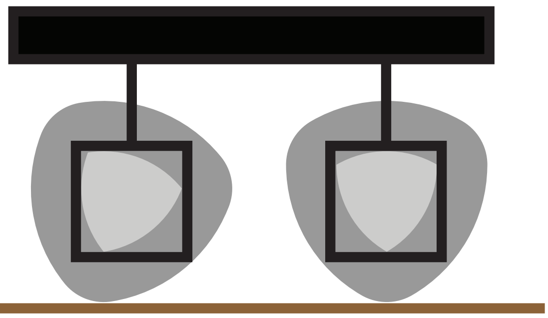

Figure 13: A cart with non-circular wheels that avoids up and down movements.

With the touching property of SCWs it is immediately clear that we can arrange for the wheels to touch at all times. We may even build a tank with caterpillar tracks whose supporting wheels form a touching chain of these strange wheels.

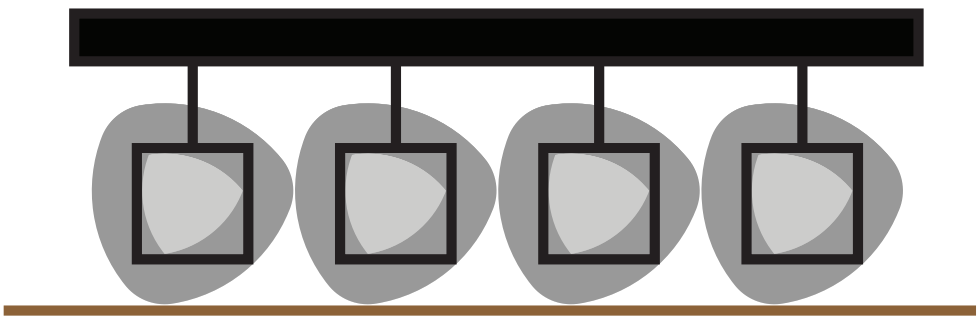

Figure 14: Touching wheels.

In general, all the tricks that were possible with rotating SCWs around their centers are also possible with parallel shapes that rotate in this new way. To make sure that corresponding points of the different rotating SCWs always are the same distance apart, we just have to make sure that all the square hubs are oriented the same way. For example, Figure 15 shows a new water wheel using this kind of “square suspension”.

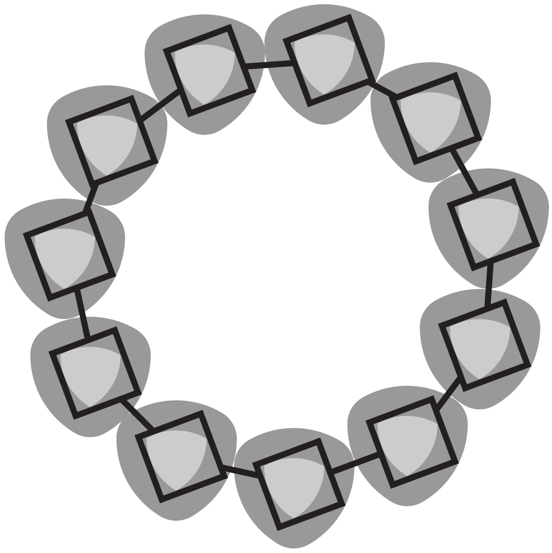

Figure 15: A water wheel with the buckets suspended on square hubs.

However, with the new suspension it is also possible to perform some “touching” tricks that are not possible with the simpler kind of suspension that we used before.

As an example consider the gadget shown in Figure 16 on the right which, in terms of crazyness, is at least one level up from the “standard” multiwheel unicycle shown on the left. In it, instead of all wheels rotating in the same direction, we actually have the one in the middle running in the opposite direction. So what we are dealing with is a simple transmission using non-circular sprockets.

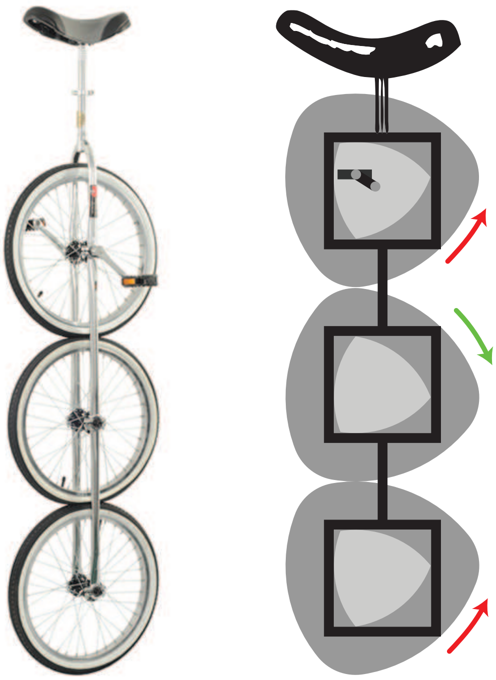

Figure 16: An unusual tall unicycle (left) and its even more unusual relative (right).

It is clear that for this transmission to work it is essential that: (1) the bottom and top parts of the wheels are at constant distance from the square hubs (this is automatically the case); and that (2) the square hubs are aligned such that corresponding vertical sides of the square hubs are part of the same lines.

Condition (2) is quite restrictive when it comes to putting together dance formations in which adjacent participants rotate in opposite directions (while always touching). Here some SCWs that rotate inside regular polygons show a way forward. The constructions of such special SCW is described in the [3], a very nice article about drilling holes in the shape of regular polygons using SCWs. For, example, Figure 17 shows a hexagonal "transmission" based on a SCW that rotates inside a regular hexagon.

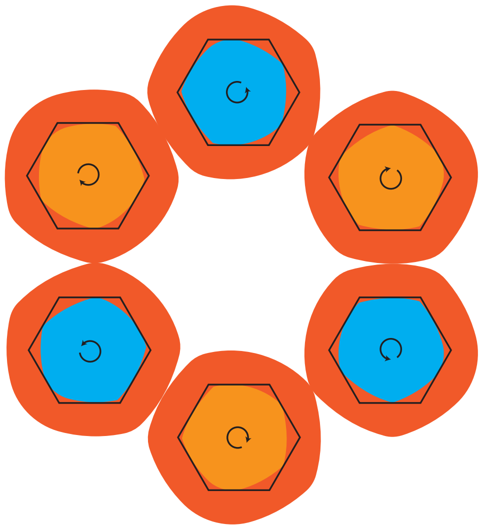

Figure 17: A hexagon-shaped transmission based on a SCW and its parallel curves.

Kenichi Miura once again

When I met Kenichi Miura again at the 2012 Gathering for Gardner he mentioned that nobody has built his water wheel yet. So, to end, here are two challenges for you: be the first to build Kenichi Miura's water wheel, or be the first to build and ride a multi-Reuleaux unicycle!

References

[1] Bogomolny, A. Article and java applet demonstrating a very general method of constructing shapes of constant width, `www.cut-the-knot.org/Curriculum/Geometry/CWStar.shtml`

[2] Bryant, J. and Sangwin, C. *How Round Is Your Circle?* Princeton University Press, Princeton, 2008.

[3] Cox, B. and Wagon, S. Mechanical Circle-Squaring, *The College Mathematics Journal* **40** (2009), 238–247.

[4] Masferrer León, C. and Von Wuthenau Mayer, S. Reinventing the Wheel: Non-Circular Wheels, *The Mathematical Intelligencer* **27** (2005), 7–13.

[5] Yaglom, I.M. and Blotyanskii, V.G. *Convex Figures*, Holt, Rinehart and Winston, New York, 1961.