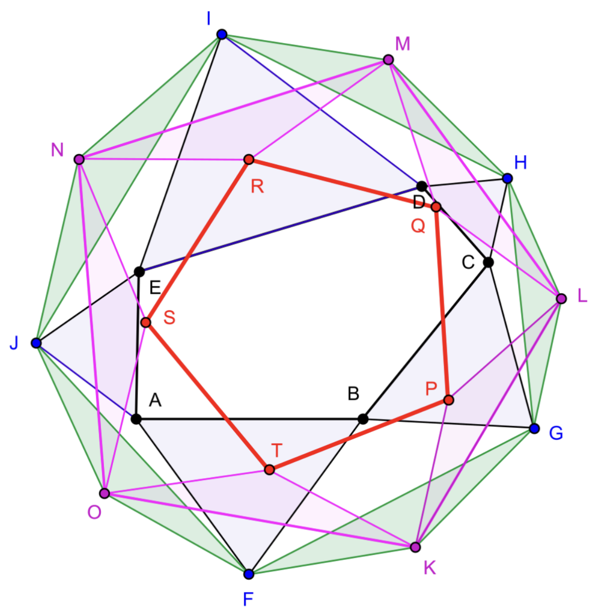
The Petr-Douglas-Neumann Theorem [1, 2, 3]. For any closed planar n-sided polygon P_0, define a sequence of closed n-gons $P_1, \ldots, P_{n-2}$, such that the vertices of the jth polygon P_j are the apices of the isosceles triangles placed on the exterior sides of the previous one P_{j-1}, with apex angles α_{σ_j} chosen to be a multiple, $\frac{2\pi\sigma_j}{n}$, of $\frac{2\pi}{n}$, where $\{\sigma_j\}_{j=1,\ldots,n-2}$ is any permutation of the numbers $(1, \ldots, n-2)$. Then P_{n-2} is a convex regular n-gon, whose centroid coincides with that of P_0 and all the other polygons $P_1, \ldots, P_{n-3}$ in the sequence.

The diagram below is for $n = 5$, and illustrates the theorem for pentagons, with σ chosen as the identity permutation: $\{\sigma_j = j\}_{j=1,2,3}$. (See ref. [4].)



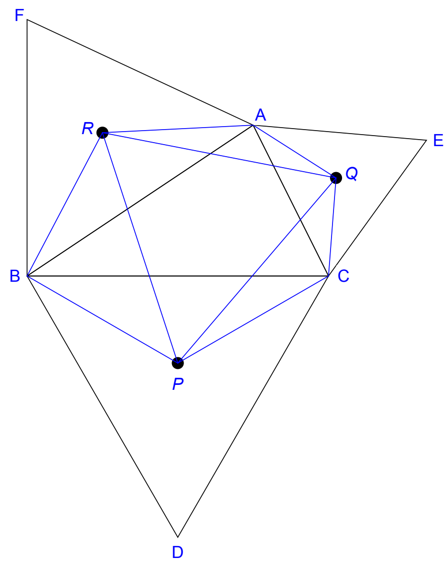
Petr-Neumann-Douglas Theorem applied to a pentagon

ABCDE : Arbitrary pentagon

PQRST : Generated regular pentagon

The initial pentagon P_0 is $ABCDE$. $P_1 = FGHIJ$ is constructed by adding five isosceles triangles with apex angles $\frac{2\pi}{5} = 72°$ onto the sides of P_0. P_2, is then constructed from P_1 by adding isosceles triangles with apex angles $\frac{4\pi}{5} = 144°$ onto the sides of P_1 and $P_3 = PQRST$ by adding isosceles triangles with apex angles $\frac{6\pi}{5} = 216°$ onto the sides of P_2, resulting in a regular pentagon.

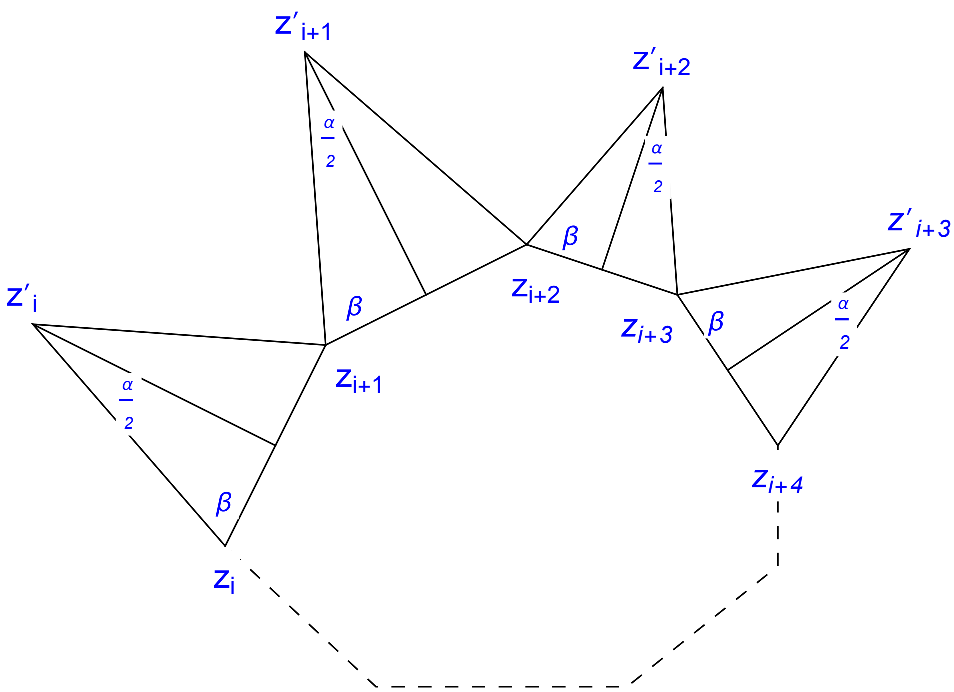
The case $n = 3$, where P_0 is a triangle, reduces to **Napoleon's theorem**, which says that: **The centroids of the external equilateral triangles erected on an arbitrary triangle are the vertices of an equilateral triangle**. This is illustrated in the following diagram, where $P_0 = ABC$ is the original triangle and $P_1 = PQR$ is the triangle formed from the centroids P, Q, R of the external equilateral triangles on each of its sides.



The isosceles triangles of the Petr-Douglas-Theorem join the pair of vertices bounding each side of ABC to the corresponding centroid, and the apex angles are $\frac{2\pi}{3} = 120°$.

The general case is illustrated in the next figure, which shows four of the eight sides of an octagon, and the isosceles triangles placed on them have apex angles $\frac{2\pi}{8} = \pi/4$. (The indices σ_j on the apex angles α_{σ_j} and base angles β_{σ_j} have been omitted for simplicity.) The perpendicular bisectors of each of the sides is joined to the apices of the triangles, dividing them into a pair of oppositely oriented congruent right triangles whose angles (restoring the indices) $(\frac{\alpha_{\sigma_j}}{2}, \beta_{\sigma_j}, \frac{\pi}{2})$ add up to π, so

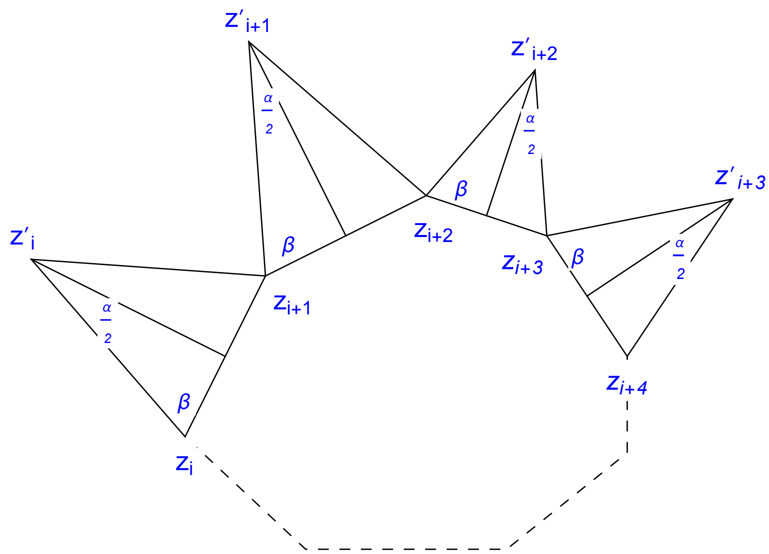
$$\beta_{\sigma_j} = \frac{\pi - \alpha_{\sigma_j}}{2}.$$



Proof: We proceed by first identifying the plane as the complex plane, relative to an arbitrarily chosen origin and Cartesian coordinate system, in which (x, y) denote the real and imaginary parts of complex numbers $z = x + iy$. The vertices of P_0 are identified as a cyclic sequence of complex numbers $(z_1, \dots, z_n, z_{n+1} := z_1)$, in which all successive triplets (z_i, z_{i+1}, z_{i+2}) are distinct, and the orientation corresponds to increasing indices. We similarly identify the successive vertices of the n-gon P_j, for $j = 1, \dots, n-2$, with a cyclic sequence of complex numbers $(z_1^j, \dots, z_n^j, z_{n+1}^j := z_1^j)$ and set $(z_1^0, \dots, z_n^0, z_{n+1}^0)$ $:=(z_1, \dots, z_n, z_{n+1})$ for P_0. By convention, we define the “exterior” side of each oriented edge (z_i^j, z_{i+1}^j) to be on its left. (If the apex angle α_{σ_j} is a reflex angle however, this places the apex vertex z_i^{j+1} on the opposite side.) The isosceles triangle whose base is (z_i^j, z_{i+1}^j) thus has vertices $(z_i^{j+1}, z_i^j, z_{i+1}^j)$. For $j = 1, \ldots, n$, let

$$\alpha_j := \frac{2\pi j}{n}, \quad e^{i\alpha_j} := \omega^j, \quad \omega := e^{2\pi i/n},$$

where ω is the primitive nth root of unity and $\{\omega^j\}_{j=1,\ldots,n}$ is the complete set of nth roots of unity. The apex angle of the triangle $(z_i^{j+1}, z_i^j, z_{i+1}^j)$ is thus $\alpha_{\sigma_j} = \frac{2\pi\sigma_j}{n}$.



The apex vertex z_i^{j+1} is obtained from the pair (z_i^j, z_{i+1}^j) of base vertices by rotating the difference $z_{i+1}^j - z_i^j$, viewed as a 2-dimensional vector, by the angle β_{σ_j} in the counterclockwise direction, keeping the point z_i^j fixed. This means: multiplying $z_{i+1}^j - z_i^j$ by the unit length complex number $e^{i\beta_{\sigma_j}}$, dividing by the factor $2\cos\beta_{\sigma_j}$ to obtain the correct length $|z_i^{j+1} - z_i^j|$ for the hypotenuse of the right triangle formed from the vertices (z_i^{j+1}, z_i^j) and the midpoint of the side (z_i^j, z_{i+1}^j), and adding this to z_i^j, to give

$$z_i^{j+1} = z_i^j + \frac{e^{i\beta_{\sigma_j}}(z_{i+1}^j - z_i^j)}{2\cos\beta_{\sigma_j}} = \frac{1}{1-\omega^{\sigma_j}} z_{i+1}^j - \frac{\omega^{\sigma_j}}{1-\omega^{\sigma_j}} z_i^j. \qquad (*)$$

The second equality follows from

$$\cos\beta_{\sigma_j} = \frac{e^{i\beta_{\sigma_j}} + e^{-i\beta_{\sigma_j}}}{2}$$

and

$$e^{-2i\beta_{\sigma_j}} = -e^{i\alpha_{\sigma_j}} = -\omega^{\sigma_j}.$$

The equation

$$z_i^{j+1} = \frac{1}{1-\omega^{\sigma_j}} z_{i+1}^j - \frac{\omega^{\sigma_j}}{1-\omega^{\sigma_j}} z_i^j, \qquad (*)$$

can be more simply expressed as a single n-component vector equation as follows. For $j = 1, \dots, n-2$, let $\mathbf{z}^j \in \mathbf{C}^n$ be the complex n-component column vector

$$\mathbf{z}^j := \begin{pmatrix} z_1^j \\ \vdots \\ z_n^j \end{pmatrix}, \quad \mathbf{z}^0 := \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} =: \mathbf{z},$$

and

$$\Lambda := \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \dots & \dots & \dots & \dots & 1 \\ 1 & 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

the cyclic permutation matrix, which satisfies

$$\Lambda^n = \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. Then the equation (*) can be written in n-component vector form as

$$\mathbf{z}^{j+1} = \frac{1}{1-\omega^{\sigma_j}} (\Lambda - \omega^{\sigma_j}\mathbf{I})\, \mathbf{z}^j. \qquad (*)$$

Composing these successively for $j = 0, \dots, n-3$ gives

$$\mathbf{z}^{n-2} = \prod_{j=1}^{n-2} \left(\frac{1}{1-\omega^{\sigma_j}} \right) (\Lambda - \omega^{\sigma_j}\mathbf{I})\, \mathbf{z}.$$

Whatever the permutation σ: $(1, \dots, n-2) \to (\sigma_1, \dots, \sigma_{n-2})$ chosen, this can be written simply as

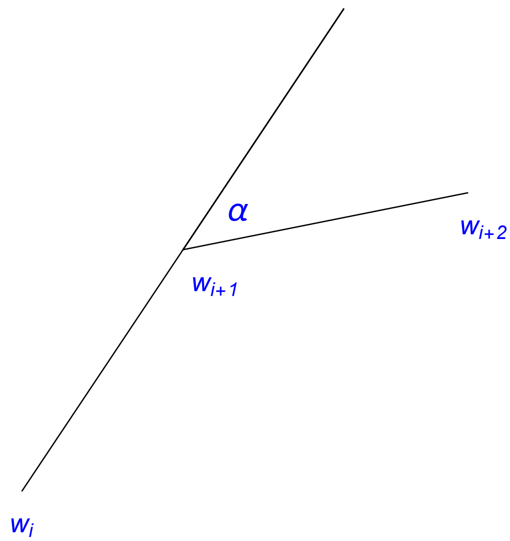
$$\mathbf{z}^{n-2} = \prod_{j=1}^{n-2} \left(\frac{1}{1-\omega^j} \right) \prod_{j=1}^{n-2} (\Lambda - \omega^j \mathbf{I})\, \mathbf{z}, \qquad (**)$$

since all the factors commute.

We now proceed to proving the regularity of the last polygon P_{n-2}. Let

$$\mathbf{w} := \mathbf{z}^{n-2} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

denote the vector with components equal to the vertices of P_{n-2}.



As illustrated in the figure, the condition, for all $i = 1, \dots n$, that (w_i, w_{i+1}, w_{i+2}) be the successive vertices of a regular polygon is equivalent to requiring that a rotation of the side (w_i, w_{i+1}) clockwise by the angle $\alpha := 2\pi/n$ gives the next side (w_{i+1}, w_{i+2}). Rotation of (w_i, w_{i+1}) by α in the clockwise direction is equivalent to multiplying the complex number $w_{i+1} - w_i$ by $\omega^{-1} = e^{-i\alpha}$, so the property of regularity is expressed by the equations

$$w_{i+2} - w_{i+1} = \omega^{-1}(w_{i+1} - w_i), \quad i = 1, \dots n, \quad (\text{with } w_{n+1} := w_1, w_{n+2} := w_2).$$

In terms of the vector of vertices $\mathbf{w}$ and the cyclic permutation matrix Λ, this means

$$\Lambda^2 \mathbf{w} - \Lambda \mathbf{w} - \omega^{-1} \Lambda \mathbf{w} + \omega^{-1} \mathbf{w} = 0,$$

or equivalently

$$(\Lambda - \mathbf{I})(\Lambda - \omega^{-1} \mathbf{I})\, \mathbf{w} = 0. \qquad (***)$$

Since $\omega^n = 1$ and $\omega^{n-1} = \omega^{-1}$, the condition (***) of regularity is equivalent to

$$(\Lambda - \omega^n \mathbf{I})(\Lambda - \omega^{n-1}\mathbf{I})\,\mathbf{w} = 0. \qquad (***)$$

To prove that the components of $\mathbf{z}^{n-2}$ are the vertices of a regular polygon, we must therefore show that (***) holds for all $\mathbf{w}$ of the form

$$\mathbf{w} = \mathbf{z}^{n-2} = \prod_{j=1}^{n-2}\left(\frac{1}{1-\omega^j}\right)\prod_{j=1}^{n-2}(\Lambda - \omega^j \mathbf{I})\,\mathbf{z}. \qquad (**)$$

Combining (***) with (**) gives

$$\prod_{j=1}^{n-2}\left(\frac{1}{1-\omega^j}\right)\prod_{j=1}^{n}(\Lambda - \omega^j \mathbf{I})\,\mathbf{z} = 0,$$

which indeed is satisfied, because the polynomial identity

$$z^n - 1 = \prod_{j=1}^{n}(z - \omega^j)$$

expressing $z^n - 1$ as a product of its elementary factors also holds if z is replaced by the matrix Λ, and 1 by the identity matrix $\mathbf{I}$, giving

$$\prod_{j=1}^{n}(\Lambda - \omega^j \mathbf{I}) = \Lambda^n - \mathbf{I} = 0,$$

since

$$\Lambda^n = \mathbf{I}.$$

To compute the centroids $(c_{\mathbf{z}^0}, \dots, c_{\mathbf{z}^{n-2}})$ of the polygons $P_0, \dots, P_{n-2}$, recall that, as complex numbers, these are just the average values of the vertices. This means they are obtained by multiplying the corresponding column vectors $\{\mathbf{z}^j\}_{j=0,\dots,n-2}$ on the left by the row vector $E := \frac{1}{n}(1, \dots, 1)$

$$c_{\mathbf{z}^j} = E\mathbf{z}^j.$$

Since E is invariant under any permutation of its entries, it is a left eigenvector of the cyclic permutation matrix Λ with eigenvalue 1

$$E\Lambda = E.$$

Therefore, multiplying the recursion relations

$$\mathbf{z}^{j+1} = \frac{1}{1-\omega^{\sigma_j}}(\Lambda - \omega^{\sigma_j}\mathbf{I})\mathbf{z}^j, \quad j = 0, \dots, n = 3 \qquad (*)$$

for the successive vectors of vertices on the left by E gives

$$c_{\mathbf{z}^{j+1}} = E\mathbf{z}^{j+1} = \frac{1}{1-\omega^{\sigma_j}}(E - \omega^{\sigma_j}E)\mathbf{z}^j = \frac{1-\omega^{\sigma_j}}{1-\omega^{\sigma_j}}E\mathbf{z}^j = c_{\mathbf{z}^j},$$

proving that all the centroids are equal.

Remark. A slight generalization of this result is also true, extending it beyond convex regular n-gons, to include regular n-grams, with vertex angles $2(n-p)\pi/n$, for $p = 1, \dots, n-1$.

The generalized Petr-Douglas-Neumann Theorem [2].

For any closed planar n-sided polygon P_0, define a sequence of closed n-gons $P_1, \dots, P_{n-2}$, such that the vertices of the jth polygon P_j are the apices of the isosceles triangles placed on the exterior sides of the previous one P_{j-1}, with apex angles α_{σ_j} chosen to be a multiple, $\frac{2\pi\sigma_j}{n}$, of $\frac{2\pi}{n}$, where $\{\sigma_j\}_{j=1,\dots,n-2}$ is any permutation of the numbers $(1, \dots, \widehat{n-p}, \dots n-1)$, with $\widehat{n-p}$ omitted, for $p = 1, \dots n-1$. Then P_{n-2} is a regular n-gram, with apex angles equal to $2(n-p)\pi/n$, whose centroid coincides with that of P_0 and all the other polygons $P_1, \dots, P_{n-3}$ in the sequence.

Proof: The proof is almost exactly the same; the only change is that eq. (**) is replaced by

$$\mathbf{w} = \mathbf{z}^{n-2} = \prod_{\substack{j=1 \\ j\neq n-p}}^{n-1} \left(\frac{1}{1-\omega^j}\right)\left(\Lambda - \omega^j \mathbf{I}\right) \mathbf{z}. \qquad (**')$$

and (***) is replaced by

$$(\Lambda - \omega^n \mathbf{I})(\Lambda - \omega^{n-p}\mathbf{I})\,\mathbf{w} = 0. \qquad (***')$$

References

[1] K. Petr, "Ein Satz über Vielecke", *Arch. Math. Phys.* **13**, 29-31 (1908).

[2] J. Douglas, "On linear polygon transformations", *Bull. Amer. Math. Soc.* **46** (6), 551-561 (1940).

[3] B.H. Neumann, "Some remarks on polygons", *J. Lond. Math. Soc.* **s1-16** (4), 230-245 (1941).

[4] PDN pentagon diagram by: Krishnachandranvn, Wikipedia article: **Petr-Douglas-Neumann theorem**.